1. (a) Using \( p(z) = \sqrt{2 + \sin \pi z} \), the parameterization is given by

\[
\mathbf{r}(\theta, z) = \langle p(z) \cos \theta, p(z) \sin \theta, z \rangle
\]

with \( 0 \leq \theta \leq 2\pi \), \( 0 \leq z \leq 10 \). We calculate

\[
\mathbf{N}(\theta, z) = p(z) \langle \cos \theta, \sin \theta, -p'(z) \rangle.
\]

Since \( (p(z))^2 = 2 + \sin \pi z \), we have \( 2p(z)p'(z) = \pi \cos \pi z \) and so

\[
p(z)p'(z) = \frac{\pi}{2} \cos \pi z.
\]

Hence,

\[
|\mathbf{N}(\theta, z)| = \sqrt{(p(z))^2 + (p(z)p'(z))^2}
\]

\[
= \sqrt{2 + \sin \pi z + \frac{\pi^2}{4} \cos^2 \pi z}
\]

\[
= \frac{1}{2} \sqrt{8 + 4 \sin \pi z + \pi^2 \cos \pi z^2}.
\]

(b) On \( S \), we have \( f(x, y, z) = x^2 + y^2 = p(z)^2 \). So,

\[
I = \iiint_S f \, dS = \iint_D (p(z))^2 |\mathbf{N}(\theta, z)| \, dA(\theta, z)
\]

\[
= \int_0^{2\pi} \int_0^{10} (2 + \sin \pi z) \frac{1}{2} \sqrt{8 + 4 \sin \pi z + \pi^2 \cos \pi z^2} \, dz \, d\theta
\]

\[
= \pi \int_0^{10} (2 + \sin \pi z) \sqrt{8 + 4 \sin \pi z + \pi^2 \cos \pi z^2} \, dz.
\]

(c) Write \( \mathbf{F} = \langle x, y, z \rangle = \langle P, Q, R \rangle \). Using \( x = p(z) \cos \theta \) and \( y = p(z) \sin \theta \), we have

\[
\mathbf{N}(\theta, z) = \langle x, y, -(\pi/2) \cos \pi z \rangle = \langle N_1, N_2, N_3 \rangle
\]
and so
\[ PN_1 + QN_2 + RN_3 = x^2 + y^2 - \frac{\pi}{2} z \cos \pi z \]
\[ = p(z)^2 - \frac{\pi}{2} z \cos \pi z \]
\[ = 2 + \sin \pi z - \frac{\pi}{2} z \cos \pi z. \]

Hence,
\[ I = \iint_D PN_1 + QN_2 + RN_3 \, dA(\theta, z) = \int_0^{2\pi} \int_0^{10} 2 + \sin \pi z - \frac{\pi}{2} z \cos \pi z \, dz \, d\theta. \]

(d) We have
\[ I = 2\pi \int_0^{10} 2 + \sin \pi z - \frac{\pi}{2} z \cos \pi z \, dz. \]

Since
\[ \int_0^{10} 2 + \sin \pi z \, dz = 2z - \frac{1}{\pi} \cos \pi z \bigg|_0^{10} = \left(20 - \frac{1}{\pi}\right) - \left(0 - \frac{1}{\pi}\right) = 20 \]
and since
\[ \int_0^{10} z \cos \pi z \, dz = \frac{1}{\pi^2} (\cos \pi z + \pi z \sin \pi z) \bigg|_0^{10} = \frac{1}{\pi^2} \left((1 + 0) - (1 + 0)\right) = 0, \]

we have
\[ I = 2\pi(20 + 0) = 40\pi. \]

We have just used the definition of surface integral here. This problem can also be solved by use of the divergence theorem since \( \text{div}(F) = 3 \). Let \( T \) be the solid region \( x^2 + y^2 \leq 2 + \sin \pi z, 0 \leq z \leq 10 \). Calculate the volume of \( T \) (using cylindrical coordinates). It is \( V = 20\pi \). Next let \( S_1 \) be the top disk and \( S_2 \) be the bottom disk, oriented with outward unit normal vectors. On \( S_1 \), \( F \cdot n = z = 10 \) and so the flux through \( S_1 \) is \( I_1 = 10 \cdot 2\pi = 20\pi \). On \( S_2 \), \( F \cdot n = -z = 0 \) and so the flux through \( S_2 \) is \( I_2 = 0 \cdot 2\pi = 0 \). Hence, \( I = 3V - I_1 - I_2 = 60\pi - 20\pi - 0 = 40\pi \). This is pretty easy, but the direct calculation is not hard.
(e) $I = 0$. We can use either a direct calculation (from the definition of surface integral), use Stokes’ theorem, or use the divergence theorem (since $\text{div}(\mathbf{G}) = 0$). All are easy.

2. (a) From the definition of line integral,

$$I = \int_C P \, dx + Q \, dy + R \, dz = \int_0^{2\pi} yx' + (x^2 + y^2)y' + 1 \cdot z' \, dt$$

$$= \int_0^{2\pi} \sin t(-\sin t) + (\cos^2 t + \sin^2 t) \cos t + (-\sin t) + 2 \cos 2t \, dt$$

$$= \int_0^{2\pi} -\frac{1}{2} (1 - \cos 2t) + \cos t - \sin t + 2 \cos 2t \, dt$$

$$= -\frac{1}{2} t + \frac{1}{4} \sin 2t + \sin t + \cos t + \sin 2t \bigg|_0^{2\pi}$$

$$= (-\frac{1}{2} (2\pi) + 0 + 0 + 0 + 1 + 0) - (0 + 0 + 0 + 0 + 1 + 0) = -\pi.$$

(c) Write $\mathbf{G} = \sqrt{z^2 + 1}(x, y, 0)$ and $\mathbf{H} = (0, 0, 1)$. Note that $\mathbf{G} \cdot \mathbf{T} = 0$ at every point on curve $C$. So, $I_1 = \int_C \mathbf{G} \cdot \mathbf{dr} = \int_C \mathbf{G} \cdot \mathbf{T} \, ds = 0$. Also, note that $\mathbf{H}$ has potential function $g(x, y, z) = z$ and that $C$ is a closed curve. So, $I_2 = \int_C \mathbf{H} \cdot \mathbf{dr} = 0$. Since $\mathbf{F} = \mathbf{G} + \mathbf{H}$, we have $I = I_1 + I_2 = 0 + 0 = 0$.

3. (a)

$$x^2 + y \leq 1, \quad z^2 + y \geq 1, \quad x + 2z \leq 1, \quad z \geq 0.$$

(b) The projection of $T$ into the $xz$-plane is given by

$$R : \quad x + 2z \leq 1, \quad z \geq -x, \quad z \geq x.$$

We subdivide $R$ into two $z$-simple regions.

$$R_1 : \quad -1 \leq x \leq 0, \quad -x \leq z \leq \frac{1}{2} (1 - x)$$

$$R_2 : \quad 0 \leq x \leq 1/3, \quad x \leq z \leq \frac{1}{2} (1 - x)$$
So,

\[ I = \iiint_R f(x, y, z) \, dy \, dA(x, z) \]

\[ = \iiint_{R_1} f(x, y, z) \, dy \, dA(x, z) + \iiint_{R_2} f(x, y, z) \, dy \, dA(x, z) \]

\[ = \int_{-1}^{0} \int_{-1}^{(1/2)(1-x)} \int_{1-z^2}^{1-x^2} f(x, y, z) \, dy \, dz \, dx + \int_{1/3}^{1} \int_{(1/2)(1-x)}^{1-x^2} f(x, y, z) \, dy \, dz \, dx. \]

4. \( p(x, y, z) = \frac{1}{2} y^2 + k(z) \) where \( k(z) \) is an arbitrary function of \( z \).

6. \( \mathbf{T} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \, \mathbf{N} = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle, \, \kappa = \frac{\sqrt{2}}{3}. \)

7. area = \( \frac{1}{168} \).

8. \( I = 8\pi. \)

9. (a) We have \( I = \iiint f \, dV \) where \( R \) is the \( y \)-simple region

\[ R : \quad 1 \leq x \leq 2, \quad \sqrt{2 - x} \leq y \leq \sqrt{x}. \]

Subdivide \( R \) into the two \( x \)-simple regions

\[ R_1 : \quad 1 \leq y \leq \sqrt{2}, \quad y^2 \leq x \leq 2 \]

\[ R_2 : \quad 0 \leq y \leq 1, \quad 2 - y^2 \leq x \leq 2 \]

So,

\[ I = \iiint_{R_1} f \, dV + \iiint_{R_2} f \, dV \]

\[ = \int_{1}^{\sqrt{2}} \int_{y^2}^{2} f(x, y) \, dx \, dy + \int_{0}^{1} \int_{2-y^2}^{2} f(x, y) \, dx \, dy \]
(b) We have \( I = \iiint_R f \, dV \) where \( R \) is the \( y \)-simple region

\[
R : \quad -3 \leq x \leq -2, \quad \frac{1}{2} (x + 1)^2 \leq y \leq x^2.
\]

Subdivide \( R \) into the three \( x \)-simple regions

\[
R_1 : \quad \frac{1}{2} \leq y \leq 2, \quad -1 + \sqrt{2y} \leq x \leq -2
\]

\[
R_2 : \quad 2 \leq y \leq 4, \quad -3 \leq x \leq -2
\]

\[
R_3 : \quad 4 \leq y \leq 9, \quad -3 \leq x \leq \sqrt{y}
\]

So,

\[
I = \iiint_{R_1} f \, dV + \iiint_{R_2} f \, dV + \iiint_{R_3} f \, dV
\]

\[
= \int_{1/2}^{2} \int_{-1+\sqrt{2y}}^{2} f(x, y) \, dx \, dy + \int_{2}^{4} \int_{-3}^{2} f(x, y) \, dx \, dy + \int_{4}^{9} \int_{-3}^{\sqrt{y}} f(x, y) \, dx \, dy
\]

10. (a) First of all Green’s theorem only applies to curves in the \( xy \)-plane. Secondly, they must be closed curves, in other words, the starting point is the same as the ending point. In more detail, there must a bounded region \( R \) and the curve \( C \) appearing in the line integral must give the entire boundary of \( R \), traveling in a counter-clockwise direction.

The easiest example of a line integral that can’t be part of Green’s theorem is to pick a non-closed curve, for example a straight line from \((0, 0)\) to \((1, 0)\).

(b) The divergence theorem applies when we have a solid bounded region \( T \) with a surface integral on \( S \), the entire boundary of \( T \) with an outer unit normal vector. In particular, the surface \( S \) must separate 3-dimensional space into an “inside” and an “outside”.

Therefore, an example of a surface that cannot be used with the divergence theorem would be a surface that does not separate in this sense like the top half of the unit sphere: \( x^2 + y^2 + z^2 = 1, \, z \geq 0 \).

We can sometimes use the divergence theorem with such nonqualifying surfaces by using trickery. Suppose that \( S \) is only part of the boundary of some solid region \( T \). Let’s
suppose that that entire boundary consists of $S$ plus $S_1$. If we know the surface integral on $S_1$ and the triple integral of the divergence over $T$, then we can deduce the surface integral on $S$.

In the above example, the entire boundary of the half-ball $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ consists of $S$, the top half of the sphere, and $S_1$, the disk that spans the equator of the sphere, $x^2 + y^2 \leq 1$, $z = 0$. The integral of the divergence of $\mathbf{F}$ over $T$ equals the surface integral on $S$ plus the integral on $S_1$. If the latter is easy to calculate, then we can obtain the surface integral on $S$.

(c) The surface integral appearing the Stokes’ theorem can be just about any surface integral. The only requirement is that the surface is bounded and has a unit normal vector $\mathbf{n}$. (The Möbius strip does not qualify. Don’t worry if you have never heard about this surface. It isn’t part of our course.) The line integral is over a curve $C$ which bounds $S$. In particular, $C$ must be a closed curve. (Stokes’ theorem is a generalization of Green’s theorem; so it has most of the same restrictions.) Of course, the direction of $C$ must be compatible with the normal vector on $S$. In some weird cases, a closed curve does not bound any surface.

The easiest way to construct a non-qualifying curve is the same as for Green’s theorem. Pick any non-closed curve.