PROBLEM SET 3 SOLUTIONS

Remember for surface integrals that \( dS = |N(u,v)|dudv \). Anyway...

Problem 1. Let \( S \) be the surface obtained by rotating the curve
\[
z = \sin(2t), \quad r = \cos(t), \quad -\pi/2 \leq t \leq \pi/2
\]
about the \( z \)-axis. Find the volume of the solid \( T \) whose boundary is \( S \).

Solution: Parametrize \( S \) by \( r = \mathbf{r}(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(2u)) \) for \( u \in [-\pi/2, \pi/2] \) and \( v \in [0,2\pi] \). Here \( u \) is chosen to be \( t \), and one sees that \( x^2 + y^2 = r^2 \) (also the range of \( v \) ensures that the rotating curve traverses \( S \) around the \( z \)-axis exactly once). Now
\[
\mathbf{N} = \mathbf{N}(u,v) = \mathbf{r}_u \times \mathbf{r}_v = \frac{\partial}{\partial u} \mathbf{r} \times \frac{\partial}{\partial v} \mathbf{r} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(u)\cos(v) & -\sin(u)\sin(v) & 2\cos(2u) \\ -\cos(u)\sin(v) & \cos(u)\cos(v) & 0 \end{bmatrix} = \langle -2\cos(2u)\cos(u)\cos(v), -2\cos(2u)\cos(u)\sin(v), -\cos(u)\sin(u) \rangle.
\]
To check whether or not this is the outward normal vector we note that since \( T \) is symmetric about the \( z \)-axis, \( \mathbf{N} \) must be pointing away from the origin when \( t = u = 0 \) (sketch what the rotating curve looks like in the \( rz \)-plane). Restricting to the \( yz \)-plane (take \( v = \pi/2 \)) we see that \( \mathbf{N}(0, \pi/2) = (0, -2, 0) \) points towards the origin from the point \( r(0, \pi/2) = (0, 1, 0) \) on \( S \). Thus we take \( \mathbf{n} = -\mathbf{N}/|\mathbf{N}| \).

Finally by looking at the components of \( \mathbf{N}(u,v) \) it’s clear that the choice of \( \mathbf{F} \) satisfying \( \nabla \cdot \mathbf{F} = 1 \), which yields the least complicated surface integral, should be \( \mathbf{F} = (0, 0, z) \). So by Gauss’ divergence theorem we find the volume of \( T \) to be
\[
\iiint_T dV = \iint_S \mathbf{F} \cdot \mathbf{n}dS = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin(2u)\cos(u)\sin(u)dudv = \frac{\pi}{8} [4u - \sin(4u)]^{\pi/2}_{-\pi/2} = \pi^2/2.
\]

Problem 2. Let \( S \) be the outward oriented surface \( x^2 + y^2 + z^2 = 4 \) such that \( z \leq 1 \). Calculate the outward flux \( \Phi \) given \( \mathbf{F} = \langle e^z, xz, x^2 + y^2 \rangle \).

Solution: Currently \( S \) is an open surface. Gauss’ theorem only applies for closed surfaces since a solid \( T \) must have a boundary. So add a lid \( \tilde{S} \) consisting of points \((x,y,z)\) such that \( x^2 + y^2 \leq 3 \) and \( z = 1 \). Then, for the boundary \( S_c = S \cup \tilde{S} \) of \( T \), we have
\[
\left( \iint_S + \iint_{\tilde{S}} \right) \mathbf{F} \cdot \mathbf{n}dS = \iint_{S_c} \mathbf{F} \cdot \mathbf{n}dS = \iint_T \nabla \cdot \mathbf{F}dV = 0.
\]
Hence \( \iint_{\tilde{S}} \mathbf{F} \cdot \mathbf{n}dS = -\iint_S \mathbf{F} \cdot \mathbf{n}dS \).

While the integral over \( S \) is hard to compute, the integral over \( \tilde{S} \) is easy in comparison: parametrize \( \tilde{S} \) by \( r(u,v) = (u, v, 1) \) subject to \( u^2 + v^2 \leq 3 \), and note that \( \mathbf{F} \cdot \mathbf{n} = u^2 + v^2 \) since \( \mathbf{n} = \mathbf{N} = (0, 0, 1) \) is the upward (outward) pointing normal from \( \tilde{S} \) (from \( T \)). Letting \( R \) represent the disk \( u^2 + v^2 \leq 3 \) in the \( uv \)-plane, we find the flux to be
\[
\Phi = \iint_{\tilde{S}} \mathbf{F} \cdot \mathbf{n}dS = -\iint_S \mathbf{F} \cdot \mathbf{n}dS = -\iint_R (u^2 + v^2)dudv = -\int_0^{2\pi} \int_0^{\sqrt{3}} r^2drd\theta = -\frac{\pi}{2} [r^3]_0^{\sqrt{3}} = -9\pi/2.
\]