MATH 252. PROBLEM SET 1

1. a) Let $k$ be an uncountable algebraically closed field and let $V$ be a vector space over $V$ of countable dimension. Let $T : V \to V$ be a linear transformation. Prove that there exists $\lambda \in k$ such that $T - \lambda$ is not invertible (hint: use the fact that the field $k(t)$ of rational functions over $k$ has uncountable dimension over $k$).

b) Use a) to show that there are no non-trivial division algebras over $k$ of countable dimension.

2. Let $k$ be a field of characteristic 0. Let $D$ be the subalgebra of $\text{End}_k(k[x])$ generated by the derivative $d$ and by multiplication by $x$ ($D$ is called the algebra of differential operators with polynomial coefficients in one variable).

a) Show that the elements of the form $x^i d^j$ form a basis of $D$.

b) Show that $D$ can be described as the algebra generated by $x$ and $d$ with the only relation $dx - xd = 1$.

c) Show that $D$ is a central simple algebra over $k$.

d) Assume that $k$ has positive characteristic. Show that all the statements a), b), c) fail in this case.

3. Let $G$ be a group. Show that if $G$ has more than one element, then for any commutative ring $k$ the group algebra $k[G]$ is not simple.

4. Let $G$ be a finite field. Let $k$ be a field whose characteristic does not divide the order of $G$. Show that the algebra $k[G]$ is semi-simple.

5. Let $R$ be a ring and let $\text{Mat}_n(R)$ be a matrix algebra over $R$. Prove that every module over $\text{Mat}_n(R)$ has the form $M^n$ for some module $M$ over $R$. 

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