Assignment 29 — Assigned Mon Nov 15

Section 16.1, Problem 7.
The differential equation to solve is $2y'' - y' - 3y = 0$. Hence we look at the equation $2r^2 - r - 3 = 0$. Now, the roots of this equation are $-1$ and $\frac{3}{2}$. Hence the general solution is $y = c_1 e^{-x} + c_2 e^{\frac{3}{2}x}$.

Section 16.1, Problem 15.
The differential equation to solve is $y'' - 2y' + 5y = 0$. We look at the equation $r^2 - 2r + 5 = 0$. The roots are $1 + 2i$ and $1 - 2i$. Hence the general solution is $y = e^x (c_1 \cos 2x + c_2 \sin 2x)$.

Section 16.1, Problem 25.
The differential equation to solve is $y'' + 6y' + 9y = 0$. We look at the equation $r^2 + 6r + 9 = 0$. This has a double root at $-3$. Hence the general solution is $y = c_1 e^{-3x} + c_2 xe^{-3x}$.

Section 16.1, Problem 28.
The differential equation to solve is $4y'' - 4y' + y = 0$. We look at the equation $4r^2 - 4r + 1 = 0$. This has a double root at $\frac{1}{2}$. Hence the general solution is $y = c_1 e^{x/2} + c_2 xe^{x/2}$.

Section 16.1, Problem 41.
The differential equation to solve is $y'' - 2y' - 3y = 0$. We look at the equation $r^2 - 2r - 3 = 0$. This has roots $3$ and $-1$. Hence the general solution is $y = c_1 e^{3x} + c_2 e^{-x}$.

Section 16.1, Problem 46.
The differential equation to solve is $y'' + 2y' + 2y = 0$. We look at the equation $r^2 + 2r + 2 = 0$. This has roots $-1 + i$ and $-1 - i$. Hence the general solution is $y = e^{-x}(c_1 \cos x + c_2 \sin x)$.

Section 16.1, Problem 55.
The differential equation to solve is $6y'' - 5y' - 4y = 0$. We look at the equation $6r^2 - 5r - 4 = 0$. This has roots $-\frac{1}{2}$ and $\frac{4}{3}$. Hence the general solution is $y = c_1 e^{-x/2} + c_2 e^{4x/3}$.
Assignment 30 — Assigned Weds Nov 17

Section 16.1, Problem 31.
\( r^2 + 6r + 5 = 0 \) has solutions \( r = -1, -5 \) so the solution to this second order homogeneous differential equation is

\[
y = c_1 e^{-5x} + c_2 e^{-x}.
\]

The initial value \( y(0) = 0 \) gives that \( c_1 + c_2 = 0 \) and \( y'(0) = 3 \) gives that \(-5c_1 - c_2 = 3\). From here we get \( c_1 = \frac{3}{4} \) and \( c_2 = -\frac{3}{4} \), so the solution is

\[
y = \frac{3}{4} e^{-5x} - \frac{3}{4} e^{-x}.
\]

Section 16.1, Problem 32.
\( r^2 + 16 = 0 \) has roots \( r = \pm 4i \), so the solution to this second order homogeneous differential equation is

\[
y = c_1 \cos 4x + c_2 \sin 4x.
\]

Initial value \( y(0) = 2 \) gives that \( c_1 = 2 \), and \( y'(0) = -2 \) gives that \( 4c_2 = -2 \). From here we get \( c_2 = -\frac{1}{2} \), so the solution is

\[
y = 2 \cos 4x - \frac{1}{2} \sin 4x.
\]

Section 16.1, Problem 36.
\( r^2 + 4r + 4 = 0 \) has the double root \( r = -2 \) so the solution to this second order homogeneous differential equation is

\[
y = c_1 e^{-2x} + c_2 xe^{-2x}.
\]

Initial value \( y(0) = 0 \) gives that \( c_1 = 0 \), and \( y'(0) = 1 \) gives that \(-2c_1 + c_2 = 1\). From here we get \( c_2 = 1 \), so the solution is

\[
y = xe^{-2x}.
\]
Section 16.2, Problem 3.
The homogenous part of this equation gives \( r^2 - r = 0 \) with roots \( r = 0 \) and \( r = 1 \), so the solution is 

\[
y_c = c_1 + c_2 e^x.
\]

A particular solution will have the form \( y_p = a \cos(x) + b \sin(x) \). Substituting \( y_p \) into the differential equation gives 

\[
y_p'' - y_p' = (-a \cos(x) - b \sin(x)) - (-a \sin(x) + b \cos(x)) = (-a - b) \cos(x) + (a - b) \sin(x).
\]

This is supposed to equal \( \sin(x) \), so we need 

\[-a - b = 0 \quad \text{and} \quad a - b = 1.\]

Solving gives \( a = \frac{1}{2} \) and \( b = -\frac{1}{2} \). So the general solution of the equation is 

\[
y = y_c + y_p = c_1 + c_2 e^x + \frac{1}{2} \cos(x) - \frac{1}{2} \sin(x).
\]

Section 16.2, Problem 4.
The homogenous part of this equation gives \( r^2 + 2r + 1 = 0 \), so the solution to the homogeneous equation is 

\[
y_c = c_1 e^{-x} + c_2 xe^{-x}.
\]

A particular solution to the nonhomogeneous equation will have the forms 

\[
y_p = ax^2 + bx + c.
\]

Substituting \( y_p \) into the differential equation gives 

\[
y_p'' + 2y_p' + y_p = 2a + 2(2ax + b) + (ax^2 + bx + c) = ax^2 + (4a + b)x + 2a + 2b + c.
\]

This is supposed to equal \( x^2 \), so we need 

\[a = 1, \quad 4a + b = 0, \quad 2a + 2b + c = 0.\]

So \( a = 1, b = -4, \) and \( c = 6 \). Then the general solution is 

\[
y = y_c + y_p = c_1 e^{-x} + c_2 xe^{-x} + x^2 - 4x + 6.
\]
Section 16.2, Problem 9.
The homogenous part of this equation gives $r^2 - 1 = 0$, so the solution to the homogeneous differential equation is
\[ y_c = c_1 e^{-x} + c_2 e^x. \]
A particular solution will look like
\[ y_p = axe^x + be^x + cx^2 + dx + e. \]
Substituting $y_p$ into the differential equation gives
\[
y''_p - y_p = (axe^x + 2ae^x + be^x + 2c) - (axe^x + be^x + cx^2 + dx + e)
= 2ae^x - cx^2 - dx + (2c - e).\]
This is supposed to equal $e^x + x^2$, so we get
\[ a = \frac{1}{2}, \quad c = -1, \quad d = 0, \quad e = -2. \]
(Notice that $b$ has completely disappeared. The reason is because $e^x$ is already a solution to the homogeneous equation. So we really didn’t need to use the $be^x$ term.) Then the general solution is
\[ y = y_c + y_p = c_1 e^{-x} + c_2 e^x + \frac{1}{2}xe^x - x^2 - 2. \]

Section 16.2, Problem 31.
We substitute in the differential equation
\[
[A \cos(x) + Bx \sin(x)]'' + Ax \cos(x) + Bx \sin(x) = 2 \cos(x) + \sin(x).
\]
After taking two derivatives and combining and canceling terms, we get
\[ -2A \sin(x) + 2B \cos(x) = 2 \cos(x) + \sin(x). \]
Equating the coefficients of the sin and cos terms, we get that $A = -\frac{1}{2}$ and $B = 1$. So the particular solution turns out to be
\[ y_p = -\frac{1}{2} \cos(x) + x \sin(x). \]
The general solution comes from solutions of homogeneous part $r^2 + 1 = 0$, which is $y_c = c_1 \cos(x) + c_2 \sin(x)$. So the general solution to the inhomogeneous equation is
\[ y = y_c + y_p = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2} x \cos(x) + x \sin(x). \]
Section 16.2, Problem 19.
We follow the example $y'' + y = \tan x$ in Chapter 16. Using $\tan x$ instead of $\sin x$, we get
$$v_1' = -\sin^2 x$$
and
$$v_2' = \cos x \sin x = \frac{1}{2} \sin 2x.$$
Integrating these functions gives us
$$v_1 = \frac{\sin 2x}{4} - \frac{x}{2},$$
and
$$v_2 = -\frac{\cos 2x}{2}.$$
The particular solution is therefore
$$y_p = \frac{\cos x \sin 2x}{4} - \frac{x \cos x}{2} + \frac{\sin x \sin 2x}{2}.$$
Thus, the general solutions is
$$y = c_1 \cos x + c_2 \sin x + \frac{\cos x \sin 2x}{4} - \frac{x \cos x}{2} + \frac{\sin x \sin 2x}{2}.$$
Note that there are other equivalent ways to write the solution. For example, we could have integrated $v_2' = \cos x \sin x$ directly to get $v_2 = \frac{1}{2} \sin^2 x$, in which case the general solution would look like
$$y = c_1 \cos x + c_2 \sin x + \frac{\sin x \cos^2 x}{2} - \frac{x \cos x}{2} + \frac{\sin^2 x \cos x}{2}.$$
Section 16.2, Problem 20.
The solution to the associated homogeneous equation $y'' + 2y' + y = 0$ is

$$c_1y_1 + c_2y_2 \quad \text{with} \quad y_1 = xe^{-x} \text{ and } y_2 = e^{-x}.$$ 

By using the same idea as in the previous exercise, we find that

$$v'_1 = e^{2x} \quad \text{and} \quad v'_2 = -xe^{2x}.$$ 

Integrating we find that

$$v_1 = \frac{1}{2} e^{2x} \quad \text{and} \quad v_2 = -\frac{1}{2} xe^{2x} + \frac{1}{4} e^{2x}.$$ 

The particular solution is thus

$$y_p = v_1y_1 + v_2y_2 = \frac{1}{4} e^x.$$ 

Section 16.2, Problem 32.
We have

$$y' = Ax^2e^x + 2Axe^x + Bxe^x + Be^x$$

and

$$y'' = Ax^2e^x + 4Axe^x + 2Ae^x + Bxe^x + 2Be^x.$$ 

Thus,

$$y'' + y - 2y = 6Axe^x + (2A + 3B)e^x = xe^x.$$ 

Solving, we find that

$$A = \frac{1}{6} \quad \text{and} \quad B = -\frac{1}{9}.$$ 

The solution to the homogenous equation is given by $y_1 = e^x$ and $y_2 = e^{-2x}.$

The general solution is thus

$$y = c_1e^x + c_2e^{-2x} + \frac{x^2}{6}e^x - \frac{x}{9}e^x.$$
Section 16.2, Problem 37.
Using the solutions

\[ y_1 = \cos x \quad \text{and} \quad y_2 = \sin x, \]

we find the functions

\[ v_1' = -\cos x \quad \text{and} \quad v_2' = \frac{\cos^2 x}{\sin x}. \]

Using integration we find

\[ v_1 = -\sin x \quad \text{and} \quad v_2 = -\ln|\csc x + \cot x| - \cos x, \]

where the last integral was found by writing

\[ \frac{\cos^2 x}{\sin x} = \frac{1 - \sin^2 x}{\sin x} = \csc x - \sin x. \]

Thus the particular solution is given by

\[ y_p = y_1 v_1 + y_2 v_2 = -\sin x \ln|\csc x + \cot x|. \]

The general solution is thus

\[ y = c_1 \cos x + c_2 \sin x - (\sin x) \ln(\csc x + \cot x), \]

where we have dropped the absolute value sign in the natural logarithm because of the restriction \( 0 < x < \pi. \)
Section 16.2, Problem 43.
In this problem, the auxiliary equation \( r^2 + 2r = 0 \) tells us that
\[
y_1 = 1 \quad \text{and} \quad y_2 = e^{-2x}.
\]
By using the determinant trick, we find
\[
\begin{align*}
v'_1 &= \frac{x^2}{2} - \frac{e^x}{2} \quad \text{and} \quad v'_2 = \frac{e^{2x}}{2}(x^2 - e^x) = \frac{1}{2}x^2e^{2x} - \frac{1}{2}e^{3x}.
\end{align*}
\]
Integration gives
\[
\begin{align*}
v_1 &= \frac{x^3}{6} - \frac{e^x}{2} \quad \text{and} \quad v_2 = -\left(\frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{8}\right)e^{2x} + \frac{1}{6}e^{3x}.
\end{align*}
\]
(To integrate \( x^2e^{2x} \), you can either use the integral table in the back of the book, or integrate by parts twice.) The particular solution is
\[
y_p = v_1y_1 + v_2y_2 = \frac{1}{6}x^2 - \frac{1}{4}x^2 + \frac{1}{4}x + \frac{3}{8} - \frac{1}{3}e^x,
\]
so the general solution is
\[
y = c_1 + c_2e^{-2x} + \frac{1}{6}x^2 - \frac{1}{4}x^2 + \frac{1}{4}x + \frac{3}{8} - \frac{1}{3}e^x.
\]

Section 16.2, Problem 59.
Using
\[
y_1 = x^{-2} \quad \text{and} \quad y_2 = x,
\]
we find
\[
\begin{align*}
v_1 &= -\frac{x^4}{12} \quad \text{and} \quad v_2 = \frac{x}{3}.
\end{align*}
\]
A particular solution is thus
\[
y_p = v_1y_1 + v_2y_2 = -\frac{x^2}{12} + \frac{x^2}{3} = \frac{x^2}{4}.
\]