Problem 1. (10 points) Write the following iterated integral as a sum of one or more iterated integrals with the order of integration reversed, i.e., each inner integral should be with respect to $x$ and each outer integral should be with respect to $y$.

$$\int_0^\pi \int_{\sin x}^2 f(x, y) \, dy \, dx.$$ 

Also sketch the region.

Solution. The region lies over the curve $y = \sin x$ and under the curve $y = 2$. In order to integrate in the other order, we split it into three regions. (Note that the values of $\sin^{-1}(y)$ are between $-\frac{1}{2} \pi$ and $\frac{1}{2} \pi$.)

\begin{align*}
D_1 &= \{(x, y) : 1 \leq y \leq 2, \ 0 \leq x \leq \pi\} \\
D_2 &= \{(x, y) : 0 \leq y \leq 1, \ 0 \leq x \leq \sin^{-1}(y)\} \\
D_3 &= \{(x, y) : 0 \leq y \leq 1, \ \pi - \sin^{-1}(y) \leq x \leq \pi\}
\end{align*}

Hence

$$\int_0^\pi \int_{\sin x}^2 f(x, y) \, dy \, dx = \int_1^2 \int_0^\pi f(x, y) \, dx \, dy$$

$$+ \int_0^1 \int_0^{\sin^{-1}(y)} f(x, y) \, dx \, dy + \int_0^1 \int_{\pi - \sin^{-1}(y)}^\pi f(x, y) \, dx \, dy$$

Problem 2. (10 points) Evaluate the following integral.

$$\int \int_D \cos(x^2 + y^2) \, dA,$$ 

where $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 9\}$.

Solution. We use polar coordinates, so

$$x = r \cos \theta, \quad y = r \sin \theta, \quad dA = dx \, dy = r \, dr \, d\theta, \quad x^2 + y^2 = r^2.$$ 

Also $D^* = \{(r, \theta) : 1 \leq r \leq 3 \text{ and } 0 \leq \theta \leq 2\pi\}$. So

$$\int \int_D \cos(x^2 + y^2) \, dx \, dy = \int \int_{D^*} \cos(r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^3 \cos(r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \sin(r^2) \bigg|_{r=1}^{r=3} \, d\theta$$
\[ \int_{0}^{2\pi} \frac{1}{2} (\sin(9) - \sin(1)) \, d\theta = \pi (\sin(9) - \sin(1)). \]

**Problem 3.** (10 points) The curve
\[ c(t) = (t^{-3}, t^{-2}, t^{-1}) \]
for \( t > 0 \) is a flow line for the vector field
\[ F(x, y, z) = (axz + y^2, byz - x, cy + z^2). \]
What are the values of \( a, b, \) and \( c? \)

**Solution.** We set \( F(c(t)) = c'(t). \) We have
\[
F(c(t)) = F(t^{-3}, t^{-2}, t^{-1}) \\
= (at^{-4} + t^{-4}, bt^{-3} - t^{-3}, ct^{-2} + t^{-2}) \\
= ((a + 1)t^{-4}, (b - 1)t^{-3}, (c + 1)t^{-2}),
\]
\[ c'(t) = (-3t^{-4}, -2t^{-3}, -t^{-2}). \]
These need to be equal for all \( t > 0, \) so we get
\[ a + 1 = -3, \quad b - 1 = -2, \quad c + 1 = -1, \]
which gives
\[ a = -4, \quad b = -1, \quad c = -2 \]

**Problem 4.** (10 points)

**NOTE:** Grading for each part of this True/False question is +2 for the correct answer, 0 if left blank, and -1 if incorrect.

Indicate whether each of the following statements is true or false by circling the appropriate answer. You do not need to give a reason for your answer. For (a), (b), and (c), we write \( F(x, y, z) \) for a 3-dimensional vector field that is assumed to be sufficiently differentiable.

**Solution.**

(a) \( \text{curl(curl}(F)) = 0 \) for all \( F \) \hspace{1cm} \text{FALSE}

(b) \( \text{curl(div}(F)) = 0 \) for all \( F \) \hspace{1cm} \text{FALSE}

(c) \( \text{div(curl}(F)) = 0 \) for all \( F \) \hspace{1cm} \text{TRUE}

(d) \( F = (y, x, z) \) is a gradient vector field \hspace{1cm} \text{TRUE}

(e) \( F = (z, x, y) \) is a gradient vector field \hspace{1cm} \text{FALSE}
Here are the reasons.

(a) For cross products of vectors, $a \times (a \times b)$ is a vector in the plane spanned by $a$ and $b$ and perpendicular to $a$, but it won’t in general be zero. Similarly for the curl of a curl. It’s easy enough to find an example that gives a non-zero result. For example,

$$F(x, y, z) = (y^2, 0, 0)$$

has

$$\text{curl } F = (0, 0, -2y),$$

so

$$\text{curl}(\text{curl}(F)) = (-2, 0, 0)$$

is non-zero.

(b) The quantity $\text{curl}(\text{div}(F))$ isn’t even defined, since the divergence $\text{div}(F)$ is a real-valued function, not a vector field, so you can’t take its curl.

(c) This one is true, as is the fact that the curl of a gradient is zero.

(d) This passes the partial derivative tests. And it’s easy enough to find that $F$ is the gradient of the function $f(x, y, z) = xy + \frac{1}{2}z^2$.

(e) This is not a gradient, since for example,

$$\frac{\partial F_1}{\partial z} = \frac{\partial z}{\partial z} = 1$$

is not equal to

$$\frac{\partial F_3}{\partial x} = \frac{\partial y}{\partial x} = 0.$$

**Problem 5.** (10 points) Let $D^*$ be the square $[0, 1] \times [0, 1]$ in the $uv$-plane, and let $T$ be the function

$$T(u, v) = (u^2 - v^2, 2uv).$$

(a) Sketch the region $T(D^*)$ in the $xy$-plane. (*Hint. Figure out where the sides of the square are sent.*)

(b) If

$$\int \int_{T(D^*)} \frac{1}{\sqrt{x^2 + y^2}} \, dx \, dy \int \int_{D^*} f(u, v) \, du \, dv,$$

what is the function $f(u, v)$? (You may assume that $T : D^* \rightarrow \mathbb{R}^2$ is one-to-one.)

(c) Evaluate the integral in (b).

**Solution.** (b) The Jacobian is

$$\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} = 4u^2 + 4v^2,$$
so
\[
\iint_{T(D^*)} \frac{1}{\sqrt{x^2 + y^2}} \, dx \, dy = \int \int_{D^*} \frac{1}{\sqrt{(u^2 - v^2)^2 + (2uv)^2}} (4u^2 + 4v^2) \, du \, dv.
\]

This gives an answer to (b) with
\[
f(u, v) = \frac{4u^2 + 4v^2}{\sqrt{(u^2 - v^2)^2 + (2uv)^2}}.
\]

However, for answering (c), we should simplify by noticing that
\[
(u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2
= u^4 + 2u^2v^2 + v^4
= (u^2 + v^2)^2.
\]

So
\[
f(u, v) = \frac{4u^2 + 4v^2}{\sqrt{(u^2 + v^2)^2}} = 4.
\]

(c)
\[
\iint_{T(D^*)} \frac{1}{\sqrt{x^2 + y^2}} \, dx \, dy = \int \int_{D^*} 4 \, du \, dv = 4 \text{Area}(D^*) = [4].
\]