Problem 1. (10 points) For each of the following functions, circle what kind of singularity it has at $z = 0$, and compute the residue at $z = 0$.

(a) $f(z) = \frac{1}{z \sin(z)}$.

Removable  Pole  Essential

(b) $f(z) = \frac{1}{\sin(1/z)}$.

Removable  Pole  Essential

Solution. (a) $f(z)$ has an **pole** at $z = 0$. To see this, we note that

$$\lim_{z \to 0} z^2 f(z) = \lim_{z \to 0} \frac{z^2}{z \sin(z)} = 1,$$

so in fact, $f(z)$ has a pole of order 2. The residue can be computed using the derivative formula that we often use, or alternatively using the power series for $\sin z$. Thus

$$\frac{1}{z \sin(z)} = \frac{1}{z \left(z - \frac{z^3}{6} + \cdots\right)}$$

$$= \frac{1}{z^2} \cdot \frac{1}{1 - \frac{z^2}{6} + \cdots}$$

$$= \frac{1}{z^2} \cdot \left(1 + \frac{z^2}{6} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{6} + \cdots.$$

Hence

$$\text{Res} \left[ \frac{1}{z \sin(z)} ; 0 \right] = 0.$$

But the easiest way to compute the residue is to note that $f(z) = f(-z)$, i.e., the function $f(z)$ is even, so in its Laurent expansion

$$\sum_{k=-\infty}^{\infty} c_k z^k,$$

we have $c_k = 0$ for every odd $k$. In particular, we have $c_{-1} = 0$, so the residue is 0.
(b) $f(z)$ has an **essential singularity** at $z = 0$. We can see this by observing that $\lim_{z \to 0} z^m f(z)$ does not exist for any integer $m \geq 0$. To ease notation, let $w = 1/z$, so we’re looking at

$$f(w) = \frac{1}{\sin(w)}.$$

If we write $f(w)$ as a Laurent series in $w$, then the residue is the coefficient of $w$. Thus

$$f(w) = \frac{1}{\left( w - \frac{w^3}{6} + \cdots \right)} = \frac{1}{w} \cdot \frac{1}{1 - \frac{w^2}{6} + \cdots} = \frac{1}{w} \cdot \left( 1 + \frac{w^2}{6} + \cdots \right) = \frac{1}{w} + \frac{w}{6} + \cdots$$

Thus the Laurent series of $f(z)$ looks like

$$f(z) = z + \frac{1}{6z} + \frac{a}{z^3} + \frac{b}{z^5} + \cdots$$

so

$$\text{Res} \left[ \frac{1}{\sin(1/z)}, 0 \right] = \frac{1}{6}.$$

**Problem 2.** (15 points) Compute the values of each of the following integrals.

(a) $\int_{|z|=1} \frac{e^{3z}}{z^5} \, dz$.

(b) $\int_{|z|=1} \frac{1}{z(e^z - 1)} \, dz$.

(c) $\int_{\gamma} z \, dz$, where $\gamma$ is the line segment from 0 to 1 + i.
Solution. (a) There is a one pole at $z = 0$, and it is a pole of order 3. The residue theorem gives
\[
\frac{1}{2\pi i} \int_{|z|=1} \frac{e^{3z}}{z^2} \, dz = \text{Res} \left[ \frac{e^{3z}}{z^2}, 0 \right]
\]
\[
= \text{Res} \left[ \frac{1}{z^3} \left( 1 + 3z + \frac{(3z)^2}{2!} + \frac{(3z)^3}{3!} + \cdots \right), 0 \right]
\]
\[
= \text{Res} \left[ \frac{1}{z^3} + \frac{3}{z^2} + \frac{9}{2z} + \frac{9}{2} + \cdots, 0 \right]
\]
\[
= \frac{9}{2}.
\]
So
\[
\int_{|z|=1} \frac{e^{3z}}{z^2} \, dz = 9\pi i.
\]

(b) Again there is one pole at $z = 0$, but this time it is a pole of order 2. Letting $f(z) = \frac{1}{z(e^z - 1)}$, we have
\[
\text{Res}[f(z), 0] = \lim_{z \to 0} \frac{d}{dz} (z^2 f(z))
\]
\[
= \lim_{z \to 0} \frac{d}{dz} \left( \frac{z}{e^z - 1} \right)
\]
\[
= \lim_{z \to 0} \frac{(e^z - 1) - ze^z}{(e^z - 1)^2}.
\]
One way to proceed now is to use L’Hopital’s rule a couple of times. An easier(?) way is to use the Taylor series expansion of $e^z$. Thus
\[
\frac{e^z - 1 - ze^z}{(e^z - 1)^2} = \frac{1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots - 1 - z \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \right)}{(z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots)^2}
\]
\[
= \frac{\frac{z^2}{2} + \frac{z^3}{6} + \cdots - \left( \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots \right)}{z^2 + z^3 + \cdots}
\]
\[
= -\frac{\frac{z^2}{2} + \cdots}{z^2 + \cdots}.
\]
So the limit as $z \to 0$ is $-\frac{1}{2}$. Therefore
\[
\int_{|z|=1} \frac{1}{z(e^z - 1)} \, dz = 2\pi i \text{Res} \left[ \frac{1}{z(e^z - 1)}, 0 \right] = 2\pi i \left( -\frac{1}{2} \right) = -\pi i.
\]
(c) This line integral needs to be done directly from the definition. The curve $\gamma$ is parametrized by

$$z(t) = (1 + i)t \quad \text{for } 0 \leq t \leq 1.$$  

So

$$\int_\gamma z \, dz = \int_0^1 (1 + i)t d((1 + i)t)$$

$$= \int_0^1 (1 - i)t (1 + i) \, dt$$

$$= 2 \int_0^1 t \, dt$$

$$= 2 \cdot \frac{1}{2}$$

$$= 1.$$  

---  

Problem 3. (10 points) Use residue theory to compute the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2} \, dx.$$  

(I expect you to use complex analysis for this problem, although it can also be done using continued fractions as you learned in first-year calculus.)

Solution. Let

$$f(z) = \frac{z^2}{(z^2 + 1)^2},$$

and let

$$D_R = \{re^{i\theta} : -R < r < R \text{ and } 0 < \theta < \pi\}.$$  

(This is the usual half-disk in the upper half-plane.) The function $f(z)$ has one pole in $D_R$, namely $z = i$, and it’s a double pole, since

$$f(z) = \frac{z^2}{(z^2 + 1)^2} = \frac{z^2}{(z + i)^2(z + i)^2}.$$
The residue at $z = i$ is

$$\text{Res} [f(z), i] = \lim_{z \to i} \frac{d}{dz} \left( (z - i)^2 f(z) \right)$$

$$= \lim_{z \to i} \frac{d}{dz} \left( \frac{z^2}{(z + i)^2} \right)$$

$$= \lim_{z \to i} 2 \left( \frac{z}{z + i} \right) \frac{d}{dz} \left( \frac{z}{z + i} \right)$$

$$= \lim_{z \to i} 2 \left( \frac{z}{z + i} \right) \left( \frac{i}{(z + i)^2} \right)$$

$$= \frac{i}{2i} \cdot \frac{i}{-4}$$

$$= -\frac{i}{4}.$$  

So the residue theorem says that

$$\int_{\partial D_R} f(z) \, dz = 2\pi i \text{Res} [f(z), i] = \frac{\pi}{2}.$$  

Let $L_R$ be the line segment $-R < x < R$, and let $\Gamma_R$ be the semicircle of radius $R$ in the upper halfplane. Then

$$\int_{L_R} f(z) \, dz = \int_{-R}^{R} \frac{x^2}{(x^2 + 1)^2} \, dx.$$  

Next we estimate

$$\left| \int_{\Gamma_R} f(z) \, dz \right| \leq \sup_{z \in \Gamma_R} |f(z)| \cdot \text{Length}(\Gamma_R) \quad \text{(ML estimate),}$$

$$= \sup_{z \in \Gamma_R} \left| \frac{z^2}{(z^2 + 1)^2} \right| \cdot 2\pi R$$

$$\leq \frac{R^2}{(R^2 - 1)^2} \cdot 2\pi R \quad \text{(note it’s } R^2 - 1, \text{ not } R^2 + 1,)$$

$$\xrightarrow{R \to \infty} 0.$$  

So letting $R \to \infty$ and combining these calculations gives

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2} \, dx = \lim_{R \to \infty} \left( \int_{\partial D_R} f(z) \, dz - \int_{\partial \Gamma_R} f(z) \, dz \right) = \frac{\pi}{2}.$$  

**Problem 4.** (10 points) For each of the following functions, describe the Taylor series expansion about the indicated point, and compute the radius of convergence.
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(a) \( f(z) = \log(z) \) centered at \( z_0 = 2 \).

(b) \( f(z) = \frac{1}{(1 - z)^2} \) centered at \( z_0 = 0 \).

Solution. In general the Taylor series expansion of \( f(z) \) centered at \( a \) is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k.
\]

Each part of this problem can be done by computing the derivatives at the indicated point. Alternatively, one can use related series and differentiate or integrate them.

(a) We have \( f'(z) = \frac{1}{z} \), so

\[
f'(z) = \frac{1}{z} = \frac{1}{2 + (z - 2)} = \frac{1}{2} \cdot \frac{1}{1 + \frac{z - 2}{2}}.
\]

Now we can expand using the geometric series to get

\[
f'(z) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \left( \frac{z - 2}{2} \right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k+1} (z - 2)^k.
\]

Integrating gives

\[
f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)2^{k+1}} (z - 2)^{k+1} + C.
\]

The constant is obtained by setting \( z = 2 \), so \( \log(2) = f(2) = C \). Finally, relabeling, we get

\[
f(z) = \log(z) = \log(2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k2^k} (z - 2)^k.
\]

The radius of convergence \( \rho \) may be computed using the ratio test or the root test. The latter gives

\[
\rho^{-1} = \lim_{k \to \infty} \left| \frac{(-1)^{k+1}}{k2^k} \right|^{1/k} = \lim_{k \to \infty} \frac{1}{2k^{1/k}} = \frac{1}{2},
\]

so \( \rho = 2 \).

(b) Again, it’s not very hard to compute the derivatives. But even easier to note that \( f(z) \) is the derivative of \( \frac{1}{1-z} \), which is just a geometric series. So

\[
f(z) = \frac{d}{dz} \left( \frac{1}{1-z} \right) = \frac{d}{dz} \sum_{k=0}^{\infty} z^k = \sum_{k=1}^{\infty} k z^{k-1} = \sum_{k=0}^{\infty} (k - 1) z^k.
\]
The radius of convergence is
\[ \rho = \lim_{k \to \infty} \frac{1}{(k - 1)^{1/k}} = 1. \]

**Problem 5.** (10 points) Let
\[ f(z) = \frac{1}{z^2 - 2z}. \]

(a) Find the Laurent series of \( f(z) \) centered at 0 in the domain \( |z| < 2 \).
(b) Find the Laurent series of \( f(z) \) centered at 0 in the domain \( |z| > 2 \).

**Solution.** (a) We note that \( f(z) \) has a pole at \( z = 0 \), but that’s okay. The partial fraction expansion of \( f(z) \) is
\[ f(z) = \frac{1/2}{z - 2} - \frac{1/2}{z}. \]

We leave the second term alone and expand the first using the geometric series
\[ \frac{1/2}{z - 2} = \frac{1/4}{1 - \frac{z}{2}} = \frac{1}{4} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n = \sum_{n=0}^{\infty} \frac{-1}{2^{n+2}} z^n. \]

This converges on \( |z| < 2 \). Further, if we also include \( n = -1 \), we get the other term, so the Laurent series of \( f \) on the domain \( |z| < 2 \) is
\[ f(z) = \sum_{n=-1}^{\infty} \frac{-1}{2^{n+2}} z^n. \]

(b) For \( |z| > 2 \), we want an expansion in the variable \( 1/z \), so
\[ \frac{1/2}{z - 2} = \frac{1}{2z} \cdot \frac{1}{1 - \frac{2}{z}} = \frac{1}{2z} \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n = \sum_{n=0}^{\infty} \frac{2^{n-1}}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{2^{n-2}}{z^n}. \]

This gives
\[ f(z) = -\frac{1}{2z} + \sum_{n=1}^{\infty} \frac{2^{n-2}}{z^n}. \]

We can simplify by noting that the \( n = 1 \) term cancels the \(-1/2z\), so
\[ f(z) = \sum_{n=2}^{\infty} \frac{2^{n-2}}{z^n}. \]

**Problem 6.** (15 points) Let \( D \) be a bounded domain with nice boundary.
(a) Suppose that $f(z)$ is analytic on $D$, continuous on $D \cup \partial D$, and does not vanish on $D \cup \partial D$. Let

$$m = \inf_{z \in \partial D} |f(z)|$$

be the smallest value of $|f(z)|$ on the boundary of $D$. Prove that

$$|f(z)| \geq m \quad \text{for all } z \in D.$$

(This is a minimum principle that complements the maximum principle.)

**Problem 6.** (continued)

(b) Let $D$ be the unit disk. Find a function that is analytic on $D \cup \partial D$ and satisfies $f(0) = 0$, and such that $f$ does not satisfy the minimum principle.

(c) Suppose that $f(z)$ is analytic and non-constant on $D$ and continuous on $D \cup \partial D$. Assume further that $|f(z)|$ is constant for $z \in \partial D$. Prove that $f(z)$ must have a zero in $D$.

**Solution.** (a) For any function $h$, we write

$$M(h) = \sup_{z \in \partial D} |h(z)| \quad \text{and} \quad m(h) = \inf_{z \in \partial D} |h(z)|.$$

Since $f(z)$ does not vanish on $D$, we know that $g(z) = 1/f(z)$ is analytic on $D$. So we can apply the maximum principle to $g(z)$ to conclude that

$$|g(z)| \leq M(g) \quad \text{for all } z \in D.$$

Since $g = 1/f$, this implies that

$$\frac{1}{|f(z)|} \leq M(1/f) \quad \text{for all } z \in D.$$

But if $T$ is any set of positive real numbers, we have

$$\sup \left\{ \frac{1}{t} : t \in T \right\} = \frac{1}{\inf \{t : t \in T\}}.$$

This implies that $M(1/f) = 1/m(f)$. Substituting this in above gives

$$m(f) = 1/|f(z)| \quad \text{for all } z \in D.$$

(b) The simplest example is $f(z) = z$. Then $m(f) = 1$, but $|f(z)|$ is not larger than $m(f)$. In fact, we have $|f(z)| < m(f)$ for all $z$ in the unit circle.

(c) The maximum principle says that

$$|f(z)| \leq M(f) \quad \text{for all } z \in D.$$
Suppose that $f$ does not vanish. Then the minimum principle (proven in (a)) says that
\[ |f(z)| \geq m(f) \quad \text{for all } z \in D. \]
However, we’re given that $|f(z)|$ is constant for $z \in \partial D$, so $m(f) = M(f)$ directly from the definitions of $m(f)$ and $M(f)$. So our two inequalities imply that
\[ |f(z)| = M(f) = m(f) \quad \text{for all } z \in D. \]
In particular, there are points $z \in D$ for which $|f(z)| = M(f)$, so the other half of the maximum principle tells us that $f$ is constant.

**Problem 7.** (10 points) Let $f(z)$ be the polynomial $f(z) = z^4 + 5z + 1$.

(a) Prove that $f(z)$ has exactly one root inside the disk $|z| < 1$.
(b) How many roots does $f(z)$ have inside the annulus $1 < |z| < 2$? Prove that your answer is correct.

**Solution.** (a) For $|z| = 1$ we have
\[ |5z| = 5 \geq 2 = |z^4| + 1 \geq |z^4 + 1|. \]
So from Rouché’s theorem, the polynomial $f(z)$ and the polynomial $5z$ have the same number of zeros in the disk $|z| < 1$. Since $5z$ clearly has one zero in the disk, so does $f(z)$.
(b) On the circle $|z| = 2$ we have
\[ |z^4| = 16 \geq 6 = |5z| + 1 \geq |5z + 1|, \]
so $f(z)$ and $z^4$ have the same number of zeros in the disk $|z| < 2$. Since $z^4$ has four zeros (counted with multiplicity), so does $f(z)$. That’s the number of zeros in the disk $|z| < 2$, and we know from (a) that there is one zero in the disk $|z| < 1$, so $f(z)$ has three zeros in the annulus $1 < |z| < 2$.

**Problem 8.** (10 points) Let $f(z)$ be analytic in a domain $D$, and suppose that $f$ satisfies
\[ \Re(f(z)) = \Im(f(z)) \quad \text{for all } z \in D. \]
Prove that $f$ is constant in $D$. 

Solution. There are probably lots of ways to do this problem. Here’s one. Write $f(z) = u(x, y) + iv(x, y)$ as usual. The assumption that $\text{Re}(f(z)) = \text{Im}(f(z))$ says that

$$u(x, y) = v(x, y).$$

Now the Cauchy–Riemann equations yield

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$  \hspace{1cm} (Cauchy–Riemann equation)

$$= \frac{\partial u}{\partial y}$$  \hspace{1cm} (since $v = u$)

$$= -\frac{\partial v}{\partial x}$$  \hspace{1cm} (Cauchy–Riemann equation)

$$= -\frac{\partial u}{\partial x}$$  \hspace{1cm} (since $v = u$).

It follows that

$$\frac{\partial u}{\partial x} = 0.$$

A similar calculation gives

$$\frac{\partial u}{\partial y} = 0.$$

Alternatively, we can use $u_x = 0$ and compute

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}.$$

Thus $u_x = 0$ and $u_y = 0$, which implies that $u$ is a constant. And since $v = u$, we find that $f = u + iv$ is also a constant.

Problem 9. (10 points) Prove that there exists a function $f(z)$ with the following properties:

- $f(z)$ is meromorphic on $\mathbb{C}$.
- $f(z)$ has simple poles at the points $\{1, 2, 3, 4, \ldots\}$ and no other poles.
- For $k \in \{1, 2, 3, \ldots\}$, the residue of $f(z)$ at $k$ is equal to $k$.

Be sure to prove that the function that you define is meromorphic, as well as having the indicated poles and residues.

Solution. We’d like to use

$$\sum_{k=1}^{\infty} \frac{k}{z - k}.$$
but it doesn’t converge. The function \( k/(z-k) \) looks like \(-1\) when \( k \) is large, so we might try summing
\[
\frac{k}{z-k} + 1 = \frac{z}{z-k}.
\]
But \( z/(z-k) \) looks like \(-z/k\) when \( k \) is large, so its sum won’t converge, either. So we add on \( z/k \) to compensate,
\[
\frac{k}{z-k} + 1 + \frac{z}{k} = \frac{z}{z-k} + \frac{z}{k} = \frac{z^2}{(z-k)k}.
\]
Note that this calculation shows that \( z^2/((z-k)k) \) has a simple pole at \( z = k \) with residue \( k \).

Then we define
\[
f(z) = \sum_{k=1}^{\infty} \left( \frac{k}{z-k} - 1 - \frac{z}{k} \right) = \sum_{k=1}^{\infty} \frac{z^2}{(z-k)k}.
\]
The usual argument shows that \( f \) is meromorphic with the correct poles. We briefly indicate. Choose any \( R \) (not an integer) and break up \( f \) as
\[
f(z) = f_1(z) + f_2(z) = \sum_{k<2R} \frac{z^2}{(z-k)k} + \sum_{k>2R} \frac{z^2}{(z-k)k}.
\]
Let \( D_R = \{ |z| < R \} \) be a disk of radius \( R \). Then \( f_1(z) \) is meromorphic on \( D_R \) with simples poles at the integers \( k < R \) and residue \( k \) at \( k \). On the other hand, for \( z \in D_R \) and \( k > 2R \) we have
\[
\left| \frac{z^2}{(z-k)k} \right| \leq \frac{R^2}{(k-R)k};
\]
so
\[
\sum_{k>2R} \left| \frac{z^2}{(z-k)k} \right| \leq \sum_{k>2R} \frac{R^2}{(k-R)k} < \infty.
\]
The Weierstrass \( M \)-test implies that the series defining \( f_2(z) \) converges to an analytic function on \( D_R \). Hence \( f(z) \) is meromorphic on \( D_R \) with the desired poles and residues at \( \{ k < R \} \). Since \( R \) is arbitrary, this shows that \( f(z) \) is entire with the desired poles and residues.

**Problem 10.** (10 points) Let \( f(z) \) be an analytic function that maps the unit disk conformally to a domain \( D \). In other words, if we denote the unit disk by \( \mathbb{D} = \{ |z| < 1 \} \), then
\[
f : \mathbb{D} \rightarrow D \quad \text{is analytic, one-to-one, and onto.}
\]
Also let
\[
m = \inf_{w \in \partial D} |f(0) - w|
\]
be the distance from \( f(0) \) to the boundary of \( D \). Prove that
\[
|f'(0)| \geq m.
\]

(Hint. Consider the inverse function \( f^{-1} : D \rightarrow \mathbb{D} \), note that the disk around \( f(0) \) of radius \( m \) is contained in \( D \), and use Schwarz’s lemma.)

**Solution.** We consider the inverse function
\[
f^{-1} : D \rightarrow \mathbb{D}.
\]
Our choice of \( m \) tells us that the disk
\[
B = \{ w \in \mathbb{C} : |f(0) - w| < m \}
\]
is contained in \( D \), so \( f^{-1} \) is analytic on \( B \); and since the image of \( f^{-1} \) is in \( \mathbb{D} \), we know that
\[
|f^{-1}(w)| \leq 1 \quad \text{for all } w \in D.
\]
We want to shift \( B \) to be the unit disk. The map \( z \mapsto mz + f(0) \) send the unit disk to \( B \), so we should look at the function
\[
g(z) = f^{-1}(mz + f(0)).
\]
Then \( g : \mathbb{D} \rightarrow \mathbb{D} \) with \( g(0) = 0 \), so the derivative version of Schwarz’s lemma says that \( |g'(0)| \leq 1 \). Note that
\[
g'(0) = (f^{-1})'(f(0))m.
\]
So we find that
\[
|(f^{-1})'(f(0))| \leq \frac{1}{m}.
\]
Okay, now we differentiate the identity
\[
f^{-1}(f(z)) = z
\]
to get
\[
(f^{-1})'(f(z)) \cdot f'(z) = 1.
\]
Evaluating at \( z = 0 \) gives
\[
(f^{-1})'(f(0)) \cdot f'(0) = 1.
\]
So
\[
(f^{-1})'(f(0)) = \frac{1}{f'(0)},
\]
and substituting this above gives
\[
\left| \frac{1}{f'(0)} \right| \leq \frac{1}{m}.
\]
Cross-multiplying gives
\[
m \leq |f'(0)|,
\]
which is the desired result.
Problem 11. (10 points) Compute the value of the integrals
\[ \int_0^\infty \cos(x^2) \, dx \quad \text{and} \quad \int_0^\infty \sin(x^2) \, dx \]

*Hint #1.* Integrate the function \( f(z) = e^{iz^2} \) around the boundary of the region
\[ D_R = \{ re^{i\theta} : 0 < r < R \text{ and } 0 < \theta < \frac{\pi}{4} \} . \]

*Hint #2.* The following integral from 3rd semester calculus may be useful:
\[ \int_0^\infty e^{-t^2} \, dt = \frac{\sqrt{\pi}}{2} . \]

*Warning.* Don’t spend too much time on this problem until you’ve worked on the other problems.

*Solution.* The function \( f(z) = e^{iz^2} \) is entire, so Cauchy’s theorem tells us that
\[ \int_{\partial D_R} f(z) \, dz = 0. \]

The boundary of \( D_R \) consists of 3 pieces:
\[ L_1 = \{ x : 0 \leq x \leq R \}, \]
\[ L_2 = \{ t\sqrt{i} : 0 \leq t \leq R \} \quad (\text{in reverse direction}), \]
\[ \Gamma_R = \{ Re^{i\theta} : 0 \leq \theta \leq \pi/4 \} . \]

Here \( \sqrt{i} \) is the square root in the first quadrant, i.e., \( \sqrt{i} = \frac{1+i}{\sqrt{2}} \).

The integral along \( L_1 \) gives the integrals that we’re trying to compute,
\[ \int_{L_1} f(z) \, dz = \int_0^R e^{ix^2} \, dx = \int_0^R \cos(x^2) \, dx + i \int_0^R \sin(x^2) \, dx. \]

For \( L_2 \), we have
\[ \int_{L_2} f(z) \, dz = \int_0^R e^{i(\sqrt{t})^2} \, d(\sqrt{it}) \]
\[ = -\sqrt{i} \int_0^R e^{-t^2} \, dt \]
\[ = -\frac{1+i}{\sqrt{2}} \int_0^R e^{-t^2} \, dt. \]
Hence
\[
\lim_{R \to \infty} \int_{L_2} f(z) \, dz = -\frac{1 + i}{\sqrt{2}} \int_0^\infty e^{-t^2} \, dt = -\frac{1 + i}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}.
\]

Finally, for \( \Gamma_R \) we use Jordan’s lemma, which says that if \( C_R \) is the semicircle of radius \( R \) in the upper halfplane, then
\[
\int_{C_R} |e^{iz}| \, |dz| < \pi.
\]

Our curve \( \Gamma_R \) is not equal to \( C_R \). By making the change of variables \( w = z^4 \), we could map \( \Gamma_R \) to \( C_R \), but then we wouldn’t get the integral in Jordan’s Lemma. So instead we use the change of variables \( w = z^2 \), which maps \( \Gamma_R \) to the quarter-circle
\[
B_R = \{ Re^{i\theta} : 0 \leq \theta \leq \pi/2 \}.
\]

Then
\[
\int_{\Gamma_R} f(z) \, dz = \int_{B_R} f(w^{1/2}) \, d(w^{1/2})
= \int_{B_R} e^{iw} \, \frac{dw}{2w^{1/2}}.
\]

Hence
\[
\left| \int_{\Gamma_R} f(z) \, dz \right| = \left| \int_{B_R} e^{iw} \, \frac{dw}{2w^{1/2}} \right|
\leq \int_{B_R} |e^{iw}| \frac{|dw|}{2|w|^{1/2}}
= \frac{1}{2R^{1/2}} \int_{B_R} |e^{iw}| |dw|.
\]

In order to use Jordan’s lemma, we note that \( C_R \) is the union of the quarter-circle \( B_R \) and the quarter-circle \( B'_R = \{ -\overline{z} : z \in B_R \} \). In other words, the map \( z \to -\overline{z} \) maps \( B_R \) to \( B'_R \). We also note that if \( z = x + iy \in B_R \), then \( -\overline{z} = -x + iy \), so
\[
|e^{-\pi i}| = |e^{(-x+iy)i}| = |e^{-y-ix}| = e^{-y} = |e^{(x+yi)i}| = |e^z|,
\]
and similarly
\[
|d(\overline{z})| = |dz|,
\]
so
\[
\int_{B_R} |e^{iz}| \, |dz| = \int_{B'_R} |e^{iz}| \, |dz|.
\]