INSTRUCTIONS—Read Carefully

- Exam is due at:
  5:00pm on Thursday December 11 in the Math Dept Office.
  Give your exam to Audrey or Lori. And don’t be late, because the office closes at 5:00pm.
- There are 7 problems.
- You may use your class notes and the textbook (Lang’s Algebra). You may not work with other people, and you most definitely may not look up material and/or solutions on the web. If you have any questions, for example of a definition, send me an email: jhs@math.brown.edu.
- Please write your proofs clearly, which generally means writing a rough draft and then re-writing it for clarity. Use the guidelines that I listed for the midterm exam.
- I would prefer that you write up your solutions using \LaTeX, but this is not required. If you use \LaTeX, please use 12 point fonts. If you do not use \LaTeX, be sure that you write up your solutions very neatly and legibly.
Problem 1. (15 points) Let $R$ be a Noetherian ring and let $S \subset R$ be a multiplicatively closed set.

(a) Prove that $S^{-1}R$ is a Noetherian ring.

(b) Is the converse true, i.e., if $S^{-1}R$ is Noetherian, is it true that $R$ must be Noetherian?

Problem 2. (40 points) Let $R$ be a ring and let $I$ be an ideal. Then we get a homomorphism

$$GL_d(R) \longrightarrow GL_d(R/I)$$

by sending a matrix $A$ to the matrix $\bar{A}$ obtained by reducing its coefficients modulo $I$.

(a) Let $A \in GL_d(R)$ be an element of exact order $n$. Assume that $R$ is an integral domain and that $n \notin I$. Prove that $\bar{A}$ has exact order $n$ in $GL_d(R/I)$.

(b) Let $p$ be a prime. Prove that

$$\# GL_d(\mathbb{F}_p) = (p^d - 1)(p^d - p) \cdots (p^d - p^{d-1}).$$

(c) Let $\ell$ be an odd prime and let $r \geq 1$. Prove that the unit group $(\mathbb{Z}/\ell^r\mathbb{Z})^*$ is cyclic. What happens when $\ell = 2$?

(d) Suppose that $A \in GL_d(\mathbb{Z})$ has exact order $\ell^r$ for some prime $\ell$ and $r \geq 0$. Use (a), (b), and (c) to give a bound for $\ell^r$ in terms of $d$. (You may use Dirichlet’s theorem on primes in arithmetic progressions, i.e., if $\gcd(a,m) = 1$, then there are infinitely many primes $p$ satisfying $p \equiv 1 \pmod{m}$.)

Problem 3. (15 points) Construct a ring $R$ that has non-zero elements $a, b \in R$ with the following two properties:

(i) Neither $a$ nor $b$ is a zero divisor. (Even better, construct an $R$ that is an integral domain with $a, b \neq 0$.)

(ii) For every $n \geq 1$ there exists an element $c \in R$ such that

$$a = b^n c.$$  

(Equivalently, for every $n \geq 1$, the element $a$ is in the ideal generated by $b^n$.)

Also prove that if $R$ is such a ring, then it is cannot be Noetherian.
Problem 4. (25 points) Let $R$ be a ring. An element $e$ of $R$ that satisfies $e^2 = e$ is called an idempotent.

(a) Let $e$ be an idempotent of $R$ that is not equal to 0 or 1. Prove that the set

$$eR = \{er : r \in R\}$$

can be given the structure of a ring using the addition and multiplication inherited from $R$ and letting $e$ be the multiplicative identity.

(b) If $e \neq 0, 1$ is an idempotent, prove that $1-e$ is also an idempotent, and thus $(1-e)R$ is a ring with identity element $1-e$.

(c) Prove that the map

$$\alpha : R \longrightarrow eR \times (1-e)R, \quad r \longmapsto (er, (1-e)r),$$

is an isomorphism of rings by constructing a ring homomorphism in the opposite direction and showing that it is the inverse of $\alpha$.

(d) Conversely, if

$$\beta : R \sim R' \times R''$$

is an isomorphism of rings, prove that there is an idempotent $e \in R$ and ring isomorphisms

$$\beta' : R' \sim eR \quad \text{and} \quad \beta'' : R'' \sim (1-e)R$$

so that the following diagram commutes, where $\alpha$ is the isomorphism from (c):

$$\begin{array}{ccc}
R & \xrightarrow{\beta} & R' \times R'' \\
\| & & \downarrow{\beta' \times \beta''} \\
R & \xrightarrow{\alpha} & eR \times (1-e)R
\end{array}$$

(e) Let $R$ be a local ring, i.e., a ring with a unique maximal ideal. Prove that the only idempotents of $R$ are 0 and 1.
Problem 5. (10 points) Let $R$ be an integral domain and $M$ an $R$-module. Recall that an element $m \in M$ is a torsion element if $rm = 0$ for some non-zero $r \in R$, and that the set of torsion elements

$$M_{\text{tors}} = \{m \in M : m \text{ is a torsion element}\}$$

is a submodule of $M$. Prove that the following are equivalent:

(a) $M_{\text{tors}} = 0$.
(b) $(M_p)_{\text{tors}} = 0$ for all prime ideals $p$ of $R$.
(c) $(M_{\mathfrak{m}})_{\text{tors}} = 0$ for all maximal ideals $\mathfrak{m}$ of $R$.

Problem 6. (20 points)

(a) Prove that the map

$$\mathcal{F} : \text{Rings} \rightarrow \text{Groups},$$

$$R \mapsto R^*,$$

that sends a ring $R$ to its unit group is a functor from the category of rings to the category of groups.

(b) Prove that the functor in (a) may also be described as the map

$$R \mapsto \text{Hom} \left( \frac{\mathbb{Z}[x,y]}{(xy - 1)}, R \right),$$

where $\text{Hom}(R_1, R_2)$ denotes the set of ring homomorphisms from $R_1$ to $R_2$. 
Problem 7. (50 points)

(a) Let $R$ be a ring, let $I$ be an ideal of $R$, and let $M$ be an $R$-module. Prove that

$$\frac{R}{I} \otimes_R M \cong \frac{M}{IM}$$

by showing directly that $M/IM$ satisfies the universal property of the indicated tensor product.

(b) Let $R$ be a local ring and let $M$ and $N$ be finitely generated $R$-modules. Suppose that $M \otimes_R N = 0$. Prove that either $M = 0$ or $N = 0$. (Hint. Use (a) and apply Nakayama’s lemma.)

(c) Give an example of finitely generated $\mathbb{Z}$-modules $M$ and $N$ such that $M \neq 0$ and $N \neq 0$ and $M \otimes_{\mathbb{Z}} N = 0$. (This shows that in proving (b), it is essential that $R$ be a local ring.)

(d) Let $M$ and $N$ be free, finitely generated $R$-modules. Prove that $M \otimes_R N$ is free and satisfies

$$\text{rank}(M \otimes_R N) = (\text{rank } M)(\text{rank } N).$$

(e) Let $M$ be a finitely generated $\mathbb{Z}$-module. Describe the kernel of the homomorphism

$$M \rightarrow M \otimes_{\mathbb{Z}} \mathbb{Q}, \quad m \mapsto m \otimes 1.$$

(What about if $M$ is not assumed to be finitely generated?)