ETHIER & KURTZ NOTES 1:

**Definition.** Let $A$ be a closed linear operator on $L$. A subspace $D$ is said to be a core for $A$ if $A|_D = A$.

Recall: $\overline{B}$ (the closure of $B$) is the minimal closed extension of $B$. We proved last time that $\overline{B}$ exists ("$B$ is closable") if and only if the closure $\overline{G(B)}$ in $L \times L$ of $G(B)$ is the graph of a function, and characterised $\overline{B}$ as the operator whose graph is $\overline{G(B)}$.

In particular, Lemma 2.11 stated that if $B$ is dissipative and $D(B)$ is dense, then $B$ is closable.

In the next proposition, we will use the Hille-Yoshida theorem to characterize cores of generators of sccsg’s.

**3.1 Proposition.** Let $A$ be the generator of a strongly continuous contraction semigroup on $L$. Then a subspace $D$ of $D(A)$ is a core for $A$ if and only if $D$ is dense in $L$ and $R(\lambda - A|_D)$ is dense in $L$ for some $\lambda > 0$.

**Proof.** Suppose $D$ is dense and $R(\lambda - A|_D)$ is dense for some $\lambda > 0$. Then, since dissipativity of $A$ implies dissipativity of $A|_D$, $A|_D$ satisfies the hypotheses of the Hille-Yoshida theorem (Theorem 2.12), hence $\overline{A|_D}$ exists and generates a semigroup $T^D(t)$. On the other hand, $A$ generates a semigroup $T(t)$. We claim $\overline{A|_D}$ is extended by $A$ (i.e., $D(\overline{A|_D}) \subset D(A)$ and $\overline{A|_D} f = Af$ for $f \in D(\overline{A|_D})$). This is clear since the Hille-Yoshida theorem and Lemma 2.2 imply that $A$ is closed, and obviously the graph of $A$ contains that of $A|_D$.

This result (that $A$ extends $\overline{A|_D}$) is enough to show $\overline{A|_D} = A$ in the following way. We will see that it implies that the semigroups of these generators are the same. In fact, Proposition 2.10 shows that $T(t) = T^D(t)$ on $D(\overline{A|_D})$, hence by density of $D(\overline{A|_D})$ and strong continuity of $T(t)$ and $T^D(t)$, on all of $L$. From the definition of generator, it then follows that $A = \overline{A|_D}$, in the sense that their domains are equal and they agree on this domain. Hence $D$ is a core for $A$.

Conversely, assume that $D$ is a core for $A$, so that $A = \overline{A|_D}$. Obviously, $\overline{A|_D}$ is therefore the generator of a sccsg, so by the Hille-Yoshida theorem (2.12), $D(\overline{A|_D}) = D$ is dense and $R(\lambda - A|_D)$ is dense for some $\lambda > 0$. I have no clue why the authors cite Lemma 2.11 as being relevant to this!
For a sccsg $T(t)$ with generator $A$, the above theorem provides conditions for a subspace $D$ to be a generator for $A$ based on direct relations between $A$ and $D$. On the other hand, the next theorem gives a sufficient condition for $D$ to be a core for $A$ based on a relation between $T(t)$, rather than $A$, and $D$.

(3.3 Proposition). Let $A$ be the generator of a strongly continuous contraction semigroup $\{T(t)\}$ on $L$. Let $D_0$ and $D$ be dense subspaces of $L$ with $D_0 \subset D \subset D(A)$. (Usually, $D_0 = D$.) If $T(t) : D_0 \to D$ for all $t \geq 0$, then $D$ is a core for $A$.

Remark: What is $D_0$ doing here? The presence of $D_0$ allows us to verify the condition $T : D_0 \to D$ rather than $T : D \to D$, for a smaller space $D_0$ than $D$, which is technically easier to do. For example, if $D$ is some Sobolev space, instead of verifying that $T(t)$ preserves $D$ (in case $D$ may consist of poorly behaved functions which are hard to manipulate), we may instead take some subspace $D_0$ consisting of better behaved or very smooth functions for which it is easier to verify that $T(t) : D_0 \to D$. In particular, the theorem states that this will work with any $D_0$ which is dense and contained in $D$, so it may be to our advantage in applications to take $D_0$ as small as possible.

Proof. We shall reduce this to Proposition 3.1 above. Here is the idea. Since we already know that $D$ is dense, by that proposition, we just have to show that $R(\lambda - A|_D)$ (which equals $R(\lambda - A)$ on $D$) is dense in $L$ for some $\lambda > 0$. Proposition 2.1,

$$(\lambda - A)^{-1}g = \int_0^\infty e^{-\lambda t}T(t)g dt$$

tells us that the way to $\lambda - A$ is via the above integral, and by considering this integral as a limit of simple functions we will be able to use the conditions $T(t) : D_0 \to D$.

Now given $f \in D_0$, we define the simple approximants to the above integral by $f_m$ by

$$f_n = \sum_{k=0}^{n^2} e^{-\lambda t/n}T\left(\frac{k}{n}\right)f \in D.$$ 

Applying $(\lambda - A)$ to both sides, using the linearity of $T$, taking limits, and applying the monotone convergence theorem, we get

$$\lim_{n \to \infty} (\lambda - A)f_n = \lim_{n \to \infty} \sum_{k=0}^{n^2} e^{-\lambda t/n}T\left(\frac{k}{n}\right)(\lambda - A)f \in D.$$