M1410 Homework Worksheet:

1. Define two points $p, q \in \mathbb{R}^2$ to be equivalent if $p - q \in \mathbb{Z}^2$, the set of integer points. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Prove that $T^2$ is a smooth manifold by finding coordinate charts to $\mathbb{R}^2$ such that the overlap functions are translations. $T^2$ is the square torus.

2. Prove that there exists a closed 1-form on $T^2$ which is not exact. This means that $H^1(T^2)$ is nontrivial.

3. Let $S^2$ denote the 2-sphere, namely the solution to the equation

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. Let $\Delta \subset S^2$ be any round disk. Prove that there is a diffeomorphism from $S^2 - \Delta$ to an open disk in $\mathbb{R}^2$.

4. Suppose that $\omega$ is a closed 1-form on $S^2$. Let $\Delta$ be some disk in $S^2$ which does not contain $(0, 0, -1)$. Prove that there is a unique smooth function $f$ on $S^2 - \Delta$ such that $f(0, 0, -1) = 0$ and $df = \omega$ on $S^2 - \Delta$. Hint: Problem 3 might be helpful.

5. Suppose that $\omega$ is a closed 1-form on $S^2$. Prove that there is some function $f$ on all of $S^2$ such that $df = \omega$. Hint: Problem 4 might be helpful. Conclude that $H^1(S^2)$ is trivial.

6. Suppose that $M_1$ and $M_2$ are two compact manifolds and $F : M_1 \to M_2$ is a diffeomorphism. Prove that $H^k(M_1)$ and $H^k(M_2)$ are isomorphic vector spaces for any $k$. Deduce from this that there is no diffeomorphism from $T^2$ to $S^2$. These are two compact 2-manifolds which are not diffeomorphic.

7. Generalize the argument above to show that the $n$-torus $T^n$ is not diffeomorphic to the $n$-sphere $S^n$, for all $n > 2$.

8. (Bonus) Using $H^1$, show that the product manifold $(S^2 \times T^2)$ is not diffeomorphic to $S^4$. 