The Poincare Lemma

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The purpose of these notes is to explain the proof of Poincare’s lemma from the book in somewhat less compressed form.

1 The Main Result

A domain $U \subset \mathbb{R}^n$ is star shaped with respect to $p \in \mathbb{R}^n$ if, for each $q \in U$, the entire segment $pq$ lies in $U$. We say that $U$ is star-shaped if $U$ is star-shaped with respect to some point. Here is the main result. Recall that $\Omega^r(U)$ is the space of smooth $r$-forms on $U$.

Lemma 1.1 (Poincare) Let $U$ be an open star-shaped subset of $\mathbb{R}^n$ and let $\omega \in \Omega^r(U)$. Suppose that $d\omega = 0$. Then there is some $\alpha \in \Omega^{r-1}(U)$ such that $d\alpha = \omega$.

The proof in the book is stated in terms of convex domains, but it really just uses the star-shaped property.

2 The Algebra Behind the Result

By symmetry, it suffices to consider the case when $U$ is star-shaped with respect to the origin. We define $\tilde{U} \subset \mathbb{R}^n \times \mathbb{R}$ to be the set of points $(u,t)$ such that $tu \in U$. Note that $\tilde{U}$ is an open set which contains $U \times [0,1]$.

There is a map $F : \tilde{U} \to U$ given by

$$F(u,t) = ut.$$ (1)
There are also two maps $g_0, g_1 : U \to \tilde{U}$ given by

$$g_i(u) = (u, i).$$  \hspace{1cm} (2)

Notice that

- $F \circ g_1(u) = F(u, 1) = u$. Hence $F \circ g_1$ is the identity map. This means that $g_1^* \circ F^* = (F \circ g_1)^*$ is the identity on differential forms.

- $F \circ g_0(u) = F(u, 0) = 0$. Therefore, $F \circ g_0$ is the constant map. Since $D(F \circ g_0) = 0$, this means that $g_0^* \circ F^* = (F \circ g_0)^*$ is the 0-map.

Note that both $g_0^*$ and $g_1^*$ are maps from $\Omega^r(\tilde{U})$ to $\Omega^r(U)$. The main step in the proof is do construct a map $J : \Omega^{r+1}(\tilde{U}) \to \Omega^r(U)$ with the property that

$$Jd - dJ = \pm(g_1^* - g_0^*).$$  \hspace{1cm} (3)

The sign depends on $r$ in a way that we don’t care about. Equation 3 makes sense, because all maps go from $\Omega^r(\tilde{U})$ into $\Omega^r(U)$.

Let’s see what Equation 3 gives us. We start with $\omega \in \Omega^r(U)$ such that $d\omega = 0$. We then define

$$\tilde{\omega} = F^*(\omega).$$  \hspace{1cm} (4)

Note that

$$d\tilde{\omega} = dF^*(\omega) = F^*d\omega = F^*(0) = 0.$$  \hspace{1cm} (5)

Hence $Jd\tilde{\omega} = 0$. We compute

$$dJ(\tilde{\omega}) =$$

$$0 + dJ(\tilde{\omega}) =$$

$$-Jd\tilde{\omega} + dJ\tilde{\omega} =$$

$$\pm(g_1^*(\tilde{\omega}) - g_0^*(\tilde{\omega})) =$$

$$\pm(g_1^*F^*\omega - g_0^*F^*\omega) =$$

$$\pm(\omega - 0) = \pm\omega.$$  \hspace{1cm}

We take

$$\alpha = \pm J(\tilde{\omega}).$$  \hspace{1cm} (6)

If we pick the sign right then $d\alpha = \omega$. 

2
3 Construction of the Main Map

Now we construct the map \( J : \Omega^{r+1}(\hat{U}) \to \Omega^r(U) \). It is convenient to set \( t = x_{n+1} \).

Any form \( \eta \in \Omega^{r+1}(\hat{U}) \) can be written in the standard basis:

\[
\eta = \sum_K c_K \, dx_K, \tag{7}
\]

The sum take place over increasing multi-indexes of length \( r + 1 \). We can write \( \eta = \eta_1 + \eta_2 \), where \( \eta_1 \) is the sum over the multi-indices which do not involve \( n + 1 \) and \( \eta_2 \) is the sum over the multi-indices which do involve \( n + 1 \).

We have

\[
\eta_1 = \sum_I a_I \, dx_I, \quad \eta_2 = \sum_J b_J \, dx_J \wedge dt. \tag{8}
\]

The first sum is taken over multi-indexes of length \( r + 1 \) which involve \( n + 1 \). The second sum is taken over multi-indexes \( J \) of length \( r \) which do not involve \( n + 1 \). Define

\[
J(\eta) = \sum_J B_J \, dx_J, \quad B_J(p) = \int_0^1 b_j(p, t) \, dt \tag{9}
\]

Notice that \( J \) really is a linear map from \( \Omega^{r+1}(\hat{U}) \) to \( \Omega^r(U) \). Note also that \( J(\eta_1) = 0 \).

4 The Calculations

Both sides of Equation 3 respect sums. So, it suffices to prove Equation 3 for a form \( \eta = adx_I \). There are 2 cases, depending on whether \( I \) involves \( n + 1 \).

4.1 Case 1

Suppose that \( I \) does not involve \( n + 1 \). Then \( J\eta = 0 \) and so

\[
dJ\eta = 0. \tag{10}
\]

On the other hand

\[
d\eta = \frac{\partial a}{\partial t} \, dt \wedge dx_I + \beta,
\]
where $\beta$ only has terms which do not involve $dt$. Since $J(\beta) = 0$, we have

$$Jd\eta = (-1)^r Adx_I, \quad A(p) = \int_0^1 \frac{\partial a(p,t)}{\partial t} dt = a(p,1) - a(p,0). \quad (11)$$

Note that $g_1^*(dx_i) = dx_i$ for $i = 1, ..., n$. For this reason

$$g_0^*(\eta) = a(\cdot, 0)dx_I, \quad g_1^*(\eta) = a(\cdot, 1)dx_I. \quad (12)$$

Equation 3 follows in this case from Equations 10, 11, and 12.

### 4.2 Case 2

Now suppose that $\eta = adx_I$ where $I$ involves $n + 1$. We can write

$$\eta = a \, dx_J \wedge dt,$$

where $J$ is obtained from $I$ by dropping the $n + 1$.

We compute

$$J\eta = Ax_J, \quad A(p) = \int_0^1 a(p,t) \, dt.$$ 

Therefore,

$$dJ\eta = dA \wedge dx_J = \sum_{i=1}^n \frac{\partial A}{\partial x_i} dx_i \wedge dx_J. \quad (13)$$

Differentiating under the integral sign, we get

$$\frac{\partial A}{\partial x_i}(p) = \int_0^1 \frac{\partial a}{\partial x_i}(p,t). \quad (14)$$

$$d\eta = \sum_{i=1}^n \frac{\partial a}{\partial x_i} \wedge dX_J \wedge dt.$$ 

We never differentiate by $t$ because every term of $\eta$ involves $dt$. Finally

$$Jd\eta = \sum_{i=1}^n \frac{\partial A}{\partial x_i} \wedge dX_J = dJ\eta.$$ 

In this case $dJ(\eta) - Jd(\eta) = 0$. On the other hand $g_j^*(dt) = 0$ for $j = 0, 1$. So, Equation 3 again holds.