1 Modules

Basic Definition: Let $R$ be a commutative ring with 1. A (unital) $R$-module is an abelian group $M$ together with a operation $R \times M \to M$, usually just written as $rv$ when $r \in R$ and $v \in M$. This operation is called scaling. The scaling operation satisfies the following conditions.

1. $1v = v$ for all $v \in M$.
2. $(rs)v = r(sv)$ for all $r, s \in R$ and all $v \in M$.
3. $(r + s)v = rv + sv$ for all $r, s \in R$ and all $v \in M$.
4. $r(v + w) = rv + rw$ for all $r \in R$ and $v, w \in M$.

Technically, an $R$-module just satisfies properties 2, 3, 4. However, without the first property, the module is pretty pathological. So, we’ll always work with unital modules and just call them modules. When $R$ is understood, we’ll just say module when we mean unital $R$-module.

Submodules and Quotient Modules: A submodule $N \subseteq M$ is an abelian group which is closed under the scaling operation. So, $rv \in N$ provided that $v \in N$. A submodule of a module is very much like an ideal of a ring. One defines $M/N$ to be the set of (additive) cosets of $N$ in $M$, and one has the scaling operation $r(v + N) = (rv) + N$. This makes $M/N$ into another $R$-module.

Examples: Here are some examples of $R$-modules.
• When $R$ is a field, an $R$-module is just a vector space over $R$.

• The direct product $M_1 \times M_2$ is a module. The addition operation is done coordinate-wise, and the scaling operation is given by

$$r(v_1, v_2) = (rv_1, rv_2).$$

More generally, $M_1 \times \ldots \times M_n$ is another $R$-module when $M_1, \ldots, M_n$ are.

• If $M$ is a module, so is the set of finite formal linear combinations $L(M)$ of elements of $M$. A typical element of $L(M)$ is

$$r_1(v_1) + \ldots + r_n(v_n), \quad r_1, \ldots, r_n \in R, \quad v_1, \ldots, v_n \in M.$$

This definition is subtle. The operations in $M$ allow you to simplify these expressions, but in $L(M)$ you are not allowed to simplify. Thus, for instance, $r(v)$ and $1(rv)$ are considered distinct elements if $r \neq 1$.

• If $S \subseteq M$ is some subset, then $R(S)$ is the set of all finite linear combinations of elements of $S$, where simplification is allowed. With this definition, $R(S)$ is a submodule of $M$. In fact, $R(S)$ is the smallest submodule that contains $S$. Any other submodule containing $S$ also contains $R(S)$. As with vector spaces, $R(S)$ is called the span of $S$.

2 The Tensor Product

The tensor product of two $R$-modules is built out of the examples given above. Let $M$ and $N$ be two $R$-modules. Here is the formula for $M \otimes N$:

$$M \otimes N = Y/Y(S), \quad Y = L(M \times N),$$

(1)

and $S$ is the set of all formal sums of the following type:

1. $(rv, w) - r(v, w)$.
2. $(w, rv) - r(v, w)$.
3. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$.
4. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$. 

2
Our convention is that \((v, w)\) stands for \(1(v, w)\), which really is an element of \(L(M \times N)\). Being the quotient of an \(R\)-module by a submodule, \(M \otimes N\) is another \(R\)-module. It is called the tensor product of \(M\) and \(N\).

There is a map \(B : M \times N \rightarrow M \otimes N\) given by the formula

\[
B(m, n) = [(m, n)] = (m, n) + Y(S),
\]

namely, the \(Y(S)\)-coset of \((m, n)\). The traditional notation is to write

\[
m \otimes n = B(m, n).
\]

The operation \(m \otimes n\) is called the tensor product of elements.

Given the nature of the set \(S\) in the definition of the tensor product, we have the following rules:

1. \((rv) \otimes w = r(v \otimes w)\).
2. \(r \otimes (rw) = r(v \otimes w)\).
3. \((v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w\).
4. \(v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2\).

These equations make sense because \(M \otimes N\) is another \(R\)-module. They can be summarised by saying that the map \(B\) is bilinear. We will elaborate below.

**An Example:** Sometimes it is possible to figure out \(M \otimes N\) just from using the rules above. Here is a classic example. Let \(R = \mathbb{Z}\), the integers. Any finite abelian group is a module over \(\mathbb{Z}\). The scaling rule is just \(mg = g + \ldots + g\) (\(m\) times). In particular, this is true for \(\mathbb{Z}/n\). Let’s show that \(\mathbb{Z}/2 \otimes \mathbb{Z}/3\) is the trivial module.

Consider the element \(1 \otimes 1\). We have

\[
2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0(1 \otimes 1) = 0.
\]

At the same time

\[
2(1 \otimes 1) = 1 \otimes 3 = 1 \otimes 0 = 0(1 \otimes 1) = 0.
\]

But then

\[
1(1 \otimes 1) = (3 - 2)(1 \otimes 1) = 0 - 0 = 0.
\]

Hence \(1 \otimes 1\) is trivial. From here it is easy to see that \(a \otimes b\) is trivial for all \(a \in \mathbb{Z}/2\) and \(b \in \mathbb{Z}/3\). There really aren’t many choices. But \(\mathbb{Z}/2 \otimes \mathbb{Z}/3\) is the span of the image of \(M \times N\) under the tensor map. Hence \(\mathbb{Z}/2 \otimes \mathbb{Z}/3\) is trivial.
3 The Universal Property

Linear and Bilinear Maps: Let $M$ and $N$ be $R$-modules. A map $\phi : M \to N$ is $R$-linear (or just linear for short) provided that

1. $\phi(rv) = r\phi(v)$.
2. $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$.

A map $\phi : M \times N \to P$ is $R$-bilinear if

1. For any $m \in M$, the map $n \to \phi(m,n)$ is a linear map from $N$ to $P$.
2. For any $n \in N$, the map $m \to \phi(m,n)$ is a linear map from $M$ to $P$.

Existence of the Universal Property: The tensor product has what is called a universal property. The name comes from the fact that the construction to follow works for all maps of the given type.

Lemma 3.1 Suppose that $\phi : M \times N \to P$ is a bilinear map. Then there is a linear map $\hat{\phi} : M \otimes N \to P$ such that $\phi(m,n) = \hat{\phi}(m \otimes n)$. Equivalently, $\phi = \hat{\phi} \circ B$, where $B : M \times N \to M \otimes N$ is as above.

Proof: First of all, there is a linear map $\psi : Y(M \times N) \to P$. The map is given by

$$\psi(r_1(v_1, w_1) + \ldots + r_n(v_n, w_n)) = r_1\psi(v_1, w_1) + \ldots + r_n\psi(v_n, w_n).$$

That is, we do the obvious map, and then simplify the sum in $P$. Since $\phi$ is bilinear, we see that $\psi(s) = 0$ for all $s \in S$. Therefore, $\psi = 0$ on $Y(S)$. But then $\psi$ gives rise to a map from $M \otimes N = Y/Y(S)$ into $P$, just using the formula

$$\hat{\phi}(a + Y(S)) = \psi(a).$$

Since $\psi$ vanishes on $Y(S)$, this definition is the same no matter what coset representative is chosen. By construction $\hat{\phi}$ is linear and satisfies $\hat{\phi}(m \otimes n) = \phi(m,n)$. ♠

Uniqueness of the Universal Property: Not only does $(B, M \otimes N)$ have the universal property, but any other pair $(B', (M \otimes N)'\prime)$ with the same property is essentially identical to $(B, M \otimes N)$. The next result says this precisely.
Lemma 3.2 Suppose that $(B', (M \otimes N)')$ is a pair satisfying the following axioms:

- $(M \otimes N)'$ is an $R$-module.
- $B' : M \times N \rightarrow (M \otimes N)'$ is a bilinear map.
- $(M \otimes N)'$ is spanned by the image $B'(M \times N)$.
- For any bilinear map $T : M \times N \rightarrow P$ there is a linear map $L : (M \otimes N)' \rightarrow P$ such that $T = L \circ B'$.

Then there is an isomorphism $I : M \otimes N \rightarrow (M \otimes N)'$ and $B' = I \circ B$.

Proof: Since $(B, M \otimes N)$ has the universal property, and we know that $B' : M \times N \rightarrow (M \otimes N)'$ is a bilinear map, there is a linear map $I : M \otimes N \rightarrow (M \otimes N)'$ such that

$$B' = I \circ B.$$  

We just have to show that $I$ is an isomorphism. Reversing the roles of the two pairs, we also have a linear map $J : (M \otimes N)' \rightarrow M \otimes N$ such that

$$B = J \circ B'.$$

Combining these equations, we see that

$$B = J \circ I \circ B.$$

But then $J \circ I$ is the identity on the set $B(M \times N)$. But this set spans $M \otimes N$. Hence $J \circ I$ is the identity on $M \otimes N$. The same argument shows that $I \circ J$ is the identity on $(M \otimes N)'$. But this situation is only possible if both $I$ and $J$ are isomorphisms. ♠

4 Vector Spaces

The tensor product of two vectors spaces is much more concrete. We will change notation so that $F$ is a field and $V, W$ are vector spaces over $F$. Just to make the exposition clean, we will assume that $V$ and $W$ are finite
dimensional vector spaces. Let \( v_1, ..., v_m \) be a basis for \( V \) and let \( w_1, ..., w_n \) be a basis for \( W \). We define \( V \otimes W \) to be the set of formal linear combinations of the \( mn \) symbols \( v_i \otimes w_j \). That is, a typical element of \( V \otimes W \) is

\[
\sum_{i,j} c_{ij} (v_i \otimes w_j).
\]

(6)

The space \( V \otimes W \) is clearly a finite dimensional vector space of dimension \( mn \). It is important to note that we are not giving a circular definition. This time \( v_i \otimes w_j \) is just a formal symbol.

However, now we would like to define the bilinear map

\[
B : V \times W \rightarrow V \otimes W.
\]

Here is the formula

\[
B \left( \sum a_i v_i, \sum b_j w_j \right) = \sum_{i,j} a_i b_j (v_i \otimes w_j).
\]

(7)

This gives a complete definition because every element of \( V \) is a unique linear combination of the \( \{v_i\} \) and every element of \( W \) is a unique linear combination of the \( \{w_j\} \). A routine check shows that \( B \) is a bilinear map.

Finally, if \( T : V \times W \rightarrow P \) is some bilinear map, we define \( L : V \otimes W \rightarrow P \) using the formula

\[
L \left( \sum_{i,j} c_{ij} (v_i \otimes w_j) \right) = \sum_{i,j} c_{ij} T(v_i, w_j).
\]

(8)

It is an easy matter to check that \( L \) is linear and that \( T = L \circ B \).

Since our definition here of \( B \) and \( V \otimes W \) satisfies the universal property, it must coincide with the more abstract definition given above.

5 Properties of the Tensor Product

Going back to the general case, here I’ll work out some properties of the tensor product. As usual, all modules are unital \( R \)-modules over the ring \( R \).

Lemma 5.1 \( M \otimes N \) is isomorphic to \( N \otimes M \).
**Proof:** This is obvious from the construction. The map $(v,w) \to (w,v)$ extends to give an isomorphism from $Y_{M,N} = L(M \times N)$ to $Y_{N,M} = L(N \times N)$, and this isomorphism maps the set $S_{M,N} \subset Y_{M,N}$ of bilinear relations set $S_{N,M} \subset Y_{N,M}$ and therefore gives an isomorphism between the ideals $Y_{M,N}S_{M,N}$ and $Y_{N,M}S_{N,M}$. So, the obvious map induces an isomorphism on the quotients. ♠

**Lemma 5.2** $R \otimes M$ is isomorphic to $M$.

**Proof:** The module axioms give us a surjective bilinear map $T : R \times M \to M$ given by $T(r,m) = rm$. By the universal property, there is a linear map $L : R \otimes M \to M$ such that $T = L \circ B$. Since $T$ is surjective, $L$ is also surjective. At the same time, we have a map $L^* : M \to R \otimes M$ given by the formula

$$L^*(v) = B(1,v) = 1 \otimes v. \tag{9}$$

The map $L^*$ is linear because $B$ is bilinear. We compute

$$L^* \circ L(r \otimes v) = L^*(rv) = 1 \otimes rv = r \otimes v. \tag{10}$$

So $L^* \circ L$ is the identity on the image $B(R \times M)$. But this image spans $R \otimes M$. Hence $L^* \circ L$ is the identity. But this is only possible if $L$ is injective. Hence $L$ is an isomorphism. ♠

**Lemma 5.3** $M \otimes (N_1 \times N_2)$ is isomorphic to $(M \otimes N_1) \times (N \otimes N_2)$.

**Proof:** Let $N = N_1 \times N_2$. There is an obvious isomorphism $\phi$ from $Y = Y_{M,N}$ to $Y_1 \times Y_2$, where $Y_j = Y_{M,N_j}$, and $\phi(S) = S_1 \times S_2$. Here $S_j = S_{M,N_j}$. Therefore, $\phi$ induces an isomorphism from $Y/YS$ to $(Y_1/Y_1S_1) \times (Y_2/Y_2S_2)$. ♠

Finally, we can prove something (slightly) nontrivial.

**Lemma 5.4** $M \otimes R^n$ is isomorphic to $M^n$.

**Proof:** By repeated applications of the previous result, $M \otimes R^n$ is isomorphic to $(M \otimes R)^n$, which is in turn isomorphic to $M^n$. ♠

As a special case,
Corollary 5.5 $R^m \otimes R^n$ is isomorphic to $R^{mn}$.

This is a reassurance that we got things right for vector spaces.

For our next result we need a technical lemma.

Lemma 5.6 Suppose that $Y$ is a module and $Y' \subset Y$ and $I \subset Y$ are both submodules. Let $I' = I \cap Y'$. Then there is an injective linear map from $Y'/I'$ into $Y/I$.

Proof: We have a linear map $\phi : Y' \to Y/I$ induced by the inclusion from $Y'$ into $Y$. Suppose that $\phi(a) = 0$. Then $a \in I$. But, at the same time $a \in Y'$. Hence $a \in I'$. Conversely, if $a \in I'$ then $\phi(a) = 0$. In short, the kernel of $\phi$ is $I'$. But then the usual isomorphism theorem shows that $\phi$ induces an injective linear map from $Y'/I'$ into $Y/I$. ⊠

Now we deduce the corollary we care about.

Lemma 5.7 Suppose that $M' \subset M$ and $N' \subset N$ are submodules. Then there is an injective linear map from $M' \otimes N'$ into $M \otimes N$. This map is the identity on elements of the form $a \otimes b$, where $a \in M'$ and $b \in N'$.

Proof: We apply the previous result to the module $Y = Y_{M,N}$ and the submodules $I = S_{M,N}$ and $M' = Y_{M',N'}$. ⊠

In view of the previous result, we can think of $M' \otimes N'$ as a submodule of $M \otimes N$ when $M' \subset N$ and $N' \subset N$ are submodules.

This last result says something about vector spaces. Let’s take an example where the field is $\mathbb{Q}$ and the vector spaces are $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$. These two vector spaces are infinite dimensional. It follows from Zorn’s lemma that they both have bases. However, You might want to see that $\mathbb{R} \otimes \mathbb{R}/\mathbb{Q}$ is nontrivial even without using a basis for both. If we take any finite dimensional subspaces $V \subset \mathbb{R}$ and $W \subset \mathbb{R}/\mathbb{Q}$, then we know $V \otimes W$ is a submodule of $\mathbb{R} \otimes \mathbb{R}/\mathbb{Q}$. Hence $\mathbb{R} \otimes \mathbb{R}/\mathbb{Q}$ is nontrivial. In particular, we can use this to show that the element $1 \otimes [\alpha]$ is nontrivial when $\alpha$ is irrational.