Math 201 Final: This final is due Tuesday, Dec 12, at noon. Please send me email if you have any questions about it. You can use the book but no other references. Do 4 of the problems.

1. Consider the paraboloid $P$ in $\mathbb{R}^3$ given by the equations $z = x^2 + y^2$. Let $\gamma \subset P$ be the curve given by $\gamma(t) = (t, 0, t^2)$. Describe the behavior of the Jacobi fields along $\gamma$ in as much detail as you can.

2. Construct a complete smooth Riemannian metric on $\mathbb{R}^2$ with the property that $\infty$ is the supremum of the sectional curvatures and $-\infty$ is the infimum of the sectional curvatures. In other words, the sectional curvature becomes arbitrarily negative at some points and arbitrarily positive at other points.

3. Consider the metric on $\mathbb{R}^3$ given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle_{x,y,t} = e^{2t}u_1v_1 + e^{-2t}u_2v_2 + u_3v_3$$

Is this a metric of non-positive curvature on $\mathbb{R}^3$? (This is known as the solvable metric, because of its close connection to a certain solvable Lie group.)

4. An ideal polyhedron in hyperbolic $n$-space $H^n$ is a polyhedron whose vertices all lie on the ideal boundary of $H^n$. Prove that any ideal polyhedron in $H^n$ has finite volume.

5. Let $SO(3)$ denote the Lie group of determinant 1 orthogonal $3 \times 3$ matrices, equipped with a bi-invariant Riemannian metric. Prove that $SO(3)$ has constant curvature.

6. Let $X$ be the manifold obtained by deleting $n \geq 3$ points from the two dimensional sphere. Prove that $X$ admits a metric of zero curvature with the following two properties: The diameter of $X$ is finite and, for any $\epsilon > 0$, each deleted point has a neighborhood of diameter less than $\epsilon$.

7. Given a positive smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ we can put a Riemannian metric on $\mathbb{R}^2$ using the formula

$$\langle v_1, v_2 \rangle_p = f(p) \cdot v_1 \cdot v_2.$$

Compute the sectional curvature of this metric in terms of $f$. 