Math 52 Sample Midterm 2
(The solutions are on the second page.)

1. Find the equation of the plane in $\mathbb{R}^3$ that consists of points which are equidistant to the point $(0, 5, 3)$ and the point $(-2, 3, 1)$.

2. Recall that a rotation of $\mathbb{R}^4$ is a map that preserves the dot product. Prove that there is a rotation of $\mathbb{R}^4$ that maps the vector $(1, 2, 3, 4)$ to the vector $(5, 0, 1, 2)$. (You don't need to write out an explicit formula.)

3. Recall that the span of a set $S$ is the set of all finite linear combinations of elements of $S$. Prove that

$$\text{Span}(\text{Span}(S)) \subset \text{Span}(S)$$

for any subset $S$ of vectors in a vector space.

4. Let $\{v_1, ..., v_n\}$ be a basis for a finite dimensional vector space $V$. Let $\{w_1, ..., w_n\}$ be another basis. We can take each $v_k$ and write it as a linear combination

$$v_k = a_{k1}w_1 + ... + a_{kn}w_n.$$ 

In this way we get an $n \times n$ matrix $A = \{a_{ij}\}$. Prove that $A$ is invertible.
Solutions:

1. Let \( v_1 = (0, 5, 3) \) and let \( v_2 = (-2, 3, 1) \). The vector pointing from \( v_1 \) to \( v_2 \) is the vector \( v_3 := v_2 - v_1 = (-2, -2, -2) \). The vector pointing from the \( v_1 \) to the midpoint of \( v_1 \) and \( v_2 \) is \( v_4 := v_1 + \frac{1}{2}v_3 = (-1, 4, 2) \). This is one of the points on the plane. Any other point on the plane has the form \( v_4 + w \), where \( w \) is perpendicular to \( v_3 \). This is to say that a point \((x, y, z)\) lies in the plane if and only if
\[
((x, y, z) - v_4) \cdot v_3 = 0.
\]
This works out to be \( x + y + z = 5 \).

2. Let \( v_1 = (1, 2, 3, 4) \) and \( w = (5, 0, 1, 2) \). Note first that \( v_1 \cdot v_1 = w \cdot w = 30 \), so we would expect this problem to work out. Let \( T_1 \) be the map
\[
T_1(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2).
\]
Then clearly \( T_1 \) preserves the dot product. Also \( T_1(v_1) = (3, 4, 1, 2) \). Since the composition of two rotations—i.e. the effect of doing one first and the second one—is also a rotation, it now suffices to find a rotation that maps \((3, 4, 1, 2)\) to \((5, 0, 1, 2)\). You can check explicitly that a linear transformation represented by the matrix
\[
\begin{pmatrix}
c & s & 0 & 0 \\
-s & c & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
is a rotation as long as \( c^2 + s^2 = 1 \). (Here \( c \) and \( s \) are really abbreviations for \( \cos(\theta) \) and \( \sin(\theta) \).) To get what we want, we take \( c = 3/5 \) and \( s = 4/5 \). Then the resulting linear transformation \( T_2 \) has the property that \( T_2(3, 4, 1, 2) = (5, 0, 1, 2) \). So, all in all, the rotation \( T_3 \) defined by the equation \( T_3(x) = T_2(T_1(x)) \) maps \( v_1 \) to \( w \).

3. An arbitrary element of \( \text{Span} (\text{Span}(S)) \) has the form
\[
x = a_1 v_1 + \ldots + a_n v_n
\]
where \( v_1, \ldots, v_n \in \text{Span}(S) \) and \( a_1, \ldots, a_n \) are real numbers. By definition each \( v_j \) has for the form
\[
v_j = b_{j1} w_1 + \ldots + b_{jm} w_m,
\]
where $w_1, ..., w_m$ is some finite list of vectors in $\text{Span}(S)$. (Note: Even though the different $v$’s might be linear combinations of different vectors in $S$, we can just make one master list of $w$’s that contains all the vectors used in any of the combinations—there will just possibly be a lot of 0s in the equations above.) So, now we can write

$$x = a_1(b_{11}w_1 + ... + b_{1m}w_m) + ... + a_n(b_{n1}w_1 + ... + b_{nm}w_m) = c_1w_1 + ...c_mw_m,$$

where the $c$’s are real numbers obtained by expanding everything out and grouping terms. This shows that $x \in \text{Span}(S)$.

4. Let $M_1$ be the matrix obtained by writing the basis vectors $v_1, ..., v_n$ as columns of a square matrix. Let $M_2$ be the matrix obtained by writing the basis vectors $w_1, ..., w_n$ as columns of a square matrix. From the very definition of matrix multiplication, $M_1A^t = M_2$. (Here $A^t$ is the transpose of $A$.) For instance, the first column of $M_1A^t$ is $a_{11}v_1 + ... + a_{1n}v_n = w_1$. Since $v_1, ..., v_n$ is a basis, the matrix $M_1$ is invertible. Likewise $M_2$ is invertible. So, we can write

$$A^t = M_1^{-1}M_2,$$

the product of invertible matrices. Hence $A^t$ is invertible. This means that $A^t$ has nonzero determinant. But $A$ and $A^t$ have the same determinant. So, $A$ is also invertible.