Some Symplectic Geometry

1 The Goal

The purpose of these notes is to explain (to myself) the three basic facts about symplectic manifolds, Hamiltonian vector fields, and the Poisson bracket. I wrote these notes by filling in the proofs of the claims made on the Lie derivatives page of Wikipedia.

Let $M$ be a smooth $(2n)$-dimensional manifold and let $\omega$ be a symplectic form on $M$. This means that $\omega$ is a closed nondegenerate 2-form. For any function $f : M \rightarrow \mathbb{R}$ we introduce the Hamiltonian $H_f$. It has the property that

$$\omega(H_f, W) = W f = df(W);$$  \hspace{1cm} (1)

for any vector field $W$. You need the nondegeneracy of $\omega$ to guarantee the existence of $H_f$. We also define the Poisson bracket

$$\{f, g\} = \omega(H_f, H_g)$$  \hspace{1cm} (2)

Here are the three basic facts.

1. The flow generated by $H_f$ preserves $f$. That is, $H_f$ is tangent to the level sets of $f$. This fact is easy: $df(H_f) = \omega(H_f, H_f) = 0$. That’s it.

2. The flow generated by $H_f$ preserves $\omega$. That is, the flow is a symplectomorphism for each time value.

3. If $\{f, g\} = 0$ then $H_f$ and $H_g$ generate commuting flows.

These three basic facts are all you need to understand the miracle of completely integrable systems. A completely integrable system on $M$ is a collection $f_1, ..., f_n$ of functions such that $\{f_i, f_j\} = 0$ for all $i, j$ and such that the vector fields $\{H_1, ..., H_n\}$ are linearly independent.

The generic common level set $L$ of $\{f_1, ..., f_n\}$ is an $n$-dimensional compact smooth manifold, and the vectors $H_1, ..., H_n$ generate pairwise commuting flows tangent to $L$. But then these flows give coordinate charts from $L$ to $\mathbb{R}^n$ in which the overlap functions are translations. This forces $L$ to be a torus, and each flow to be an isometric motion in the given coordinates.

The rest of the notes are devoted to proving Fact 2 and Fact 3.
2 The Lie Derivative

Let $M$ be a smooth manifold and let $V$ be a vector field on $M$. Suppose that $M$ generates the flow $\phi_t : M \to M$. For a function $f$, we have

$$L_V f = \frac{d}{dt}(f \circ \phi_t) = V f = df(V).$$

(3)

Here $V f$ is the directional derivative of $f$ along $V$.

If $W$ is another vector field, we define

$$L_V W = \frac{d}{dt}\left((\phi_t^{-1})_* (W_{\phi_t})\right) = [V,W].$$

(4)

So, if we are interested at the derivative at the point $p$, we evaluate the vector field $W$ at $\phi_t(p)$ and map the vector back to the tangent plane at $p$ using the tangent map of $\phi_t^{-1}$.

If $\omega$ is a differential form, we define

$$L_V \omega = \frac{d}{dt}\left((\phi_t^{-1})^*(\omega_{\phi_t})\right).$$

(5)

Suppose that $\omega$ is a 2-form and $X, Y$ are vector fields. Then $\omega(X,Y)$ is a function. From the product rule

$$L_V(\omega(X,Y)) = (L_V \omega)(X,Y) + \omega([V,X],Y) + \omega(X,[V,Y]).$$

(6)

Equation 6 is one of the key equations we will use when establishing Fact 3 about symplectic geometry.

We introduce the contraction operator $i_V$, which maps $(n+1)$-forms to $n$-forms. Here is the formula

$$(i_V \beta)(X_1, \ldots, X_n) = \beta(V, X_1, \ldots, X_n).$$

(7)

We have Cartan’s formula

$$L_V \beta = i_V (d\beta) + d(i_V \beta).$$

(8)

This holds for any differential form $\beta$. We will prove Cartan’s formula below, in the case we need. Cartan’s formula is the key equation we need to establish Fact 2 about symplectic geometry.
3 Some Cases of Cartan’s Formula

We need Cartan’s formula for 1-forms and for closed 2-forms. Here we prove these 2 cases. For closed 2-forms, Cartan’s formula reduces to

\[ L_V \omega = d(i_V \omega). \]  
(9)

**Lemma 3.1** If Cartan’s formula holds for 1-forms, then Cartan’s formula holds for closed 2-forms.

**Proof:** Let \( \omega \) be a closed 2-form. Cartan’s formula is a local calculation, and so we may assume that \( \omega = d\alpha \) where \( \alpha \) is a closed 1-form. The pullback map commutes with the \( d \)-operator. Hence \( L \) and \( d \) commute. This gives us

\[ L_V \omega = L_V (d\alpha) = d(L_V \alpha) = d(i_V d\alpha) + d(d(i_V \alpha)) = d(I_V \omega), \]  
(10)

since \( d^2 = 0 \). ♠

**Lemma 3.2** Cartan’s formula holds for 1-forms.

**Proof:** Any 1-form can be expressed as a finite sum \( \sum_i f_i dg_i \) for smooth functions \( f_i \) and \( g_i \). So, it suffices to prove Cartan’s formula for \( f dg \). Using the fact that \( d \) and \( L \) commute, we have

\[ L_V (f dg) = f L_V (dg) + (V f) dg = f d(L_V g) + (V f) dg = f d(V g) + (V f) dg. \]  
(11)

On the other hand

\[ i_V d(f dg) = i_V (df \wedge dg) = i_V (df \otimes dg - dg \otimes df) = (V f) dg - (V g) df, \]  
(12)

and

\[ d(i_V (f dg)) = d(f V g) = f d(V g) + (V g) df. \]  
(13)

Adding the last two equations, we get that

\[ i_V d(f dg) + d(I_V (f dg)) = f d(V g) + (V f) dg = L_V (f dg), \]  
(14)

so it works. ♠
4 Proof of the Facts

Fact 2: We first prove Fact 2. This amounts to showing that $L_V \omega = 0$ when $V = H_f$. Using the special case of Cartan’s formula, we have

$$L_{H_f} \omega = d(i_{H_f}(\omega)) = d(df) = 0.$$ 

The point here is that $i_{H_f}(\omega)(X) = \omega(H_f, X) = df(X)$, by definition. That’s the proof.

Fact 3: We will show that $H\{f, g\} = [H_f, H_g]$, the Lie bracket of $H_f$ and $H_g$. When $\{f, g\} = 0$ it means that $[H_f, H_g] = 0$, and this means that $H_f$ and $H_g$ generate commuting flows.

Let $V = H_f$ and $W = H_g$. Below we will derive the identity.

$$i_{[V, W]} \omega = d(i_V i_W \omega).$$ (15)

Assuming this identity, we get the following for any vector field $X$:

$$\omega([H_f, H_g], X) = \omega([V, W], X) = i_{[V, W]} \omega(X) =$$

$$d(i_V i_W \omega)(X) = X \omega(V, W) = X \{f, g\} = \omega(H\{f, g\}, X).$$ (16)

This proves what we want. It only remains to prove Equation 15. Choose $X$ to be a vector field which commutes with $V$. We have the identity

$$L_V (\omega(W, X)) = (L_V \omega)(W, X) + \omega(L_V W, X) + \omega(W, L_V X) = \omega([V, W], X).$$ (17)

Here we have used the fact that $L_V \omega = 0$ and $L_V W = [V, W]$ and $L_V X = 0$. Since Equation 17 is true for any choice of commuting $X$, and we can arrange for such a vector field to be arbitrary at a point of interest to us, we get

$$L_V (i_W \omega) = i_{[V, W]} \omega.$$ (18)

Let $\alpha = i_W (\omega)$. Note that $\alpha = dg$. Hence $d\alpha = 0$. Applying Cartan’s formula to $\alpha$, we have

$$L_V (i_W \omega) = L_V \alpha = d(i_V \alpha) = d(i_V i_W \omega).$$ (19)

Equation 15 comes from putting together Equations 18 and 19.