(1) Let $C$ be the standard middle-third Cantor set and $\mu_C$ the Cantor measure. Compute the Cantor measure of the interval $[\frac{1}{\pi}, \frac{1}{e}]$. Compute $\int_0^1 x \, d\mu_C(x)$.

Note that $1/\pi = 0.0221\ldots$ in base 3 and $1/e = 0.1\ldots$ in base 3. So $1/3 < 1/e < 2/3$ and $0.0220 < 1/\pi < 0.0222$ which makes $F(1/e) = 1/2$ and $F(1/\pi) = \frac{9}{16}$. (here $F$ is the Cantor function). Thus $\mu_C([1/\pi, 1/e]) = F(1/e) - F(1/\pi) = \frac{1}{16}$.

We have $\int_0^1 x \, d\mu_C(x) = \int_0^1 (x - \frac{1}{2}) \, d\mu_C(x) + \int_0^1 \frac{1}{2} \, d\mu_C(x)$.

By symmetry this first integral is zero: $d\mu(x) = d\mu(1 - x)$ and the second interval is $\frac{1}{2}\mu_C(C) = \frac{1}{2}$. Alternatively one can write $\int_0^1 x \, d\mu_C(x) = \int_0^1 F(x) \, dx$ and use the fact that the area under the graph of $F(x)$ is $\frac{1}{2}$ by symmetry.

(2) For $\alpha \in (0, 1)$ let $E_\alpha$ be the set of reals in $[0, 1]$ whose binary expansion has density $\alpha$ of 1s. Show that $E_\alpha$ is a Borel set of type $G_{\delta\sigma\delta}$ or simpler. (Hint: Define $E_\alpha$ using an expression with three quantifiers $\forall \ldots \exists \ldots \forall$. Each $\forall$ represents a countable intersection, each $\exists$ represents a countable union.)

We can write $E_\alpha = \left\{ x \in [0, 1] : \lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n} = \alpha \right\}$. This set is the set of $x$ for which

$$\forall \epsilon > 0 \exists N > 0 \forall n \geq N \left| \frac{\sum_{i=1}^{n} x_i}{n} - \alpha \right| < \epsilon.$$ 

In other words letting $\epsilon = 1/M$ and $M \to \infty$,

$$E_\alpha = \bigcap_{M>0} \bigcup_{N>0} \bigcap_{n \geq N} \left\{ x : \left| \frac{\sum_{i=1}^{n} x_i}{n} - \alpha \right| < \frac{1}{M} \right\}.$$ 

But the set $\left\{ x : \left| \frac{\sum_{i=1}^{n} x_i}{n} - \alpha \right| < \frac{1}{M} \right\}$ is a finite union of open intervals, hence open. Thus $E_\alpha$ is a $G_{\delta\sigma\delta}$ (or simpler—we have not proved that it is not a $G_{\delta\sigma}$ or an $F_{\sigma\delta}$).

(3) Let $\mu$ be a Lebesgue-Stieltjes measure. Show that if $E$ is $\mu$-measurable and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set $A$ that is a finite union of open intervals such that $\mu(E \Delta A) < \epsilon$. 

(Here $S_1 \Delta S_2$ is the symmetric difference: $(S_1 \cup S_2) \setminus (S_1 \cap S_2)$.)

**Proof:** Recall that $\mu(E) = \inf \{ \mu(U) \mid U \text{ open and } E \subset U \}$. Find $U$ open so that $\mu(U) - \mu(E) < \epsilon/2$. An open set in $\mathbb{R}$ is a countable disjoint union of open intervals. Let $U_1, U_2, \ldots$ be the disjoint intervals of $U$; since $U = \cup U_i$, we have $\mu(U) - \mu(\bigcup_{i=1}^{N} U_i) < \epsilon/2$ for $N$ sufficiently large. Then $\mu(E \setminus \bigcup_{i=1}^{N} U_i) \leq \mu(U \setminus \bigcup_{i=1}^{N} U_i) \leq \epsilon/2$ and $\mu(\bigcup_{i=1}^{N} U_i \setminus E) \leq \mu(U \setminus E) \leq \epsilon/2$ from which we conclude that

$$\mu(E \Delta \bigcup_{i=1}^{N} U_i) < \epsilon.$$ 

(4) Compute the following limits and justify:

(a) $\lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(x/n) \, dx.$

(b) $\lim_{n \to \infty} \int_0^1 \frac{1 + nx}{(1 + x^n)^n} \, dx.$

(a) Note first that $(1 + x/n)^{-n}$ is decreasing in $n$: this can be proved e.g. by differentiation wrt $n$ and the fact that $\log(1 + x) < x$ for $x > 0$. Now for $n \geq 2$

$$\left|(1 + \frac{x}{n})^{-n} \sin(x/n)\right| \leq (1 + \frac{x}{n})^{-n} \leq (1 + \frac{x}{2})^{-2}$$

which is integrable on $[0, \infty)$. Hence we can use the DCT to bring the limit inside the integral. The limit is zero.

(b) $\frac{1 + nx}{(1 + x^n)^n} \leq 1$ for all $x \in [0, 1]$ when $n \geq 1$. So we can use the DCT with $g = 1$ and bring the limit inside the integral. Now $\lim_{n \to \infty} \frac{1 + nx}{(1 + x^n)^n} = 0$ if $x > 0$ by e.g. l'Hopital’s rule. So the integral of the limit is 0.

(5) Let $x = 0.\alpha_1\alpha_2\alpha_3 \ldots$ be the binary expansion of a real number $x \in [0, 1)$. Let $f : [0, 1) \to [0, 1)$ be defined by $f(0.\alpha_1\alpha_2\alpha_3 \ldots) = 0.\alpha_1\alpha_3\alpha_5 \ldots$. Show that $f$ is a measurable function. Compute $\int_0^1 f \, d\mu$ where $\mu$ is Lebesgue measure.

$f$ is the increasing limit of the functions $f_n$ defined by $f_n(0.\alpha_1\alpha_2 \ldots) = 0.\alpha_1\alpha_3 \ldots \alpha_{2n-1}000 \ldots$ each of which are step functions and hence measurable. So $f$ is the limit of measurable functions hence measurable. (There are many other ways to show that $f$ is measurable.)

By the MCT we have $\int_0^1 f(x) \, dx = \lim_{n \to \infty} \int_0^1 f_n(x) \, dx$. However $f_n$ is a step function: $f_n^{-1}(0.\beta_1\beta_2 \ldots \beta_n)$ is a disjoint union of $2^{n-1}$ disjoint intervals (given by the $2^{n-1}$ choices of $\alpha_2, \alpha_4, \ldots, \alpha_{2n-2}$) each of length $2^{-2n+1}$. So $f_n$ takes the $2^n$ values $0.\beta_1\beta_2 \ldots \beta_n,$
each on a set of the same measure $2^{n-1-2n+1} = 2^{-n}$. In other words
\[
\int_0^1 f_n(x) \, dx = 2^{-n} \sum_{k=0}^{2^n-1} \frac{k}{2^n} = 2^{-2n} \frac{(2^n - 1)(2^n - 2)}{2}
\]
which tends to 1/2.

(6) Is $\frac{1}{x} \sin(1/x)$ Lebesgue integrable on $(0, 1]$?

We’ll show that the positive part has infinite integral, so that $\frac{1}{x} \sin(1/x)$ is not integrable. Note that $\sin y > 1/2$ on $(\frac{\pi}{6}, \frac{5\pi}{6})$.

Thus for integer $k > 0$,
\[
\int_{1/(2\pi k + \pi/6)}^{1/(2\pi k + 5\pi/6)} \frac{1}{x} \sin \frac{1}{x} \, dx \geq 2\pi k \mu\left(\left[\frac{1}{2\pi k + 5\pi/6}, \frac{1}{2\pi k + \pi/6}\right]\right) > \frac{C}{k}
\]
for some positive constants $C$. Sum over $k$ and use the fact that the harmonic series diverges.

(7) Show: if $f \geq 0$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ with $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Since $f$ is positive it is the increasing limit of positive simple functions $\phi_n$, and $\int f = \lim \int \phi_n$ by the MCT. Thus given $\epsilon > 0$ there is an $N$ such that $\int f - \int \phi_n < \epsilon$ for all $n \geq N$. Note that $\phi_n$ is supported on a set $E$ of finite measure, since it has finite integral ($\int \phi_n < \int f < \infty$). Thus $\epsilon > \int f - \int \phi_n \geq \int f - \int_E f$ which was what we needed to prove.