1. Evaluate the limit

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - xy + y^2}{x^2 + xy + y^2}$$

or prove that it does not exist.

The limit along the $x$ axis is obtained by setting $y = 0$; the limit is 1. The limit along the line $x = y$ is obtained by setting $x = y$: the limit is $1/3$. Since these are different, no limit exists.
2. Find the equation of the line normal to the surface $x^2 + y^3 + z^4 = 3$ through the point $(1, 1, 1)$.

The normal to $F(x, y, z) = 0$ is given by $(F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)) = (2x, 3y^2, 4z^3)$. At $(1, 1, 1)$ this normal vector is $(2, 3, 4)$. The line through $(1, 1, 1)$ and in direction $(2, 3, 4)$ is $L = (1, 1, 1) + t(2, 3, 4), t \in \mathbb{R}$. 
3. (a) Show that the space curve
\[ g(t) = (\sqrt{4 + t^2} \cos t, t, \sqrt{4 + t^2} \sin t) \]
is contained in a hyperboloid, and find an equation for this hyperboloid.

(b) Suppose that \( F(x, y, z) \) is a function satisfying \( F(0, 2, 0) = 0 \) and with the property that the tangent plane to the surface \( F(x, y, z) = 0 \) at \((0, 2, 0)\) is given by
\[ 5x + 4y - 3z = 8. \]

Let \( F(t) \) be the function restricted to the space curve in the problem above (i.e., \( F(t) = F(x(t), y(t), z(t)) \)) and \((x(t), y(t), z(t)) = g(t)\) above. Find \( \frac{dF}{dt} \) at \( t = 0 \).

(a) Let \( g(t) = (x(t), y(t), z(t)) \). Then \( x(t)^2 + z(t)^2 - y(t)^2 = (4 + t^2) \cos^2(t) + (4 + t^2) \sin^2(t) - t^2 = 4 \), so \( g(t) \) is contained in the hyperboloid \( x^2 + z^2 - y^2 = 4 \).

(b) \[
\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}.
\]

From the statement \( F_x = 5, F_y = 4, F_z = 3 \). Also \( \frac{dx}{dt} = (4 + t^2)^{-1/2} t \cos t - \sqrt{4 + t^2} \sin t \) and similarly for \( y'(t), z'(t) \). Plug in at \( t = 0 \).
4. (a) Find the equation for the tangent plane to the surface given by \( \sin z = xyz \), at the point \((x_0, y_0, z_0)\).

(b) Show that at \((0, 0, 0)\) the tangent plane is horizontal (parallel to the \(xy\)-plane).

(c) Show that where \(\cos z = xy\), the tangent plane is vertical (perpendicular to the \(xy\)-plane).

(a) The tangent plane to \(F(x, y, z)\) at \((x_0, y_0, z_0)\) is

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0).
\]

Use \(F(x, y, z) = \sin z - xyz\) above.

(c) The tangent plane is perpendicular to the \(xy\)-plane when the normal to the plane is normal to the \(xy\)-plane, that is, when \((F_x, F_y, F_z)\) is perpendicular to \((0, 0, 1)\). This happens when \(F_z = 0\), that is, \(\cos z_0 = x_0y_0\).
5. Use a linear approximation to estimate \( \sin(\pi ye^{-2x}) \) at the point \((x, y) = (0.01, 0.03)\).

The linear approximation of \(f(x, y)\) near \((x_0, y_0)\) is

\[
f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

Use \((x_0, y_0) = (0, 0)\) here with \(f_x(0, 0) = \cos(\pi ye^{-2x})\pi ye^{-2x}(-2)|_{(x,y) = (0,0)} = 0\) and \(f_x(0, 0) = \cos(\pi ye^{-2x})\pi e^{-2x}|_{(x,y) = (0,0)} = \pi\). We get

\[
sin(\pi ye^{-2x}) = 0 + 0(x - 0) + \pi(y - 0).
\]

Plug in \(x = 0.01\) and \(y = 0.03\) to find \(0.03\pi\).
6. At what points \((x, y)\) does the function \(f(x, y) = e^{-x^2-y^2}\) have the property that the length of its gradient equals the directional derivative in the direction \(\pi/4\)?

The directional derivative in direction \(\pi/4\) is \(\nabla f \cdot u\) where \(u\) is the unit vector in direction \(\pi/4\), that is, \(u = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\).

When is \(\nabla f \cdot u = |\nabla f|\)? By definition of dot product, \(\nabla f \cdot u = |\nabla f||u|\cos \theta\) where \(\theta\) is the angle between the vectors. Here \(|u| = 1\), so we need \(\theta = 0\), that is, we need to find points \((x, y)\) where the gradient points in the \(\pi/4\) direction.

The gradient is \((-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2})\) so we need \(-2x = -2y > 0\). Thus the set of points is \(\{(x, x) \mid x < 0\}\).