Math 1610
Homework 11 hints

No notes, books or calculators are allowed. Reduce answer as best you can.

Page 422: 6. Rearrange rows and columns so that R is the last row and column. \(Nc\) is the expected time to reach state \(R\). \(NR\) is the constant matrix \((1 \ 1)\) (the probability of being absorbed is 1).

9.

\[
P = \begin{pmatrix}
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}
\]

\[
N = \begin{pmatrix}
1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
0 & 1 & -\frac{3}{4} & -\frac{1}{4} \\
0 & 0 & 1 & -\frac{3}{4} \\
0 & 0 & 0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 1 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & 1 & \frac{3}{4} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Take \(Nc\).

12. The matrix \(Q\) is something like \(Q = \begin{pmatrix}
\frac{5}{18} & \frac{5}{18} & \frac{2}{9} \\
0 & \frac{1}{3} & 0 \\
0 & 0 & \frac{5}{9}
\end{pmatrix}\). Use \(N = (I - Q)^{-1}\) etc.

13. (a) Solve the equations \(p_j = .4p_{j+1} + .6p_{j-1}\) with \(p_0 = 0\) and \(p_8 = 1\) (see problem 33 below). This gives \(p_1 \approx .02\).

(b) \(A^3 = .064\)

(c) strategy (b).

18. similar to above problems

21. (b) \(w = u + uQ + uQ^2 + \cdots = u(I - Q)^{-1}\).

(c) Find \(u\) so that \(w = u(I - Q)^{-1}\), that is, \(u = w(I - Q)\).


33. Verify that \(f(i) = (q/p)^i\) satisfies \(f(i) = pf(i + 1) + qf(i - 1)\). The probability of being absorbed in state \(N\) is then \(A(q/p)^i + B\) where \(A\) and \(B\) are constants. The boundary values \(f(0) = 0\) and \(f(N) = 1\) then determine \(A\) and \(B\) uniquely.

Page 442: 2. (a) \(P^3\) is strictly positive.

(b) \((P^2)_{13}\)

(c) Solve \(w = wP\) with the sum of \(w\)'s entries equal to 1. \(w = 1/2, 1/3, 1/6\).

5. easy

12. ergodic but not regular (on even steps you are at an odd vertex).
17. If you are currently at \( i \mod 7 \) then you will be at \( i + 1, i + 2, \ldots, i + 6 \mod 7 \) with probability \( 1/6 \) each. The transition matrix is

\[
P = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

It is regular and so there is a unique stationary distribution...which is uniform. Each state has probability \( 1/7 \).

26. Suppose there is a path of length \( n \) from \( i \) to \( j \). Then there is a path of any length \( n' > n \) from \( i \) to \( j \), which consists of moving to \( j \) and then using the self-loop at \( j \) \( n' - n \) times. Thus for any sufficiently large \( n \), for all \( i, j \) there is a path from \( i \) to \( j \) of length \( n \). In other words \( P^N > 0 \) for sufficiently large \( N \).

Page 452: 7. See problem 17 above.