Show for small \( z \),

\[
\lim_{N \to \infty} \frac{1}{N} \log Z_N(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n
\]  
(1)

where \( Z_N(z) \) is the 1D hard disk partition function.

\[
Z_N(z) = \sum_{k=0}^{N} \frac{(N-k)^k}{k!} z^k
\]  
(2)

We use several formulations of Lagrange Inversion Formula (LIF), which we will prove at the end. We consider all the functions as formal Laurent series. The most basic LIF states that for \( f(z), g(z) \) such that \( f(g(z)) = z \), \( f(0) = g(0) = 0 \), \( f'(0) \neq 0 \), \( g'(0) \neq 0 \),

\[
[z^n]g(z) = \frac{1}{n} \left[ z^{-1} \right] \frac{1}{f(z)^n}
\]  
(3)

where \( [z^n]F(z) \) denotes the coefficient of the \( z^n \) \((n \neq 0)\) term in the Laurent series \( F(z) \). A direct application of this is to solve the Lambert function \( w \) implicitly defined by \( we^w = z \). In particular

\[
[z^n]w(z) = \frac{1}{n} \left[ z^{-1} \right] \frac{1}{z^n e^{nz}}
= \frac{1}{n} \left[ z^{n-1} \right] e^{-nz}
= \frac{1}{n} \frac{(-n)^{n-1}}{(n-1)!} = \frac{(-n)^{n-1}}{n!}
\]

which agrees with (1). So we need to show that the LHS of (1) is indeed the solution \( w \) of \( we^w = z \).

For \( k \in \mathbb{Z} \), a slight generalization of (3), which is proved the same way, is

\[
[z^n]g(z)^k = \frac{k}{n} \left[ z^{-k} \right] \frac{1}{f(z)^n}
\]  
(4)
In the particular case $f(z) = z/\phi(z)$, and $g(z) = z\phi(g(z))$, where $\phi(0) \neq 0$, we see that $f(g(z)) = z$, so by (4),

$$[z^n]g(z)^k = \frac{k}{n} [z^{n-k}] \phi(z)^n = \frac{k}{n} [z^{n-k}] \phi(z)^n$$

(5)

Hence if $F(z)$ is another formal Laurent series, (5) directly implies that

$$[z^n]F(g(z)) = \frac{1}{n} [z^{n-1}] F'(z) \phi(z)^n$$

(6)

From (6), one can further deduce the following:

$$[z^n] F(g(z)) = [z^n] F(z) \phi(z)^n - [z^n] F(z) \phi(z)^n (-z \phi'(z))$$

(7)

$$[z^n] \frac{F(g(z))}{1 - z \phi'(g(z))} = [z^n] F(z) \phi(z)^n$$

(8)

The formula we are left to prove are (4), (7), (8). Before proving them, let us see how (8) solves the problem.

The coefficient of $z^k$ in (2) is $\frac{(N-k)^k}{k!}$, which is $[z^k] e^{(N-k)z}$. This matches the RHS of (8) with $F(z) = e^{Nz}$ and $\phi(z) = e^{-z}$, so

$$\frac{(N-k)^k}{k!} = [z^k] e^{Nz} \left(e^{-z}\right)^k$$

$$= [z^k] \frac{e^{Ng(z)}}{1 + ze^{-g(z)}}$$

and since we know $g(z) = z\phi(g(z)) = ze^{-g(z)}$,

$$\frac{(N-k)^k}{k!} = [z^k] \frac{e^{Ng(z)}}{1 + g(z)}$$

In other words, if we let the sum run up to $\infty$ in (2), then

$$\tilde{Z}_N(z) = \sum_{k=0}^{\infty} \frac{(N-k)^k}{k!} z^k = \frac{e^{Ng(z)}}{1 + g(z)}$$

We claim that $\lim_{N \to \infty} \frac{1}{N} \log Z_N(z) = \lim_{N \to \infty} \frac{1}{N} \log \tilde{Z}_N(z)$, One way to see this is that $\log(1 + z)$ has power series expansion $z - \frac{z^2}{2} + \frac{z^3}{3} \ldots$, so the $z^n$ coefficient of $\log(1 + F(z))$ only depends on the first $n$ coefficients of $F(z)$ if $F(0) = 0$. Therefore, for a fixed $n$, $[z^n] \frac{1}{N} \log Z_N(z) = [z^n] \frac{1}{N} \log \tilde{Z}_N(z)$ if $N \geq n$.

Since $\log \tilde{Z}_N(z) = Ng(z) - \log(1 + g(z))$,

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N(z) = \lim_{N \to \infty} g(z) - \frac{\log(1 + g(z))}{N} = g(z)$$

But $g(z)$ is defined by $g(z)e^{g(z)} = z$, so $g(z)$ is precisely the Lambert function.
For completeness, we include the proofs of formula (4)(7)(8).

**Proof of (4).** We are given power series $f, g$ such that $f(0) = g(0) = 0$, $f'(0) \neq 0$, $g'(0) \neq 0$ and $f(g(z)) = z$, and want to show
\[ [z^n]g(z)^k = \frac{k}{n} [z^{-k}] \frac{1}{f(z)^n} \]
for $k \in \mathbb{Z}$ and $n \neq 0$.

Expand $g(z)^k = \sum_{i=-N}^{\infty} b_i z^i$. Since $g(f(z)) = z$, we have
\[ z^k = g(f(z))^k = \sum_{i=-N}^{\infty} b_i f(z)^i \]
and taking derivative
\[ kz^{k-1} = \sum_{i=-N,i\neq 0}^{\infty} ib_i f(z)^{i-1} f'(z) \]
Divide by $f(z)^n$ on both sides,
\[ \frac{kz^{k-1}}{f(z)^n} = \sum_{i=-N,i\neq 0}^{\infty} ib_i f(z)^{i-1-n} f'(z) \]
\[ = \sum_{i=-N,i\neq 0,i\neq n}^{\infty} ib_i (f(z)^{i-n})' + nb_n \frac{f'(z)}{f(z)} \]
Now we extract the $z^{-1}$ coefficient of both sides. The LHS is $[z^{-1}] \frac{kz^{k-1}}{f(z)^n} = k [z^{-k}] \frac{1}{f(z)^n}$. On the RHS, $[z^{-1}] (f(z)^{i-n})'$ is always 0. Since $f'(0) \neq 0$, it is easy to see that $[z^{-1}] \frac{f'(z)}{f(z)} = 1$. Therefore $[z^{-1}] \text{RHS} = nb_n$, and (4) is established. \qed

**Proof of (7).** Starting from (6),
\[ [z^n]F(g(z)) = \frac{1}{n} [z^{n-1}] F'(z) \phi(z)^n \]
\[ = \frac{1}{n} [z^{-1}] F'(z) \left( \frac{\phi(z)}{z} \right)^n \]
Since in general, $0 = [z^{-1}] (r(z) s(z))' = [z^{-1}] r'(z) s(z) + [z^{-1}] r(z) s'(z)$, we have
\[ [z^n]F(g(z)) = -\frac{1}{n} [z^{-1}] F(z) \frac{d}{dz} \left( \frac{\phi(z)}{z} \right)^n \]
\[ = -[z^{-1}] F(z) \left( \frac{\phi(z)}{z} \right)^{n-1} \frac{z \phi'(z) - \phi(z)}{z^2} \]
\[ = [z^n] F(z) \phi(z)^{n-1} (\phi(z) - z \phi'(z)) \]
Proof of (8). We are given \( g(z) = z\phi(g(z)) \), so \( z = g(z)/\phi(g(z)) \). Then we can write

\[
\frac{F(g(z))}{1 - z\phi'(g(z))} = \frac{F(g(z))}{1 - g(z)\phi'(g(z))/\phi(g(z))} = G(g(z))
\]

where

\[
G(z) := \frac{F(z)}{1 - z\phi'(z)/\phi(z)} = \frac{\phi(z)F(z)}{\phi(z) - z\phi'(z)}
\]

So according to (7),

\[
[z^n]G(g(z)) = [z^n]G(z)\phi(z)^{n-1}(\phi(z) - z\phi'(z))
\]

\[
= [z^n] \frac{\phi(z)F(z)}{\phi(z) - z\phi'(z)} \phi(z)^{n-1}(\phi(z) - z\phi'(z))
\]

\[
= [z^n]F(z)\phi(z)^n
\]