1 Overview

We are considering interactions between three fields of mathematics: combinatorics, statistics and probability. In short, combinatorics is the study on counting. Statistics is the aspect on weighted counting and probability is also the aspect on weighted counting but with the condition \( \sum \text{weights} = 1 \).

Example 1.1. Consider paths from origin to \((m, n)\) on integer lattice.

- View of combinatorics: the number of such lattice paths is \( \binom{N}{m} \) where \( N = m + n \).

- View of stat. mech.: Let \( a, b \) be the weight for stepping right and up respectively. This provides a weight for each path. Let \( Z \) be the sum of all weighted path with length \( N \). Then

\[
Z = \sum_{\text{path of length } N} \text{weight(path)} = \sum_{\text{path of length } N} a^R b^U.
\]

where \( R, U \) are the numbers of right and up steps of each path with length \( N \) respectively.

- View of probability: Let \( a, b, R, U, N \) be the same notation as the previous case. Let \( p = \frac{a}{a+b}, q = \frac{b}{a+b} \) be the probability for right and up steps respectively. The sum \( Z \) of all weighted path with length \( N \) becomes

\[
Z = \sum_{\text{path of length } N} \Pr(\text{path}) = \sum_{\text{path of length } N} p^R q^U.
\]
Remark 1.2. There is an underlying probability measure on paths. Given any path \( \gamma_0 \), its probability \( P_r(\gamma_0) \) is given by

\[
P_r(\gamma_0) = \frac{\text{weight}(\gamma_0)}{\sum_{\text{path } \gamma} \text{weight}(\gamma)}
\]

Example 1.3. Consider paths from origin to \((1, 1)\) and let \( a_i, b_i \) be the weights for \( i = 1, 2 \) as the following figure:

Take the path \( \gamma_0 = a_1 b_2 \). Then

\[
P_r(\gamma_0) = \frac{a_1 b_2}{a_1 b_2 + b_1 a_2}.
\]

2 Boltzmann Measure

Given \( N \) identical particles, each falls into one of \( k \) states. For each state, we associate a number \( E_i \) for \( i = 1, \cdots k \) which is called energy of every state. For a closed system, the total energy \( E \) and population numbers \( n_i \) for energy state \( E_i \) must satisfy the equations:

\[
\begin{align*}
E &= \sum_{i=1}^{k} E_i N_i \\
N &= \sum_{i=1}^{k} N_i
\end{align*}
\]

The number \( W \) of possible ways to place particles for a given partition \( \{N_1, N_2, \cdots , N_k\} \) is

\[
W = \frac{N!}{N_1! N_2! \cdots N_k!}.
\]

By the law of large numbers, when \( N \gg 1 \) the weight of the optimal partitioning dominates. Thus we want to determine the specific partitioning which provides a maximal \( W \). Since the logarithm is a monotonically increasing function, we may consider for the maximum of

\[
\ln W = (N \ln N - N) - \sum_{i=1}^{k} (N_i \ln N_i - N_i) + \text{err}.
\]

Apply the variational method with Lagrange multipliers. We set

\[
f = \ln W + \alpha (N - \sum_{i=1}^{k} N_i) + \beta (E - \sum_{i=1}^{k} E_i N_i).
\]
with the undetermined multipliers $\alpha, \beta$. Since
\[ 0 = \frac{\partial f}{\partial N_i} = -\ln N_i - (\alpha + \beta E_i) \]

Therefore,
\[ N_i = e^{-\alpha + \beta E_i}. \]

It follows that
\[ \frac{N_i}{N} = \frac{e^{-\beta E_i}}{\sum_{j=1}^{k} e^{-\beta E_j}}. \]

Let the particle function $Z = \sum_{j=1}^{k} e^{-\beta E_j}$. In physics, the quantity $\beta = \frac{1}{T}$ where $T$ denotes temperature (usually this is $1/kT$ where this $k$ stands for Boltzmann’s constant). The free energy $F$ is given by
\[ F = -T \ln Z \]

Consequently, we can define the Boltzmann measure as the probability $\Pr(\text{state } i)$ of each particle in state $i$
\[ \Pr(\text{state } i) = \frac{e^{-\beta E_i}}{Z}. \]

Now consider the average energy $\bar{E}$
\[ \bar{E} = \frac{E}{N} = \sum_{i=1}^{k} \frac{E_i N_i}{N} = \sum_{i=1}^{k} \frac{E_i e^{-\beta E_i}}{Z} = -\frac{\partial}{\partial \beta} \ln Z = -\frac{\partial}{\partial \beta} \ln Z. \]

Lemma 2.1. $-\ln Z$ is a convex function for $\beta$.

**Proof.** We want to show that $-\frac{\partial}{\partial \beta} \ln Z$ is monotone that is $\frac{\partial^2}{\partial \beta^2} \ln Z > 0$.

\[ \frac{\partial^2}{\partial \beta^2} \ln Z = \sum_{i=1}^{k} \frac{(-E_i)^2 e^{-\beta E_i}}{Z} - \frac{(\sum_{i=1}^{k} E_i e^{-\beta E_i})^2}{Z^2} \]
\[ = \langle E^2 \rangle - \langle E \rangle^2 \]
\[ = \text{var}(E) > 0 \quad \square \]

Definition 2.2. $\frac{1}{\beta} \ln Z$ is called the free energy.

When $T \to \infty$, $\Pr(i) \to \frac{1}{k}$. When $T \to 0$, the probability is supported on the minimum energy state(s).

## 3 Statistical models

A statistical model is a pair $(\Omega, \mathcal{P})$, where $\Omega$ is the configuration space such that particles usually interact locally and $\mathcal{P}$ is a probability measure on $\Omega$. 

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• **Dimer model**
A dimer cover or perfect matching of a graph $G = (V, E)$ is a subset of edges such that each vertex is the endpoint of exactly one edge. $\Omega$ is the set of dimer covers. If each edge has an associated energy the energy of a cover is the sum of energies of its edges.

• **Six-vertex model**
The configuration space $\Omega$ is a set of orientations of a subgraphs of $\mathbb{Z}^2$ with two incoming and two outgoing edges at each vertex. There are six energies associated to the six possible configurations at a vertex. The energy of a configuration is the sum of the energies at each vertex.

• **Gradient model**
Let $G = (V, E)$ be a graph. Let the configuration space $\Omega$ be $\mathbb{R}^{|V|}$. Let $\Phi, U : \mathbb{R} \to \mathbb{R}$ be potential functions. A Hamiltonian is defined as

$$H(f) = \sum_{v \in V} \Phi(f(v)) + \sum_{e=i,j \in E} U(f(i) - f(j)).$$

• **Gaussian free field**
As a special case of the gradient model, let $G = (V, E)$ be any graph, usually a lattice in $d$-dimensional Euclidean space. Define a Hamiltonian by

$$H(f) = \sum_{v_i, v_j \in E} (f(v_i) - f(v_j))^2.$$ 

which is the Hooke’s law with spring energy. Then, the function $f$ with probability density is $e^{-\beta H(f)}$ which is a Gaussian density.

• **Ising model**
Consider the integer lattice $\mathbb{Z}^d$. Let $G = (V, E)$ be a graph on the lattice with the set of integer vertices $V$ and the set of edges $E$. For each vertex $i$, we assign a variable $\sigma_i \in \{1, -1\}$, which provides a spin configuration. Let $\Omega_G = \{1, -1\}^{|V|}$ be the configuration space and take $\sigma \in \Omega_G$. The “energy function” or “Hamiltonian” $H$ is given by

$$H(\sigma) = -\sum_{i,j \in \Omega_G :|i-j|=1} \sigma_i \sigma_j.$$
The sum is over all pairs of sites in $G$ which are nearest neighbors. The probability measure is given by
\[
\Pr(\sigma) = \frac{e^{-\beta H(\sigma)}}{\sum_{\sigma \in \Omega_G} e^{-\beta H(\sigma)}}
\]

• **Hard disk model**

Suppose there are $k$ unit disks in a region $\Lambda$. The energy function $E$ is defined by
\[
E = \begin{cases} 
\infty & \text{if two disks overlapping} \\
0 & \text{else} 
\end{cases}
\]

Let $\Omega$ be the configuration space of $k$ disjoint disks in $\Lambda$. The underlying probability measure is Lebesgue measure on $\Omega \subset \Lambda^k \subset \mathbb{R}^k$. The partition function $Z$ is the volume of $\Omega(k)$.

**Example 3.1** (1-dimensional hard disk model). Let $\Lambda$ be the interval $[0, N]$. The configuration space $\Omega_{N,3}$ is the set of three disjoint unit intervals lying in $\Lambda$, i.e.
\[
\Omega_{N,3} = \{ (a_1, a_2, a_3) \mid 0 \leq a_1, a_2, a_3, a_1 + a_2 + a_3 \leq N - 3 \}.
\]

The volume of the configuration space $\Omega_{N,3}$ is given by formula
\[
\text{vol}(\Omega_{N,3}) = \frac{(N - 3)^3}{3!}.
\]

Generally, the configuration is defined as $\Omega_{N,k} = \{ k \text{ disjoint unit intervals in } \Lambda \}$. Then the volume of $\Omega_{N,k}$ becomes
\[
\text{vol}(\Omega_{N,k}) = \frac{(N - k)^k}{k!}.
\]
Let’s combine all the cases together by giving energy $E_0$ per particle. Define $z = e^{-\beta E_0}$ as fugacity. Then

$$Z_N(z) = \sum_{k=0}^{N} z^k \cdot \text{vol}(\Omega_{N,k})$$

$$= \sum_{k=0}^{N} \frac{(N-k)^k}{k!} z^k$$

**Exercise:** If $N \to \infty$, show that the free energy per unit length $F(z) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(z)$ is given by

$$F(z) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(Z) = z - \frac{z^2}{2} + \frac{3^2}{3!} z^3 - \frac{4^3}{4!} z^4 + \cdots$$

Note that $F(z)$ is the Lambert function also called product logarithm denoted by $PL(z)$, and satisfies

$$PL(z) e^{PL(z)} = z.$$

The derivative of $PL$ satisfies

$$PL'(z) = \frac{PL(z)}{z(1 + PL(z))} \quad \text{for} \quad z \notin \{0, -1/e\}.$$

So

$$z \frac{\partial F(z)}{\partial z} = \frac{PL(z)}{1 + PL(z)} \to 1 \quad \text{as} \quad z \to \infty.$$