We know (Problem 13) that if \((p, p) \in \Delta_M\) is a limit of double points for \(F : M \to N\) then \(p\) is a non-immersion point for \(F\). In particular, if \(F\) is an immersion then there is a neighborhood of \(\Delta_M\) in \(M \times M\) containing no double points.

Say that a double point \((p, q)\) of \(F\) is a good double point if the map \(F \times F\) is transverse to \(\Delta_N\) at \((p, q)\); otherwise call it a bad double point. Say that a non-immersion point \(p\) of \(F\) is a good non-immersion point if for every non-zero vector \(v \in \ker(D_pF)\) the map \(DF : TM \to TN\) is transverse to \(0_N\) at \(v\); otherwise call it a bad non-immersion point. Our main goal (Problem 18 below) is to show that every smooth \(F\) (with compact domain) can be approximated by a smooth map having no bad double points and no bad non-immersion points. This includes the statement that every smooth map from an \(m\)-manifold to a \((2m + 1)\)-manifold can be approximated by a smooth embedding.

16. Show that if \((p, p)\) is a limit of bad double points for \(F\) then \(p\) is a bad non-immersion point for \(F\). It follows that if every non-immersion point of \(F\) is good then there is a neighborhood of \(\Delta_M\) in \(M \times M\) in which there are no bad double points.

[Hint: This is a local question, so we may assume that \(M\) is open in \(\mathbb{R}^m\) and \(N = \mathbb{R}^n\). Define \(G(t, x, v) = \frac{F(x + tv) - F(x)}{t}\) if \(t \neq 0\) and \(G(x, 0, v) = (D_xF)(v)\). Show that for \(t \neq 0\) \(G(t, x, v) = 0\) if and only if \((x, x + tv)\) is a double point for \(F\), and that it is a good double point if and only if \(G\) is transverse to \(0\) at \((t, x, v)\). Show that there exists \(v\) such that \(G(0, x, v) = 0\) if and only if \(x\) is a non-immersion point for \(F\), and that it is a good non-immersion point if and only if for every such \(v\) the map \((x, v) \mapsto G(0, x, v)\) is transverse to \(0\).]

The next problem shows in particular that any smooth map from an \(m\)-manifold to a \(2m\)-manifold can be approximated by an immersion. Note that when a smooth map \(F : \Lambda \times M \to N\) is regarded as a family of maps \(F_\lambda : M \to N\) then it gives a smooth map \(\Lambda \times TM \to TN\) by \((\lambda, v) \mapsto (DF_\lambda)(v)\).
17. Suppose that $F_0 : M \to N$ is a smooth map from a compact $m$-dimensional manifold to an $n$-dimensional manifold. Show that for every $a \in M$ the map $F_0$ belongs to some smooth family of maps $F_\lambda$ such that the associated map $\Lambda \times (TU - 0_U) \to TN$ is transverse to $0_N$ for some neighborhood $U$ of $a$. Conclude from this and a compactness argument that $F_0$ belongs to some smooth family such that $\Lambda \times (TM - 0_M) \to TN$ is transverse to $0_N$. It follows that for almost all $\lambda \in \Lambda$ (i.e. for all $\lambda$ outside a set of measure zero) the map $F_\lambda$ has no bad non-immersion points. Conclude that in the special case $n \geq 2m$ $F_\lambda$ is an immersion for almost all $\lambda$.

18. Let $F_0$ be as in Problem 17. Show that $F_0$ belongs to some smooth family such that $\Lambda \times (TM - 0_M) \to TN$ is transverse to $0_N$ and $\Lambda \times (M \times M - \Delta_M) \to N \times N$ is transverse to $\Delta_N$. 

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