
1. True or false:
   a) The rank of a matrix equal to the number of its non-zero columns.
   b) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.
   c) Elementary row operations preserve rank.
   d) Elementary column operations do not necessarily preserve rank.
   e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix.
   f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
   g) The rank of an $n \times n$ matrix is at most $n$.
   h) An $n \times n$ matrix having rank $n$ is invertible.

2. A $54 \times 37$ matrix has rank 31. What are dimensions of all 4 fundamental subspaces?

3. Compute rank and find bases of all four fundamental subspaces for the matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\ 
0 & 1 & 1 \\ 
1 & 1 & 0
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 2 & 3 & 1 & 1 \\ 
1 & 4 & 0 & 1 & 2 \\ 
0 & 2 & -3 & 0 & 1 \\ 
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

4. Prove that if $A : X \to Y$ and $V$ is a subspace of $X$ then $\dim AV \leq \text{rank } A$. ($AV$ here means the subspace $V$ transformed by the transformation $A$, i.e. any vector in $AV$ can be represented as $Av$, $v \in V$). Deduce from here that $\text{rank}(AB) \leq \text{rank } A$.

**Remark:** Here one can use the fact that if $V \subset W$ then $\dim V \leq \dim W$. Do you understand why is it true?

5. Prove that if $A : X \to Y$ and $V$ is a subspace of $X$ then $\dim AV \leq \dim V$. Deduce from here that $\text{rank}(AB) \leq \text{rank } B$.

6. Prove that if the product $AB$ of two $n \times n$ matrices is invertible, then both $A$ and $B$ are invertible. Even if you know about determinants, do not use them, we did not cover them yet. **Hint:** use previous 2 problems.