
1. True or false
   a) Determinant is only defined for square matrices.
   b) If two rows or columns of $A$ are identical, then $\det A = 0$.
   c) If $B$ is the matrix obtained from $A$ by interchanging two rows (or columns), then $\det B = \det A$.
   d) If $B$ is the matrix obtained from $A$ by multiplying a row (column) of $A$ by a scalar $\alpha$, then $\det B = \det A$.
   e) If $B$ is the matrix obtained from $A$ by adding a multiple of a row to some other row, then $\det B = \det A$.
   f) The determinant of a triangular matrix is the product of its diagonal entries.
   g) $\det(A^T) = -\det(A)$.
   h) $\det(AB) = \det(A) \det(B)$.
   i) A matrix $A$ is invertible if and only if $\det A \neq 0$.
   j) If $A$ is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$.

2. Let $A$ be an $n \times n$ matrix. How are $\det(3A)$, $\det(-A)$ and $\det(A^2)$ related to $\det A$.

3. Using cofactor formula compute inverses of the matrices
   \[
   \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 19 & -17 \\ 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}
   \]

4. Let $D_n$ be the determinant of the $n \times n$ tridiagonal matrix
   \[
   \begin{pmatrix}
   1 & -1 & & & \\
   1 & 1 & -1 & & \\
   & 1 & \ddots & \ddots & \\
   & & \ddots & 1 & -1 \\
   0 & & & 1 & 1
   \end{pmatrix}
   \]
   Using cofactor expansion show that $D_n = D_{n-1} + D_{n-2}$. This yields that the sequence $D_n$ is the Fibonacci sequence $1, 2, 3, 5, 8, 13, 21, \ldots$. 

5. Vandermonde determinant revisited. Our goal is to prove the formula
\[
\begin{vmatrix}
1 & c_0 & c_0^2 & \ldots & c_0^n \\
1 & c_1 & c_1^2 & \ldots & c_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & c_n & c_n^2 & \ldots & c_n^n \\
\end{vmatrix} = \prod_{0 \leq j < k \leq n} (c_k - c_j)
\]
for the \((n + 1) \times (n + 1)\) Vandermonde determinant.

We will apply induction. To do this

a) Check that the formula holds for \(n = 1, n = 2\) (see the previous assignments).

b) Call the variable \(c_n\) in the last row \(x\), and show that the determinant is a polynomial
of degree \(n\), \(A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n\), with the coefficients \(A_k\) depending on
\(c_0, c_1, \ldots, c_{n-1}\).

c) Show that the polynomial has zeroes at \(x = c_0, c_1, \ldots, c_{n-1}\), so it can be represented
as \(A_n \cdot (x - c_0)(x - c_1)\ldots(x - c_n)\), where \(A_n\) as above.

d) Assuming that the formula for the Vandermonde determinant is true for \(n - 1\), compute
\(A_n\) and prove the formula for \(n\).

6. Let points \(A, B\) and \(C\) in the plane \(\mathbb{R}^2\) have coordinates \((x_1, y_1)\), \((x_2, y_2)\) and \((x_3, y_3)\)
respectively. Show that the area of triangle \(ABC\) is the absolute value of
\[
\frac{1}{2} \begin{vmatrix}
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3 \\
\end{vmatrix}
\]

**Hint:** use row operations and geometric interpretation of 2 \(\times\) 2 determinants (area).

The following problem illustrates the power of block matrix notation.

7. Let \(A\) be an \(m \times n\) matrix and \(B\) be an \(n \times m\) one. Prove that
\[
\det \begin{pmatrix}
0 & A \\
-B & I
\end{pmatrix} = \det(AB).
\]

**Hint:** While it is possible to transform the matrix by row operations to a form where the
determinant is easy to compute, the easiest way is to right multiply the matrix by
\[
\begin{pmatrix}
I & 0 \\
B & I
\end{pmatrix}.
\]
You can use the fact that \(\begin{vmatrix}
C & 0 \\
0 & I
\end{vmatrix} = \det C\).