Homework assignment, March 5, 2004.

Due Monday, 3/8 (collected)

1. Suppose the permutation $\sigma$ takes $(1, 2, 3, 4, 5)$ to $(5, 4, 1, 2, 3)$.
   a) Find sign of $\sigma$
   b) What does $\sigma^2 := \sigma \circ \sigma$ do to $(1, 2, 3, 4, 5)$?
   c) What does the inverse permutation $\sigma^{-1}$ do to $(1, 2, 3, 4, 5)$?
   d) What is the sign of $\sigma^{-1}$?

2. Let $P$ be a permutation matrix, i.e. an $n \times n$ matrix consisting of zeroes and ones and such that there is exactly one 1 in every row and every column.
   a) Can you describe the corresponding linear transformation? That will explain the name.
   b) Show that $P$ is invertible. Can you describe $P^{-1}$?
   c) Show that for some $N > 0$
      \[ P^N := PP \cdots P = I. \]
      Use the fact that there are only finitely many permutations.

3. Why is there an even number of permutations of $(1, 2, \ldots, 9)$ and why are exactly half of them odd permutations? **Hint:** this problem can be hard to solve in terms of permutations, but there is a very simple solution using determinants.

4. If $\sigma$ is an odd permutation, explain why $\sigma^2$ is even but $\sigma^{-1}$ is odd.

5. If the entries of both $A$ and $A^{-1}$ are integers, is it possible that $\det A = 3$? **Hint:** what is $\det(A) \det(A^{-1})$?

   The following problem illustrates relation between sign of determinant and the so-called orientation of a system of vectors.

6. Let $v_1$, $v_2$ be vectors in $\mathbb{R}^2$. Show that $D(v_1, v_2) > 0$ if and only if there exists a rotation $T_\alpha$ such that the vector $T_\alpha v_1$ is parallel to $e_1$ (and looking in the same direction), and $T_\alpha v_2$ is in the upper half-plane $x_2 > 0$ (the same half-plane as $e_2$).
   **Hint:** what is the determinant of a rotation matrix?
7. Compute the determinant of

$$A_4 = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.$$

Find also the determinants of smaller matrices $A_3$ and $A_2$ with the same pattern of zeroes on the diagonal and ones elsewhere. Can you predict $\det A_n$? Can you justify it?

8. Let $A$ be an $n \times n$ matrix with entries $a_{j,k} = j + k$. Compute $\det A$ for all $n$. 