1. Orthogonally diagonalize the matrix,

\[ A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \]

Find all square roots of \( A \), i.e. find all matrices \( B \) such that \( B^2 = A \).

Note, that all square roots of \( A \) are self-adjoint.


3. True or false:
   a) A product of two self-adjoint matrices is self-adjoint.
   b) If \( A \) is self-adjoint, then \( A^k \) is self-adjoint.

Justify your conclusions

4. Let \( A \) be \( m \times n \) matrix. Prove that
   a) \( A^*A \) is self-adjoint.
   b) All eigenvalues of \( A^*A \) are non-negative.
   c) \( A^*A + I \) is invertible.

5. Give a proof if the statement is true, or give a counterexample if it is false;
   a) If \( A = A^* \) then \( A + iI \) is invertible.
   b) If \( U \) is unitary, \( U + \frac{3}{4}I \) is invertible
   c) If a matrix \( A \) is real, \( A - iI \) is invertible

6. Let \( U \) be a \( 2 \times 2 \) orthogonal matrix with \( \det U = 1 \). Prove that \( U \) is a rotation matrix.
7. Let $U$ be a $3 \times 3$ orthogonal matrix with $\det U = 1$. Prove that

a) $1$ is an eigenvalue of $U$;

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember, that $1$ is an eigenvalue), then in this basis the matrix of $U$ is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{pmatrix},
$$

where $\alpha$ is some angle.

**Hint:** Show, that since $\mathbf{v}_1$ is an eigenvector of $U$, all entries below $1$ must be zero, and since $\mathbf{v}_1$ is also an eigenvector of $U^*$ (why?), all entries right of $1$ also must be zero. Then show that the lower right $2 \times 2$ matrix is an orthogonal one with determinant $1$, and use the previous problem.