Homework assignment, April 16, 2004.

1. For an operator
   \[ A = \begin{pmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{pmatrix}, \]
   find its modulus \(|A| = (A^*A)^{1/2}\).

2. Orthogonally diagonalize the matrix,
   \[ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \]
   i.e. represent it as \(A = UDU^*,\) where \(D\) is diagonal and \(U\) is unitary.
   
   Note, that one of the eigenvalues has multiplicity 2, so you need to find and orthonormal basis in the corresponding eigenspace (using Gram-Schmidt, for example).

3. Complete the system of vectors \(v_1 = (1, 1, 1, 1)^T, v_2 = (1, 2, -1, -2)^T\) to an orthogonal basis in \(\mathbb{R}^4,\) i.e. find two vectors \(v_3, v_4\) such that the system \(v_1, v_2, v_3, v_4\) is an orthogonal basis.
   
   Probably the simplest way is to find an orthogonal basis in the orthogonal complement of \(\text{span}(v_1, v_2)\)

4. Let \(N\) be a normal operator. Prove that \(\text{Ran} N = \text{Ran} N^*\) and \(\ker N = \ker N^*.\) Hint: you can use diagonalization.

5. An elementary rotation in \(\mathbb{R}^3\) is a rotation about one of the coordinate axis.
   
   Prove that a \(3 \times 3\) orthogonal matrix with the determinant 1 can be represented as a product of three elementary rotation.
   
   **Hint:** Multiplying the matrix from the left by elementary rotations, make it upper triangular. Which upper triangular matrices are orthogonal?

6. (Finding eigenvalues and eigenvectors of self-adjoint operators) Let \(A = A^*\) be positive semidefinite operator (all eigenvalues are non-negative). Let \(\lambda\) be the largest eigenvalue, and let \(k\) be its multiplicity. Prove that
   \[ \frac{1}{\|A^n\|_F} A^n \to \frac{1}{\sqrt{k}} P_E \quad \text{as} \ n \to \infty \]
   where \(P_E\) is the orthogonal peojction onto the eigenspace \(E = \ker(A - \lambda I)\).
   
   Here \(\|A\|_F\) stands for the Frobenius norm of the operator \(A, \|A\|_F = \text{trace}(A^*A) = \sum_{j,k=1}^n |A_{j,k}|^2.\)