1. Orthogonally diagonalize the matrix,

\[ A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \]

Find all square roots of \( A \), i.e. find all matrices \( B \) such that \( B^2 = A \).

Note, that all square roots of \( A \) are self-adjoint.

**Solution:** \( A = U D U^* \) where

\[ D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

Square roots:

\[ U \begin{pmatrix} \pm \sqrt{5} & 0 \\ 0 & \pm 1 \end{pmatrix} U^* \]


False, the square root of

\[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

are matrices

\[ \begin{pmatrix} \pm i & 0 \\ 0 & \pm 1 \end{pmatrix} \]

none of which is self-adjoint

3. True or false:

a) A product of two self-adjoint matrices is self-adjoint.

False, consider the product

\[ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \]

b) If \( A \) is self-adjoint, then \( A^k \) is self-adjoint.

True, \( (A^k)^* = (A^*)^k = A^k \)

Justify your conclusions

4. Let \( A \) be \( m \times n \) matrix. Prove that
a) $A^*A$ is self-adjoint.

\[(A^*A)^* = A^*A^{**} = A^*A.\]

b) All eigenvalues of $A^*A$ are non-negative.

Let $Ax = \lambda x$. Then $(A^*Ax, x) = (\lambda x, x) = \lambda \|x\|^2$. On the other hand, $(A^*x, Ax) = \|Ax\|^2 \geq 0$. So, $\lambda \|x\|^2 \geq 0$ and therefore $\lambda \geq 0$ (note that $\|x\| > 0$).

c) $A^*A + I$ is invertible. By the previous problem all eigenvalues must be non-negative, so $-1$ is not an eigenvalue.

5. Give a proof if the statement is true, or give a counterexample if it is false;

a) If $A = A^*$ then $A + iI$ is invertible.

b) If $U$ is unitary, $U + \frac{3}{4}I$ is invertible

c) If a matrix $A$ is real, $A - iI$ is invertible

6. Let $U$ be a $2 \times 2$ orthogonal matrix with $\det U = 1$. Prove that $U$ is a rotation matrix.

Let $u_1$ be the first column of $U$. Since $\|u_1\| = 1$ it can be written as $u = (\cos \alpha, \sin \alpha)^T$ for some $\alpha$. Any vector $x$ orthogonal to $u_1$ is a multiple of $(-\sin \alpha, \cos \alpha)^T$ (solve the equation $u_1^T x = 0$).

The second column $u_2$ of $U$ must be orthogonal to $u_1$ and has unit norm. So, there are only two possibilities, $u_2 = \pm (-\sin \alpha, \cos \alpha)^T$. But only $u_2 = (-\sin \alpha, \cos \alpha)^T$ gives $\det U = 1$, the other choice gives $\det U = -1$.

7. Let $U$ be a $3 \times 3$ orthogonal matrix with $\det U = 1$. Prove that

a) $1$ is an eigenvalue of $U$;

b) If $v_1, v_2, v_3$ is an orthonormal basis, such that $Uv_1 = v_1$ (remember, that $1$ is an eigenvalue), then in this basis the matrix of $U$ is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{pmatrix},
\]

where $\alpha$ is some angle.

**Hint:** Show, that since $v_1$ is an eigenvector of $U$, all entries below $1$ must be zero, and since $v_1$ is also an eigenvector of $U^*$ (why?), all entries right of $1$ also must be zero. Then show that the lower right $2 \times 2$ matrix is an orthogonal one with determinant $1$, and use the previous problem.