Problem. Let \( \mathbf{r} = (x, y, z) \) be a curve so that its velocity \( \mathbf{v} \) is \( (1, 0, 2) \) and acceleration \( \mathbf{a} \) is \( (1, 1, 1) \) at time \( t = 0 \).

Find the following quantities at time \( t = 0 \) : \( \mathbf{T}, \mathbf{N}, a_T, a_N, \kappa \). Do them in any order. Show full details.

Solution:

\[
v = ||\mathbf{v}|| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}
\]

a) The easiest way to start the problem is probably solving \( \mathbf{T} \)

\[
\mathbf{T} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{(1, 0, 2)}{\sqrt{5}}
\]

Because we can decompose \( \mathbf{a} \) into its tangential \( \mathbf{T} \)-component and its normal \( \mathbf{N} \)-component with the formula

\[
\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N},
\]

there are a few different ways to find the rest quantities. Here is one way.

b) \( a_T = \mathbf{T} \cdot \mathbf{a} = \frac{(1, 0, 2)}{\sqrt{5}} \cdot (1, 1, 1) = \frac{1 + 0 + 2}{\sqrt{5}} = \frac{3}{\sqrt{5}} \)

c) \( a_N = ||\mathbf{a} - a_T \mathbf{T}|| \)

\[
= \left|\left| (1, 1, 1) - \frac{3}{\sqrt{5}} \cdot \frac{(1, 0, 2)}{\sqrt{5}} \right|\right|
\]

\[
= \left|\left| (1, 1, 1) - \left( \frac{3}{5}, 0, \frac{6}{5} \right) \right|\right| = \left|\left| \left( \frac{2}{5}, 1, -\frac{1}{5} \right) \right|\right| = \frac{\sqrt{30}}{5}
\]

d) \( \mathbf{N} = \frac{\mathbf{a} - a_T \mathbf{T}}{||\mathbf{a} - a_T \mathbf{T}||} = \frac{(\frac{2}{5}, 1, -\frac{1}{5})}{\sqrt{30/5}} = \frac{(2, 5, -1)}{\sqrt{30}} \)

e) \( \kappa = a_N/\mathbf{v}^2 = \frac{\sqrt{30}}{5} = \frac{\sqrt{30}}{25} \)

Comment Make sure you understand what these quantities are. It helps you memorize and/or possibly derive the right formulas.

In this problem, observe that no specific curve is given. Indeed, there are many different curves that have the same given velocity and acceleration at time \( t = 0 \). Notice that all the quantities we computed above only depend on the velocity and acceleration. In particular, the curvature at a given point is completely determined by the velocity and acceleration at that point.