GLOBAL BRILL–NOETHER THEORY OVER THE HURWITZ SPACE

ISABEL VOGT

The motivating question is: Given a genus g curve C, what is the geometry of the space of maps from C to $\mathbb{P}^r$ of degree d? Since this data is equivalent to a pair (L, V), where $L \in \operatorname{Pic}^d C$ is a line bundle and $V \subset H^0(C, L)$ is a basepoint free subspace of dimension $r+1$, we are naturally led to consider Brill–Noether varieties

$$W^r_d C := \{L \in \operatorname{Pic}^d C : h^0(C, L) \geq r+1\}.$$

There are many natural geometric questions about $W^r_d C$:

(1) When is $W^r_d C$ nonempty? If $W^r_d C$ is nonempty, what is its dimension?
(2) What can be said about the singularities of $W^r_d C$?
(3) When $\dim W^r_d C = 0$, what is $\# W^r_d C$?
(4) When $\dim W^r_d C > 0$, is it irreducible?

The first two of these questions are local, but the last two require a global understanding of the Brill–Noether variety. The final question can be phrased in a different manner: what are the discrete invariants necessary in understanding maps from a curve to projective space?

While these are all subtle questions for an arbitrary curve, when the curve C is of general moduli, the work of many authors in the 1980s showed that the geometry of $W^r_d C$ is more uniform.

Theorem 1 (The Brill–Noether theorem)**.** *Let C be a general curve of genus g.*

- *(Griffiths–Harris [6]) $W^r_d C$ is nonempty if and only if the Brill–Noether number*
$$\rho(g, r, d) = g - (r+1)(g - d + r)$$
is greater than or equal to 0. When $W^r_d C$ is nonempty, it has dimension $\min(g, \rho(g, r, d))$.
- *(Gieseker [5]) $W^r_d C$ is normal, Cohen–Macaulay and smooth away from $W^{r+1}_d C$.*
- *(Kempf [7], Kleiman–Laksov [8]) When $\rho(g, r, d) = 0$, $W^r_d C$ consists of*
$$g! \prod_{\alpha=1}^{r} \frac{\alpha!}{(g - d + r - \alpha)!}$$
(reduced) points.
- *(Fulton–Lazarsfeld [4], Eisenbud–Harris [3]) When $\rho(g, r, d) > 0$, $W^r_d C$ is irreducible. Futhermore, when $\rho(g, r, d) \geq 0$, the universal $\mathcal{W}^r_d$ has a unique component dominating M_g.*

When $\rho(g, r, d) \geq 0$, the Brill–Noether theorem picks out a distinguished component of the Hilbert scheme containing general curves, enabling results about general curves in projective space [10, 12].

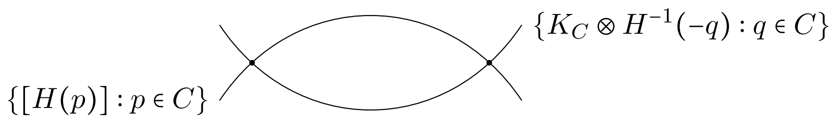
However, in nature, curves are often already encountered via some explicit realization $C \to \mathbb{P}^{r_0}$. The very existence of this map may force the curve C to be too special in moduli for the Brill–Noether theorem to apply. The first natural case is genus g curves realized as degree k covers of $\mathbb{P}^1$ for $k < \lfloor (g+3)/2 \rfloor$. The parameter space $\mathcal{H}_{k,g}$ for such covers is called the Hurwitz space.

Main Question. Let $[f: C \to \mathbb{P}^1] \in \mathcal{H}_{k,g}$ be a general point of the Hurwitz space. What is the geometry (e.g., analogues of (1) – (4)) of $W^r_d C$ in the presence of this fixed degree k map to $\mathbb{P}^1$?

Example 2 (Trigonal genus 5 curve)**.** Suppose that C is a general trigonal genus 5 curve (i.e., it is equipped with a degree 3 map $f: C \to \mathbb{P}^1$). Write $H := f^*\mathcal{O}_{\mathbb{P}^1}(1)$ for the line bundle giving rise to this map. By definition, we have $[H] \in W^1_3 C$, so

$$\dim W^1_3 C = 0 \neq \rho(5,1,3) = -1.$$

Our main question asks how this one "unexpected" linear series affects the geometry of other $W^r_d C$. Increasing the degree to 4, we can introduce an arbitrary basepoint to obtain a component $\{[H(p)] : p \in C\} \subset W^1_4 C$ isomorphic to the curve C. By Riemann–Roch, taking the Serre dual of every line bundle in this component yields another component $\{K_C \otimes H^{-1}(-q) : q \in C\}$.



These two components meet at the two points $[H(p_1)]$ and $[H(p_2)]$, where $p_1 + p_2$ is the unique effective representative of $K_C \otimes H^{-2}$. Notice that, while $W^1_4 C$ has the expected dimension, it is not irreducible, and it is singular even though $W^2_4 C = \varnothing$.

As illustrated in Example 2, when $f \colon C \to \mathbb{P}^r$, the conclusions of the Brill–Noether theorem can fail. Notably, the pair (r, d) are not the only discrete invariants of maps to projective space.

Independently, H. Larson [13] and Cook-Powell–Jensen [1] suggested that these other components may be explained by the additional discrete data of the isomorphism class of the rank k bundle $f_* L$ on $\mathbb{P}^1$ for $L \in W^r_d C$. Recall that any rank k vector bundle on $\mathbb{P}^1$ is isomorphic to a unique direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_k)$, and the tuple $\vec{e} \coloneqq (e_1, \dots, e_k)$ is called the splitting type of the vector bundle. Since cohomology behaves well under finite base change, the splitting type of a line bundle L on C refines the pair (r, d):

$$r = h^0(C, L) - 1 = h^0(\mathbb{P}^1, f_* L) - 1 = \sum_{i=1}^{k} \max(e_1 + 1, 0) - 1, \tag{1}$$

$$d = \chi(C, L) - 1 + g = \chi(\mathbb{P}^1, f_* L) - 1 + g = k + \sum_{i=1}^{k} e_i - 1 + g.$$

It therefore makes sense to define the Brill–Noether splitting loci

$$W^{\vec{e}} C \coloneqq \{[L] \in \operatorname{Pic} C : f_* L \simeq \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k) \text{ or a specialization thereof}\}.$$

Example 3 (Example 2 revisted)**.** Let C be a general trigonal curve of genus 5. For any line bundle $L \in \operatorname{Pic}^4 C$, the conditions (1) imply that $\deg f_* L = -3$. The component $\{[H(p)] : p \in C\} \subset W^1_4 C$ identified in Example 2 is characterized by the property that $L \otimes H^{-1}$ is effective. Pushing forward to $\mathbb{P}^1$, and using the push-pull formula, this is equivalent to $H^0(\mathbb{P}^1, f_* L(-1)) \geq 0$. Hence this component is the splitting locus $W^{(-2,-2,1)} C$. The Serre dual component corresponds to the Serre dual splitting locus $W^{(-3,0,0)} C$. These loci intersect in their common refinement $W^{(-3,-1,1)} C$.

In Example 3, the extra data of the splitting type explains the failure of the classical Brill–Noether theorem. For general $f \colon C \to \mathbb{P}^1$, H. Larson [13] and Cook-Powell–Jensen [1, 2] independently showed that $W^{\vec{e}} C$ is nonempty if and only if

$$\rho'(g, \vec{e}) \coloneqq g - h^1(\mathbb{P}^1, \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k)) = g - \sum_{i<j} \max(e_j - e_i - 1, 0)$$

is greater than or equal to 0. Furthermore, when nonempty, $\dim W^{\vec{e}} C = \min(g, \rho'(g, \vec{e}))$. H. Larson [13] also showed that $W^{\vec{e}} C$ is smooth away from any specializations $W^{\vec{e}'} C \subsetneq W^{\vec{e}} C$.

These results answer the two *local* Brill–Noether questions (1) and (2). However, despite the fact that the splitting type seemed poised to explain the failure of irreducibility, [13] and [1, 2] were unable to attack the *global* aspects of the Brill–Noether theorem. Both [13] and [1, 2] argue by degeneration, and the fundamental difficultly both authors encountered was the inability to prove that everything that behaved like a limit from a general curve was in fact a limit. In [11], we surmount this difficulty by proving a regeneration theorem for the types of limit linear series that occur in the context of splitting loci. We therefore obtain:

Theorem 4 (Global Brill–Noether thoery over the Hurwitz Space [11])**.** *Let $f : C \to \mathbb{P}^1$ be a general degree k genus g cover of $\mathbb{P}^1$.*

(3') *When $\rho'(g, \vec{e}) = 0$, the number of points of $W^{\vec{e}}C$ is the number of k-regular fillings of a certain k-core $\Gamma(\vec{e})$ with symbols $\{1, \ldots, g\}$.*

(4') *When $\rho'(g, \vec{e}) > 0$, $W^{\vec{e}}C$ is irreducible. Furthermore, when $\rho'(g, \vec{e}) \geq 0$, the universal $\mathcal{W}^{\vec{e}}$ over the Hurwitz space $\mathcal{H}_{k,g}$ has a unique irreducible component dominating $\mathcal{H}_{k,g}$.*

We prove this theorem by degeneration to a chain of elliptic curves whose nodes differ by k-torsion. Part (3') hints at the deep combinatorial structure underpinning our choice of specialization. Young diagrams satisfying a k-discrete convexity property are called k-cores. They are fundamental to the study of the affine symmetric group $\tilde{S}_k$, the infinite Coexeter group generated by transpositions $s_0, \ldots, s_{k-1}$ with the braid relations

$$s_i s_j = s_j s_i \text{ for } |i - j| > 1, \text{ and } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

There is a left action of $\tilde{S}_k$ on the set of k-cores and an equivariant isomorphism $\{k\text{-cores}\} \leftrightarrow \tilde{S}_k / S_k$. Using this, a k-regular (efficient) filling of a k-core is equivalent to a reduced word in the generators $s_0, \ldots, s_k$, and (using our regeneration theorem) the enumeration problem in Brill–Noether theory of k-gonal curves is equivalent to the reduced word problem in $\tilde{S}_k$ [9].

Example 5. The two 3-regular fillings of the 3-core $\Gamma((-3, -1, 1))$ with symbols $\{1, \ldots, 5\}$. In a k-regular filling, a symbol can be repeated in boxes that are lattice distance a multiple of k apart.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \leftrightarrow \quad s_0 s_1 s_2 s_1 s_0 \qquad\qquad \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \quad \leftrightarrow \quad s_0 s_2 s_1 s_2 s_0$$

These two possible fillings correspond to the two points of $W^{(-3,-1,1)}C$ observed in Example 3.

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