THE VLASOV-POISSON EQUATION

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ABSTRACT. We describe various facets of kinetic equations

1. INTRODUCTION

The Vlasov-Poisson system reads

$$\left(\partial_t + v \cdot \nabla_x\right) f + \lambda \nabla_x \phi \cdot \nabla_v f = 0, \qquad \Delta_x \phi(x,t) = \int f(x,v,t)dv \tag{1.1}$$

it represents the coupling between a kinetic density (*particle distribution*) $f(x,v,t)$ and a field (*electrostatic/gravitational potential field*) $\phi(x,t)$, where $t \in \mathbb{R}$ is the time and $(x,v) \in TM$ is a vector field (most often, we will choose $M = \mathbb{R}^d$). There are two main versions of this equation depending on whether $\lambda > 0$ (attractive forces) or $\lambda < 0$ (repulsive forces). We refer to [2] for a book reference on kinetic equations.

1.1. Origin: ODEs.

1.1.1. *Flow-map.* One can consider a general ordinary differential equation:

$$\dot{y} = \mathbf{V}(y), \qquad \mathbf{V} : M \to \Gamma(TM), \tag{1.2}$$

and the corresponding flow $\Phi^t : M \to M$ such that $y(t) = \Phi^t(y(0))$, which is a diffeomorphism of M. This explains how single trajectories evolve, but one can ask how do aggregate quantities vary? One can answer this by composition:

$$\mathbb{1}_{\Phi^t(A)} = \mathbb{1}_A \circ \Phi^{-t}.$$

On the other hand, if

$$\widetilde{g}(y,t) = g(\Phi^{-t}(y))$$

then we can verify that

$$\left(\partial_t + \mathbf{V} \cdot \nabla_y\right) \widetilde{g} = 0, \qquad \widetilde{g}(y, t = 0) = g(y).$$

1.1.2. *The Kepler problem.* An important ODE is the Kepler (inverse square) system for N point-particles: $M = T(\mathbb{R}^d)^N$ with coordinates $(x^1, x^2, \dots, x^N, v_1, v_2, \dots, v_N)$. The equation reads

$$\dot{x}^j = \mathbf{V}^j = v_j, \qquad \dot{v}_j = \mathbf{V}_j = \lambda \sum_{k \neq j} c_j c_k \frac{x^j - x^k}{|x^j - x^k|^d},$$

where

- (*plasma case*) $c_j = c_k$ and $\lambda > 0$ (this represents a gas of electrons),
- (*gravitational case*) $c_j > 0$ is the mass of each particle and $\lambda < 0$ is a gravitational constant.

Now, when $N = 2$, one has the famous 2-body problem, usually solved in Calculus. When $N \geq 3$, the system is chaotic and the system is hard to understand. The typical questions become

(1) To understand some special solutions (e.g. Lagrange points for $N = 3$),
(2) To understand special dynamics (KAM theory; Arnold diffusion),
(3) To be able to do numerical integrations of the equations (e.g. one now knows that the Solar system is stable for the next thousands of years).

1.1.3. *Statistical description.* Interestingly, when $N = \infty$ and the particles have similar characteristics ($c_j = c_k$), some aspects become easier to understand, at least as long as one adopts a statistical description (essentially work modulo permutation of the labels of each particles[1]). In this case, one focuses on the *empyrical measure*

$$\mu(t) := \sum_j c_j \delta_{(x^j(t), v_j(t))}$$

This satisfies

$$\partial_t \mu + \mathrm{div}_{x,v} \{\mu \mathbf{V}\} = 0.$$

Assuming that as $N \to \infty$, the particle are "smoothly distributed"

$$\mu_N \rightharpoonup f(x, v, t) dx dv \tag{1.3}$$

we obtain that f solves the VP equation (1.1).

1.2. **Typical questions.** The Vlasov-Poisson equation is the subject of intense current research; there are several directions of research that have proved quite fertile

(1) Understand the stationary solutions and their stability properties [1, 4, 6].
(2) Understand when one obtains a global in time solution and what are the possible blow-up scenarios [5, 8].
(3) Understand the asymptotic behavior of solutions [3].

1.3. **Modified scattering for solutions of the Vlasov-Poisson system in** $3d$**.** We can now describe a recent (2020) result [3].

Theorem 1.1 (Informally)**.** *If the initial distribution is sufficiently small, then the solution exists globally and disperses along a modified scattering.*

To understand the asymptotic behavior, one needs to proceed in several steps.

1.3.1. *The scattering mass.* Without any force, all particle would follow straight lines (*free streaming*). The total mass on each free-streaming trajectory stabilizes to the *scattering mass*:

$$M(v) := \lim_{t \to \infty} \int_{\mathbb{R}^3} f(x, v, t) dx$$

[1]This is *not* the same as using boson statistics.

1.3.2. *This creates a long range electric/gravitational field.* The field felt "on average" is then

$$\mathbf{E}(v,t) = \frac{1}{t^2}\frac{1}{4\pi}\int M(w)\frac{v-w}{|v-w|^3}dw + l.o.t. = -\frac{1}{t^2}\widetilde{\mathbf{E}}(v) + l.o.t.$$

and this leads to the ODE

$$\dot{x} = v, \qquad \dot{v} = -\frac{1}{t^2}\widetilde{\mathbf{E}}(v)$$

The force $t\mathbf{E}$ is long range (in time) because its total effects are not integrable $\int^\infty \frac{dt}{t} = \infty$, but fortunately, its nilpotent structure means that we can still understand it. We see that v should converge, and once v has stabilized, one can approximately integrate the equation:

$$x(t) = vt + \lambda \ln(t)\cdot \widetilde{\mathbf{E}}(v) + x_0$$

1.3.3. *Convergence of the pointwise particle.* The particle density converges along this dynamics:

$$f(x - tv - \lambda\ln(t)\cdot\widetilde{\mathbf{E}}(v), v, t) \rightharpoonup g_\infty(x,v), \qquad \text{as } t\to\infty.$$

1.3.4. *Particle dynamics.* Thus for large times, the electric/gravitational field stabilizes to a "fixed" field; then each particle chooses a balistic motion trajectory and follows it with a logarithmic acceleration (electron) or deceleration (gravitational case).

1.4. **Related open questions.** There are a related interesting open questions; some out of reach, some that may be approachable

(1) What happens for large initial data? In this setting, the attractive and repulsive cases lead to very different answers.
(2) What happens in lower dimensions? Higher dimensions are easier to control. In dimension $d = 2$ the equation becomes much more nonlinear.
(3) What happens for different σ-algebras? i.e. for measures which are density over a reference measure different from the Liouville measure (e.g. radial data? axisymmetric data?). This can be related to Landau damping [7].
(4) What happens for different forces? Interesting cases should be to consider an external magnetic field, to replace Newtonian gravity by General relativity, to consider point charges as well. . .

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