Evolution equations, II

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The main equations

- the defocusing pure power NLS: $u : \mathbb{R}^d \times [0, T] \to \mathbb{C}$,

$$i\partial_t u + \Delta u = u|u|^{2\sigma}, \qquad u(0) = \phi.$$

- the KdV equation: $u : \mathbb{R} \times [0, T] \to \mathbb{R}$,

$$\partial_t u + \partial_x^3 u = u\partial_x u, \qquad u(0) = \phi.$$

Local well-posedness: fixed-point argument

We rewrite the NLS in integral form (Duhamel formula)

$$u(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta} N((u(s))\,ds,$$
$$N(u) = u|u|^{2\sigma}.$$

We would like to construct the solution by the recursive scheme

$$u^{(n+1)}(t) = e^{it\Delta}\phi - i\int_0^t e^{i(t-s)\Delta} \mathcal{N}(u^{(n)}(s))\,ds,$$
$$u^{(0)}(t) = e^{it\Delta}\phi.$$

The procedure converges if

$$\Big\| \int_0^t e^{i(t-s)\Delta}\mathcal{N}(f(s))\,ds - \int_0^t e^{i(t-s)\Delta}\mathcal{N}(g(s))\,ds \Big\|_{L^\infty_T H^\rho} \ll \|f-g\|_{L^\infty_T H^\rho} \tag{1}$$

for any $f, g \in C([0,T] : H^\rho)$ with $\|f\|_{L^\infty_T H^\rho}, \|g\|_{L^\infty_T H^\rho} \leq 2R$.

If $\rho > d/2$ then H^ρ is an algebra, $\rho > d/2$, therefore

$$\|\mathcal{N}(f) - \mathcal{N}(g)\|_{L^\infty_T H^\rho} \lesssim_\rho R\|f - g\|_{L^\infty_T H^\rho}$$

If $\sigma \geq 1$ is an integer then

$$\left\|e^{i(t-s)\Delta}\{\mathcal{N}(f) - \mathcal{N}(g)\}\right\|_{H^\rho} \lesssim_{\rho,R} \|f - g\|_{L^\infty_T H^\rho}$$

for any $s \leq t \in [0, T]$. Thus, for any $t \in [0, T]$

$$\left\| \int_0^t e^{i(t-s)\Delta}\left[\mathcal{N}(f(s)) - \mathcal{N}(g(s))\right] ds\right\|_{H^\rho} \lesssim_{\rho,R} T\|f - g\|_{L^\infty_T H^\rho},$$

which gives the desired bounds (1) if $T \ll_{\rho,R} 1$.

- Conservation laws: the quantities

$$M(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx,$$
$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x u(x,t)|^2 \, dx + \frac{1}{2\sigma+2} \int_{\mathbb{R}^d} |u(x,t)|^{2\sigma+2} \, dx \tag{2}$$

are conserved.

Theorem: Assume that

$$\sigma \in \Big(0, \frac{2}{d-2}\Big).$$

If $\phi \in H^1(\mathbb{R}^d)$ then there is a unique global solution $u \in S^1 \subseteq C([0,\infty) : H^1(\mathbb{R}^d))$ of the initial-value problem

$$i\partial_t u + \Delta u = u|u|^{2\sigma}, \qquad u(0) = \phi.$$

Moreover, the mapping $\phi \to u$ is continuous from $H^1 \to S^1([0,T])$, for any $T > 0$.

Global well-posedness of the defocusing NLS

- To prove global well-posedness we need to be able to prove local well-posedness in H^1.
- The range $\sigma \in (0, \frac{2}{d-2})$ is called the energy-subcritical range.
- We would still like to construct the solution by the basic recursive scheme

$$u^{(n+1)}(t) = e^{it\Delta}\phi - \int_0^t e^{i(t-s)\Delta}\mathcal{N}(u^{(n)}(s))\,ds,$$
$$u^{(0)}(t) = e^{it\Delta}\phi.$$

in a suitable space $S^1[0, \varepsilon]$.

We define the Strichartz space of functions on $\mathbb{R}^d \times \mathbb{R}$ by

$$S^0 := L^\infty_t L^2_x \cap L^q_t L^r_x,$$
$$\|f\|_{S^0} := \max\left\{\|f\|_{L^\infty_t L^2_x}, \|f\|_{L^q_t L^r_x}\right\}$$

where (q, r) is an *admissible pair*

$$2/q + d/r = d/2, \qquad (q, r) \in (2, \infty] \times [2, \infty]. \tag{3}$$

Here

$$\|f\|_{L^q_t L^r_x} := \Big[\int_{\mathbb{R}}\Big[\int_{\mathbb{R}^d} |f(x,t)|^r\,dx\Big]^{q/r} dt\Big]^{1/q}.$$

Theorem: *(Strichartz estimates) (a) If (q, r) is admissible then*

$$\|e^{it\Delta}\phi\|_{L^q_t L^r_x} \lesssim_q \|\phi\|_{L^2}.$$

(b) If (q, r) is an admissible pair then

$$\Big\|\int_{-\infty}^t e^{i(t-s)\Delta} N(s)\,ds\Big\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} \lesssim_q \|N\|_{L^1_t L^2_x + L^{q'}_t L^{r'}_x}.$$

- By definition,

$$\|N\|_{X+Y} := \inf_{N=N_1+N_2} \|N_1\|_X + \|N_2\|_Y .$$

- Dispersive estimates. The kernel of the operator $e^{it\Delta}$ is

$$K(x,t) := C\int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-it|\xi|^2}\, d\xi = Ct^{-d/2} e^{i|x|^2/(4t)}.$$

$$\|e^{it\Delta}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim 1,$$
$$\|e^{it\Delta}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}.$$

By interpolation

$$\|e^{it\Delta}\|_{L^{r'}(\mathbb{R}^d)\to L^r(\mathbb{R}^d)} \lesssim |t|^{-d(1/2-1/r)}.$$

- The TT^* argument: we want

$$\|e^{it\Delta}\phi\|_{L^q_t L^r_x} \lesssim_q \|\phi\|_{L^2}.$$

This is equivalent to

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}} (e^{it\Delta}\phi)(x)g(x,t)\,dxdt\Big| \lesssim_q \|\phi\|_{L^2}\|g\|_{L^{q'}_t L^{r'}_x}.$$

This is equivalent to

$$\Big\|\int_{\mathbb{R}^d\times\mathbb{R}} e^{-it|\xi|^2}e^{ix\cdot\xi}g(x,t)\,dxdt\Big\|_{L^2_\xi} \lesssim_q \|g\|_{L^{q'}_t L^{r'}_x}.$$

This is equivalent to

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}}\int_{\mathbb{R}^d\times\mathbb{R}} g(x,t)\overline{g}(y,s)K(x-y,t-s)\,dxdtdyds\Big| \lesssim_q \|g\|^2_{L^{q'}_t L^{r'}_x}. \tag{4}$$

However, using the dispersive estimate,

$$\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x,t) h(y,s) K(x-y, t-s)\, dxdy \Big| \lesssim G(t)G(s)|t-s|^{-d(1/2-1/r)}$$

where

$$G(t) = \Big[\int_{\mathbb{R}^d} |g(x,t)|^{r'}\, dx \Big]^{1/r'}.$$

The desired bound follows from fractional integration:

$$\| f * |y|^{-\gamma} \|_{L^q(\mathbb{R})} \leq C_{p,q} \| f \|_{L^p(\mathbb{R})}$$

if

$$0 < \gamma < 1, \qquad 1 < p < q < \infty, \qquad 1 - \gamma = \frac{1}{p} - \frac{1}{q}.$$

We return now to the proof of the global well-posedness theorem. Assume, for simplicity, that $d = 3$ and we study the cubic NLS , corresponding to $\sigma = 1$. This is consistent with the subcritical condition

$$\sigma \in (0, 2/(d-2)).$$

We define the Strichartz space $S^1[0, T]$ as the set functions on $\mathbb{R}^d \times [0, T]$ defined by the norm

$$\|f\|_{S^1[0,T]} := \max\left\{\|\langle\nabla\rangle f\|_{L_t^\infty L_x^2}, \|\langle\nabla\rangle f\|_{L_t^4 L_x^3}\right\}.$$

with $(q, r) = (4, 3)$.

The Strichartz estimate shows that

$$\|e^{it\Delta}\phi\|_{S^1[0,T]} \lesssim \|\phi\|_{H^1}$$

To close the fixed-point argument we need to show that

$$\big\|f \cdot g \cdot \partial h\big\|_{L_t^{4/3} L_x^{3/2}[0,T]} \ll \|f\|_{S^1[0,T]} \|g\|_{S^1[0,T]} \|h\|_{S^1[0,T]} \qquad (5)$$

if T is small enough. However, using Sobolev embedding in the x variable

$$\|f\|_{L_t^4 L_x^{12}[0,T]} \lesssim \|f\|_{S^1[0,T]}, \qquad \|g\|_{L_t^4 L_x^{12}[0,T]} \lesssim \|g\|_{S^1[0,T]},$$
$$\|\partial h\|_{L_t^\infty L_x^2[0,T]} \lesssim \|h\|_{S^1[0,T]}.$$

Therefore

$$\big\|f \cdot g \cdot \partial h\big\|_{L_t^2 L_x^{3/2}[0,T]} \lesssim \|f\|_{S^1[0,T]} \|g\|_{S^1[0,T]} \|h\|_{S^1[0,T]}$$

and the desired conclusion (5) follows by taking T small.

We consider now the Korteweg-de Vries equation:
$u : \mathbb{R} \times [0, T] \to \mathbb{R}$,

$$\partial_t u + \partial_x^3 u = u\partial_x u, \qquad u(0) = \phi.$$

The equation has infinitely many conservation laws, including the conservation of mass

$$M(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx.$$

Theorem: If $\phi \in H^0(\mathbb{R})$ is real-valued then there is a unique global solution $u \in F^0 \subseteq C([0,\infty) : H^0(\mathbb{R}))$ of the KdV initial-value problem. Moreover, the mapping $\phi \to u$ is continuous from $H^0 \to F^0([0, T])$, for any $T > 0$.

Global well-posedness of the KdV equation

- The energy argument gives local well-posedness in $H^{3/2+}$.

- By scaling $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $\lambda > 0$, we may assume that the initial data is small in H^0, $\|\phi\|_{H^0} \leq \varepsilon \ll 1$, which will be propagated by the flow.

- We would still like to construct the solution by the basic recursive scheme (coming from the Duhamel formula)

$$u^{(n+1)}(t) = e^{-t\partial_x^3}\phi + \int_0^t e^{-(t-s)\partial_x^3}\mathcal{N}(u^{(n)}(s))\,ds,$$
$$u^{(0)}(t) = e^{it\Delta}\phi.$$

in a suitable space $F^0[0,1]$. In our case the nonlinearity is

$$N(u) = u\partial_x u.$$

Global well-posedness of the KdV equation

The key idea is to use a new class of spaces, called the $X^{s,b}$ spaces introduced by Bourgain, Kenig-Ponce-Vega, and Klainerman-Machedon. They are defined by the norms

$$\|f\|_{X^{s,b}} := \left\|\widehat{f}(\xi,\tau)\langle\tau-\omega(\xi)\rangle^b\langle\xi\rangle^s\right\|_{L^2} \tag{6}$$

where $\langle r\rangle := (1+r^2)^{1/2}$ and $\omega(\xi) := \xi^3$ is the KdV dispersion relation. Here $f : \mathbb{R}\times\mathbb{R}$ and $\widehat{f}$ denotes its Fourier transform in both variables.

For functions $f : \mathbb{R}\times[a,b]$ we define

$$\|f\|_{X^{s,b}[a,b]} := \inf_{Ef=f \text{ on } \mathbb{R}\times[a,b]} \|Ef\|_{X^{s,b}},$$

where the infimum is taken over all the extension Ef of f.

Global well-posedness of the KdV equation

- Notice that

$$X^{s,b} \hookrightarrow C(\mathbb{R} : H^s) \qquad \text{if } b > 1/2.$$

- To close the fixed point argument we need to prove the linear estimates

$$\|e^{-t\partial_x^3}\phi\|_{X^{s,b}[0,1]} \lesssim \|\phi\|_{H^s}, \tag{7}$$

$$\Big\| \int_0^t e^{-(t-s)\partial_x^3} N(s)\, ds \Big\|_{X^{s,b}[0,1]} \lesssim \|N\|_{X^{s,b-1}[0,1]}, \tag{8}$$

and the bilinear estimates $(b = 1/2+)$,

$$\|\partial_x(fg)\|_{X^{0,b-1}} \lesssim \|f\|_{X^{0,b}} \|g\|_{X^{0,b}}. \tag{9}$$

The linear estimates are easy, for example

$$\begin{aligned}\|e^{-t\partial_x^3}\phi\|_{X^{s,b}[0,1]} &\leq \|\eta(t)e^{-t\partial_x^3}\phi\|_{X^{s,b}}\\ &\lesssim \|\widehat{\phi}(\xi)\widehat{\eta}(\tau-\xi^3)\langle\tau-\omega(\xi)\rangle^b\langle\xi\rangle^s\|_{L^2_{\xi,\tau}}\\ &\lesssim \|\phi\|_{H^s}.\end{aligned}$$

In view of the definitions, the bilinear estimates are equivalent to

$$\int_{\mathbb{R}^4}\frac{|\xi+\eta|H(\xi+\eta,\tau+\mu)F(\xi,\tau)G(\eta,\mu)}{\langle\tau+\mu-\omega(\xi+\eta)\rangle^{1-b}\langle\tau-\omega(\xi)\rangle^b\langle\mu-\omega(\eta)\rangle^b}\,d\xi d\eta d\tau d\mu \lesssim \|F\|_{L^2}\|G\|_{L^2}\|H\|_{L^2} \tag{10}$$

where

$$\begin{aligned}F(\xi,\tau) &:= \widehat{f}(\xi,\tau)\langle\tau-\omega(\xi)\rangle^b,\\ G(\xi,\tau) &:= \widehat{g}(\xi,\tau)\langle\tau-\omega(\xi)\rangle^b.\end{aligned}$$

The point is that the denominator cannot be small, since

$$\omega(\xi+\eta)-\omega(\xi)-\omega(\eta)=(\xi+\eta)^3-\xi^3-\eta^3=3\xi\eta(\xi+\eta).$$

The bounds (10) follow by dyadic decompositions in the variables ξ,η, $\tau-\omega(\xi)$, $\mu-\omega(\eta)$. An important case is when

$$\xi\approx N\gg 1,\ |\eta|\lesssim 1/N^2,\ |\tau-\omega(\xi)|\lesssim 1,\ |\mu-\omega(\eta)|\lesssim 1.$$