Computer-assisted proofs in PDE: elliptic case

Javier Gómez-Serrano

Brown University and Universitat de Barcelona

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joint work with Joel Dahne, Kimberly Hou and Gerard Orriols

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Main takeaway message

- New method to construct (perturbations to) solutions.
- Linear operators (Friday: nonlinear).
- Constructive! Gives quantitative information on the solutions.
- Simple ideas.
- Very well suited towards counterexamples with open conditions.
- Computer-assisted proof.
- In this talk: All BCs are Dirichlet, but the method works with Neumann, Robin or Steklov as well.

Additional observations

In this talk: main operator is the Laplacian, but the method works for others. For example, following Chen-Hou-Huang (De Gregorio model):

$$\omega_t + u\omega_x = u_x\omega, \quad u_x = H\omega$$

Finite time singularities iff stability around self-similar profile $\bar{\omega}$ (+ more conditions). If one writes the linearization:

$$\omega_t = L(\omega) + N(\omega) + F$$

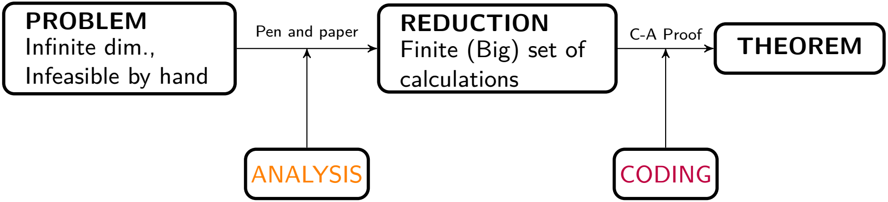
the core of the argument is to bound the spectrum and prove, for a suitable weight and $\lambda > 0$:

$$\langle L(\omega), \omega \rangle \leq -\lambda \langle \omega, \omega \rangle$$

and extract nonlinear stability out of the linear one.
$\Rightarrow$ Computer-assisted.

What is a computer-assisted proof (in analysis/PDE)?

SCHEME OF A PROOF:



KEY OBSERVATION 1: THIS IS **NOT** SIMULATION.

KEY OBSERVATION 2: EVERYTHING IS FULLY RIGOROUS.
INTERVAL ARITHMETICS ARE USED AS PART OF THE PROOF.

THE RESULT IS A THEOREM.

Part 1: Not any 3 eigenvalues determine a triangle

Joint work with Gerard Orriols.

The spectrum of the Laplace operator

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

$$\begin{cases} -\Delta u = \lambda_i u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

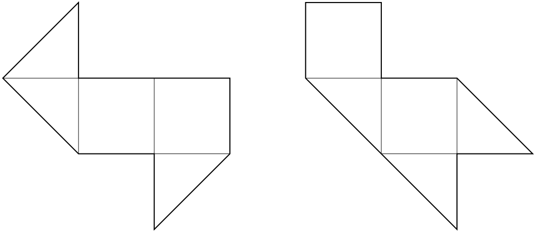
The Laplace operator has a discrete spectrum

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

"Can one hear the shape of a drum?" – Mark Kac, 1966
Can one find two non-isometric domains Ω_1, Ω_2 such that the solutions are the same λ_i in each Ω?

Theorem (Gordon-Webb-Wolpert, 1992)

In general, NO.



Can one hear the shape of a drum?

- One can't hear the shape of a drum (Gordon–Webb–Wolpert, '92).
- There is spectral determination in the class of analytic bounded planar domains with a reflection symmetry (Zelditch, '09).
- Hezari-Zelditch ('19): ellipses of small eccentricity can be heard in the class of smooth domains.
- Hezari-Zelditch ('21): Centrally symmetric analytic plane domains.
- Durso ('88), Grieser–Maronna ('13): two isospectral triangles are isometric.
- Hezari-Lu-Rowlett ('17, '20): two isospectral trapezoids are isometric.
- Enciso-JGS ('17): any (semi-)regular polygon can be heard in the class of polygons (no constraints on the number of sides).

  Fundamental: All proofs use the whole spectrum.
  $\Rightarrow$ **OVERKILL** for polygons

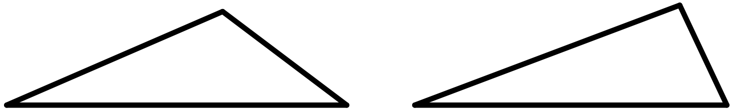
Can a *human* hear the shape of a *triangular* drum?

- It is natural to ask if finitely many eigenvalues suffice.
- Chang and DeTurck ('89) proved that it is enough to check that λ_1, λ_2 and a finite number of eigenvalues which depends on these two, but a priori unbounded.
- The dimension of the moduli space is 3.
- Antunes and Freitas conjectured, based on numerical evidence, 10 years ago, that indeed λ_1, λ_2 and λ_3 determine the shape of the triangle, and they observe that this is not the case with λ_1, λ_2 and λ_4.
- No progress since, even in this "simple" case.

Not any three eigenvalues determine a triangle

Theorem (JGS–Orriols, '20)

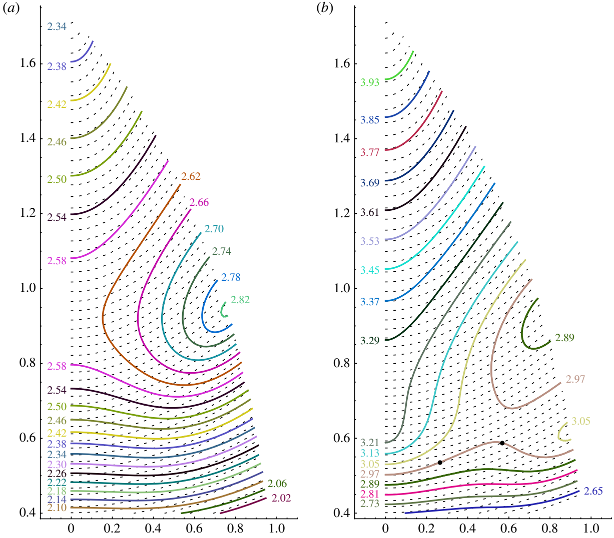
There exist two non-isometric triangles T_{A} and T_{B} for which the eigenvalues λ_1, λ_2 and λ_4 agree.



Some observations

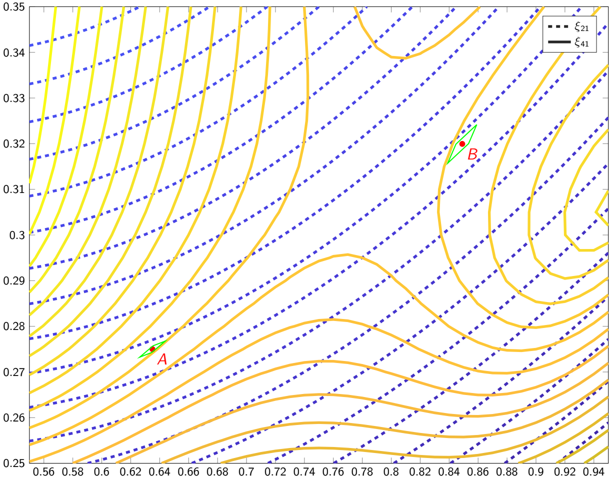
- By rescaling, it is enough to consider the functions $\xi_{21} = \frac{\lambda_2}{\lambda_1}(T), \xi_{41} = \frac{\lambda_4}{\lambda_1}(T) = (f, g)(T)$ and fix the length of one (the longest) side. This compactifies the space.
- We identify the point (x_1, x_2) with the triangle of vertices $(0,0), (1,0)$ and (x_1, x_2).
- Numerically find two candidates T_1 and T_2 with $f(T_i) \sim c_1, g(T_i) \sim c_2$.

Can a *human* hear the shape of a *triangular* drum?



Numerical evidence by Antunes–Freitas: (a) for ξ_{21}, ξ_{31}, (b) for ξ_{21}, ξ_{41}.

The regions of search



The Poincaré–Miranda theorem

Theorem (Poincaré–Miranda)

Given two continuous functions $f, g : [-1, 1]^2 \to \mathbb{R}$ such that

- $f(x, y)$ has the same sign as x when $x = \pm 1$
- $g(x, y)$ has the same sign as y when $y = \pm 1$,

there exists a point $(x, y) \in [-1, 1]^2$ such that $f(x, y) = g(x, y) = 0$.

Fix $\overline{\xi_{21}} := 1.67675$ and $\overline{\xi_{41}} := 2.99372$, and let

$$f(T) = \xi_{21}(T) - \overline{\xi_{21}}, \qquad g(T) = \xi_{41}(T) - \overline{\xi_{41}}$$

The *only* remaining thing is validating these inequalities!

Rigorous eigenvalue bounds

Finite Element Methods (FEM)

Give the index of the eigenvalue in the spectrum.
Very rough (lower) bounds.
Hard to validate.

Method of Particular Solutions (MPS)

Very good precision (depending on the geometry of the problem).
Exponential convergence using the *lightning method* (Gopal–Trefethen '19).
No information about the position in the spectrum.

We use a combination of both families in two passes, together with good stability bounds by Barnett-Hassell (2011).

Guaranteed lower bound

Theorem (Guaranteed lower bound, Liu '15)

Let λ_k be the kth eigenvalue and $\lambda_{h,k}$ the kth eigenvalue of the discrete FEM system. We can give a lower bound of λ_k by

$$\frac{\lambda_{h,k}}{1 + C\lambda_{h,k}/n^2} \leq \lambda_k$$

$\Rightarrow$ Linear algebra problem: Bound rigorously the eigenvalues of a (big) Matrix.

Proof that there are no gaps

It is enough to bound rigorously the first eigenvalues of a matrix.

1. Find numerically $\tilde{S}$ with $\tilde{S}^T\tilde{S} \approx I$ and $\tilde{S}^T M\tilde{S}$ almost diagonal.
2. There exists S near $\tilde{S}$ such that $S^T S = I$.
3. Therefore M has the same eigenvalues as $S^{-1}MS = S^T MS \approx \tilde{S}^T M\tilde{S}$.
4. Apply Gershgorin's disks theorem to this interval of matrices to isolate the first eigenvalues.

Gershgorin's intervals theorem

The eigenvalues of a symmetric matrix $A = (a_{ij})$ lie in the intervals $[a_{ii} - \sum_{j\neq i} |a_{ij}|, a_{ii} + \sum_{j\neq i} |a_{ij}|]$.

Conclusion

- This validates the condition on a triangle.
- Use monotonicity arguments to validate the condition on triangles nearby.
- Cover the segments with (finitely many) triangles and validate them.

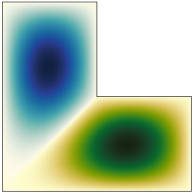
Part 2: Counterexamples to Payne's conjecture

Joint work with Joel Dahne and Kimberly Hou.

Payne's conjecture (1967)

The nodal line of u_2 on a bounded domain in $\mathbb{R}^2$ must touch the boundary.

Extended to higher dimensional case by Yau (1993).



Partial positive results

- Convex domain:
  - Melas (1992)
  - Alessandrini (1994)
- Varius symmetries and/or convexity assumptions
  - Payne (1973),
  - Lin (1987)
  - Pütter (1990)
  - Damascelli (2001)
  - Yang and Gou (2013)
  - ...

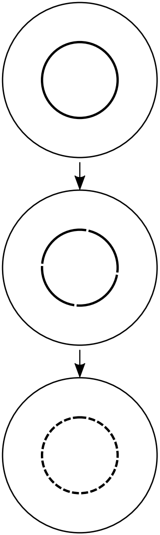
A counterexample

Counterexample constructed by Hoffmann-Ostenhof, Hoffmann-Ostenhof and Nadirashvili in 1997.

Extended to higher dimensions by Fournais (2001) and Kennedy (2013).

No explicit lower bound on the number of boundary components.

"Delicate to bound, astronomical" *"Of the order* 10^9*"*



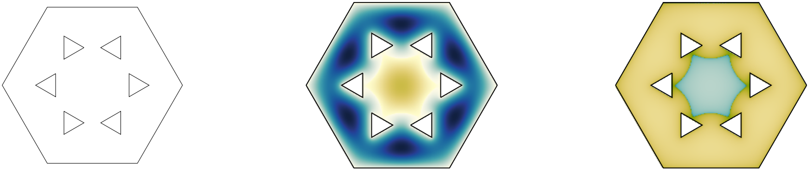
Question (Hoffmann-Ostenhof, Hoffmann-Ostenhof, Nadirashvili 1997)

What is the smallest number, N_0, of boundary components for a counterexample?

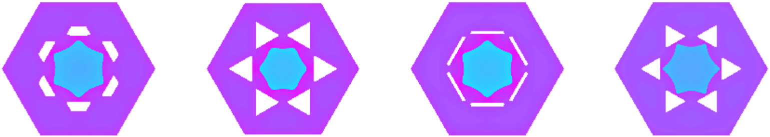
Our counterexample

Theorem (Dahne, JGS, Hou 2021)

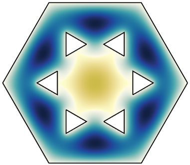
There exists a planar domain with 6 holes ($N_0 = 7$) for which the nodal line of u_2 does not touch the boundary.



Alternative counterexamples



Construction of the counterexample



1. Find good numerical approximations $\tilde{u}_2$ and $\tilde{\lambda}_2$ of u_2 and λ_2
2. Compute rigorous error bounds for the approximation
3. Certify the index of the approximation

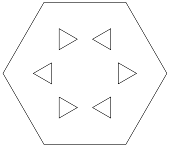
$$\lambda_2 \in [\tilde{\lambda}_2 \pm 6.89 \cdot 10^{-3}]$$
$$|u_2(x) - \tilde{u}_2(x)| \leq 4.2162 \cdot 10^{-5}$$

Similar to Dahne-Salvy (2020) and JGS-Orriols (2020)

Determining the index

Approximate eigenvalues:

$$\lambda_1 = 31.0432,\ \lambda_2 = 63.2104,\ \lambda_3 = 63.7259,\ \lambda_4 = 63.7259,\ \lambda_5 = 68.2629.$$



Proceed as in JGS-Orriols (2020).

1. Separate the first four eigenvalues by lower bounding λ_5
2. Isolate the first four eigenvalues - complicated by the double eigenvalue

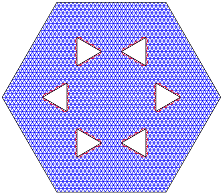
Determining the index - lower bounding λ_5

Theorem (Liu, 2015)

Consider a polygonal domain Ω with a triangulation so that each triangle has diameter at most h. Let λ_k be the k-th solution of Dirichlet Laplacian in Ω and $\lambda_{h,k}$ the k-th eigenvalue of the corresponding Crouzeix-Raviart discretized problem in Ω. Then

$$\frac{\lambda_{h,k}}{1 + C_h^2 \lambda_{h,k}} \leq \lambda_k,$$

where $C_h \leq 0.1893h$ is a constant.



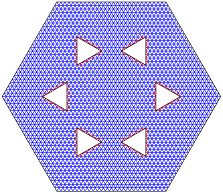
See also Liu, Oishi (2013) and Liu (2020).

Determining the index - lower bounding λ_5

We want a lower bound for λ_5

$$\frac{\lambda_{h,5}}{1 + C_h^2 \lambda_{h,5}} \leq \lambda_5,$$

Enough to find a lower bound for $\lambda_{h,5}$



Determining the index - lower bounding λ_5

Weak formulation: $\int_\Omega \nabla u \cdot \nabla \psi = \lambda \int_\Omega u\psi$

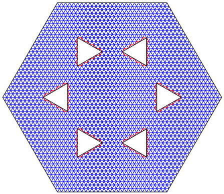
Define stiffness and mass matrices
$A = (a_{EF}), B = (b_{EF})$

$Ax = \lambda Bx$

$Mx = \lambda x$ with $M = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$

Want to lower bound the fifth eigenvalue, $\lambda_{h,5}$, of M

M is of size 6048×6048



Determining the index - lower bounding λ_5

Lemma

Let $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^m$ and $s > 0$ such that $|\langle v_i, v_j \rangle - \delta_{ij}| \leq s$ and suppose that $8ms < 1$. Then there exists an orthonormal set of vectors $w_1, \ldots, w_m \in \mathbb{R}^m$ such that $\|v_i - w_i\| \leq \sqrt{3s}$.

Let $\tilde{Q}$ consist of columns forming an (numerically) approximate orthonormal basis of eigenvectors of M.

$\tilde{D} = \tilde{Q}^T M \tilde{Q}$: almost diagonal and approximately similar to M.

By lemma exists Q orthogonal close to $\tilde{Q}$.

$$D = Q^T M Q$$

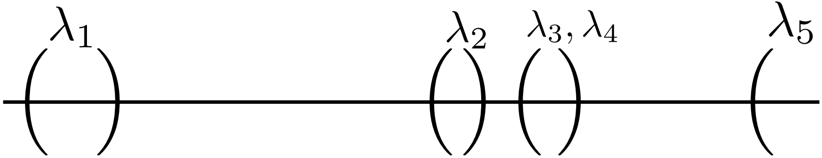
Conclude using Gershgorin's theorem.

Determining the index - isolating λ_2

Construct four approximate eigenfunctions $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$ and $\tilde{u}_4$.

Compute enclosures of their eigenvalues $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$.

Double eigenvalue ($\tilde{\lambda}_3$ and $\tilde{\lambda}_4$) needs special care.



Construction of the approximations - Method of particular solutions

Introduced by Fox, Henrici and Moler (1967) and improved by Betcke and Trefethen (2005) in "Reviving the Method of Particular Solutions".

Finds $(\tilde{\lambda}, \tilde{u})$ satisfying

$$-\Delta \tilde{u} = \tilde{\lambda}\tilde{u} \text{ in } \Omega$$
$$\tilde{u} \approx 0 \text{ on } \partial\Omega$$

$$\tilde{u} = \sum_{k=1}^{N} c_k \phi_k \quad \text{with} \quad -\Delta \phi_k = \tilde{\lambda}\phi_k \text{ in } \Omega$$

Examples of ϕ_k: $J_{\alpha k}(\sqrt{\lambda} r) \sin \alpha k\theta$, $Y_1\left(r\sqrt{\lambda}\right) \sin\theta$, $J_k\left(r\sqrt{\lambda}\right) \sin k\theta$

Construction of the approximations - choice of basis functions

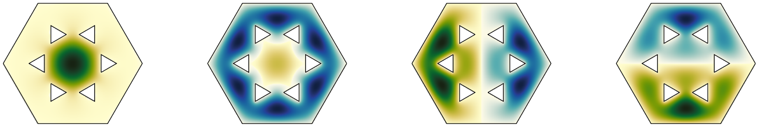
Extremely important for good (root-exponential) convergence.

Combines ideas mainly from:

- Fox, Henrici and Moler (1967)
- Betcke and Trefethen, Reviving the method of particular solutions (2005)
- Gopal and Trefethen, Solving laplace problems with corner singularities via rational functions (2019) - *The Lightning Method*
- JGS-Orriols (2020)
- Dahne-Salvy (2020)

Construction of the approximations

	Eigenvalue	Free coefficients	Collocation points
$\tilde{u}_1$	$\tilde{\lambda}_1 = 31.0432$	17	51
$\tilde{u}_2$	$\tilde{\lambda}_2 = 63.20833598626884$	476	7616
$\tilde{u}_3$	$\tilde{\lambda}_3 = 63.7259$	270	2160
$\tilde{u}_4$	$\tilde{\lambda}_4 = 63.7259$	252	2016



Rigorous error bounds

Theorem (Moler, Payne 1968)

$(\tilde{\lambda}, \tilde{u})$ approximate eigenpair. Let $\mu = \frac{\sqrt{|\Omega|}\,\sup_{x\in\partial\Omega}|\tilde{u}(x)|}{\|\tilde{u}\|_2}$.
Then there exists an eigenpair (λ_k, u_k) satisfying

$$\frac{|\tilde{\lambda} - \lambda_k|}{\lambda_k} \leq \mu,$$

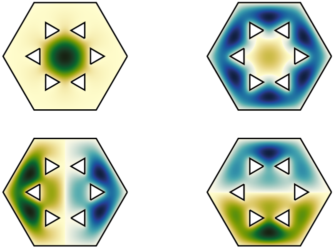
$$|\tilde{u}(x) - u_k(x)| \leq \left(\sup_{x\in\partial\Omega} |\tilde{u}(x|)\right)\left(1 + g(x)\tilde{\lambda}\left(\frac{1}{1-\mu} + \frac{1}{\alpha}\left(1 + \frac{\mu^2}{\alpha^2}\right)\right)\right).$$

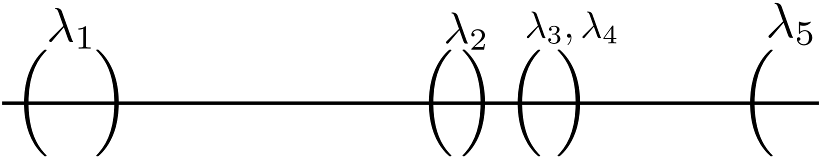
With

$$\alpha = \min_{\lambda_n \neq \lambda_k} \frac{|\lambda_n - \tilde{\lambda}|}{|\lambda_n|}, \quad g(x) = \left(\int_\Omega G(x,y)^2 dy\right)^{\frac{1}{2}} \leq \frac{1}{4\pi}\sqrt{2|\Omega|}.$$

Rigorous error bounds

	$\mu \leq$	Enclosure of λ
$\tilde{u}_1$	0.14	$[30 \pm 6.1]$
$\tilde{u}_2$	$8.26 \cdot 10^{-5}$	$[63.21 \pm 6.89 \cdot 10^{-3}]$
$\tilde{u}_3$	0.00186	$[64 \pm 0.393]$
$\tilde{u}_4$	0.00215	$[64 \pm 0.411]$





Handling the double eigenvalue

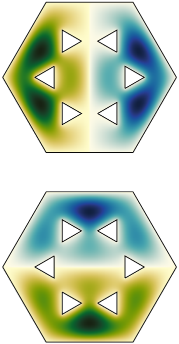
Goal: Prove that $\tilde{u}_3$ and $\tilde{u}_4$ correspond to two different eigenfunctions.

Assume $\tilde{u}_3$ corresponds to u_k and $\tilde{u}_4$ to u_l. Want to prove $k \neq l$.

Enough to show that $\frac{u_k(x_1)}{u_l(x_1)} \neq \frac{u_k(x_2)}{u_l(x_2)}$ for some $x_1 \neq x_2$.

Problem: no lower bound for $\alpha \implies$ no upper bound for error.

$$\alpha = \min_{\lambda_n \neq \lambda_k} \frac{|\lambda_n - \tilde{\lambda}|}{|\lambda_n|}$$



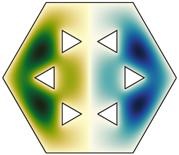
Handling the double eigenvalue

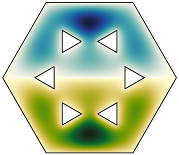
Revised goal: Prove that there are two eigenvalues in $\Lambda' = [64 \pm 0.419] \supset [64 \pm 0.393] \cup [64 \pm 0.411] = \Lambda$.

By contradiction, assume only one eigenvalue in Λ'.

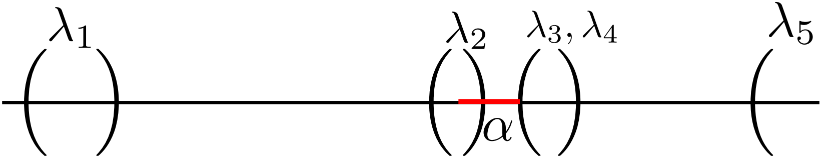
Gives us a lower bound for α, so we can prove $\frac{u_k(x_1)}{u_l(x_1)} \neq \frac{u_k(x_2)}{u_l(x_2)}$.

This implies $k \neq l$ and $\lambda_k, \lambda_l \in \Lambda \subset \Lambda'$ gives a contradiction.





Isolating λ_2



Isolating the nodal line

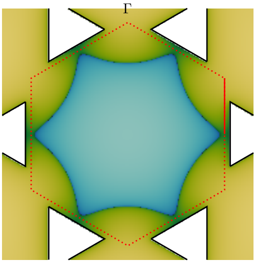
$$|\tilde{u}_2(x) - u_2(x)| \le 4.2162 \cdot 10^{-5} \quad \forall x \in \Omega.$$

Step 1 Prove u_2 strictly negative on Γ.

$\tilde{u}_2 \le -4.4929 \cdot 10^{-5}$ on Γ

Step 2 Prove u_2 positive on a pt. inside Γ.

$\tilde{u}_2(1/10, 0) \in [0.01342 \pm 10^{-6}]$



Conclusion: The nodal line is fully contained inside Γ.

Computational details

Code written in Julia using Arb for rigorous computations through Nemo and Arblib.jl

Lower bounding $\lambda_{5,h}$ took 12 minutes for computing $\tilde{Q}$ and 1 hour for verification with Arb

Construction of approximate eigenfunctions

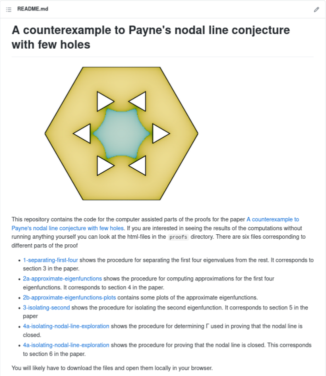
- $\tilde{u}_1$: 10 seconds
- $\tilde{u}_2$: 6 hours
- $\tilde{u}_3, \tilde{u}_3$: 5 minutes each

Computation of rigorous bounds

- $\tilde{u}_1$: 10 seconds
- $\tilde{u}_2$: 6 hours
- $\tilde{u}_3, \tilde{u}_3$: 30 minutes each

Code

Source code as well as notebooks containing all the proofs available at `https://github.com/Joel-Dahne/PaynePolygon.jl`



README.md

A counterexample to Payne's nodal line conjecture with few holes

This repository contains the code for the computer assisted parts of the proofs for the paper A counterexample to Payne's nodal line conjecture with few holes. If you are interested in seeing the results of the computations without running anything yourself you can look at the html-files in the `proofs` directory. There are six files corresponding to different parts of the proof

- 1-separating-first-four shows the procedure for separating the first four eigenvalues from the rest. It corresponds to section 3 in the paper.
- 2a-approximate-eigenfunctions shows the procedure for computing approximations for the first four eigenfunctions. It corresponds to section 4 in the paper.
- 2b-approximate-eigenfunctions-plots contains some plots of the approximate eigenfunctions.
- 3-isolating-second shows the procedure for isolating the second eigenfunction. It corresponds to section 5 in the paper
- 4a-isolating-nodal-line-exploration shows the procedure for determining Γ used in proving that the nodal line is closed.
- 4a-isolating-nodal-line-exploration shows the procedure for proving that the nodal line is closed. This corresponds to section 6 in the paper.

You will likely have to download the files and open them locally in your browser.

Bonus: Rotating solutions of the SQG equation

Joint work with Angel Castro and Diego Córdoba.

Surface Quasi-geostrophic equations

$$\theta_t + u \cdot \nabla\theta = 0, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$$

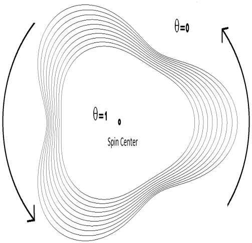
$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1/2}\psi,$$

$$u(x) = (-R_2\theta, R_1\theta) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} (-\frac{y_2}{|y|^3}, \frac{y_1}{|y|^3})\, \theta(x+y,t) dy$$

Theorem

There exists a non trivial global smooth solution for the SQG equation with finite energy.

- The level sets of this solution rotate with constant angular velocity
- The solution is actually C^2 and compactly supported
- The solution has 3-fold symmetry
- The initial data is close to a radial function. Every radial function is a stationary solution of the SQG equation



SQG as an equation for the level sets of θ

Let's assume that $z(\alpha,\rho,t) = (z_1(\alpha,\rho,t), z_2(\alpha,\rho,t))$, with $(\alpha,\rho) \in \mathbb{T}\times\mathbb{R}^+$ are the level sets of $\theta(x,t)$, in such a way that

$$\theta(z(\alpha,\rho,t),t) = f(\rho)$$

Then

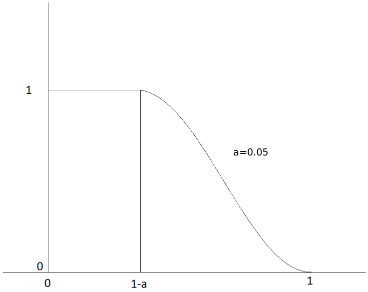
$$(-u(z(\alpha,\rho,t),t) + z_t(\alpha,\rho,t))\cdot z_\alpha^\perp(\alpha,\rho,t)\frac{f_\rho(\rho)}{z_\alpha^\perp\cdot z_\rho(\alpha,\rho,t)} = 0 \qquad (1)$$

In addition, if the level sets rotate with constant angular velocity we can write

$$z(\alpha,\rho,t) = \mathcal{O}(t)x(\alpha,\rho), \quad \mathcal{O}(t) = \begin{pmatrix} \cos(\lambda t) & -\sin(\lambda t) \\ \sin(\lambda t) & \cos(\lambda t) \end{pmatrix}$$

1. The function f is given. Indeed, it is something we choose through the initial data
2. The unknowns are the curve $x(\alpha,\rho)$ and the angular velocity λ
3. We only need to solve (1) in the support of $f_\rho(\rho)$.

The profile $f(\rho)$



- $f \in C^4$
- Monotone decreasing
- f_ρ compactly supported in $(1-a, 1)$
- $f_\rho \to -\delta_1$ when $a \to 0$

SQG as an equation for the rotating level sets of θ

We have to solve for the pair $(x(\alpha,\rho),\lambda)$

$$-\lambda x(\alpha,\rho)\cdot x_\alpha(\alpha,\rho) + \frac{1}{2\pi}x_\alpha^\perp(\alpha,\rho)\cdot\int_0^\infty\int_{-\pi}^{\pi}\frac{f_\rho(\rho')}{|x(\alpha,\rho)-x(\alpha',\rho')|}x_\alpha(\alpha',\rho')d\alpha' d\rho' = 0.$$

We still have some freedom to choose the parametrization of the level sets. We choose radial coordinates:

$$x(\alpha,\rho) = r(\alpha,\rho)(\cos(\alpha),\sin(\alpha))$$

with $r:\mathbb{T}\times\mathbb{R}^+\to\mathbb{R}^+$. And we need to solve for the pair $(r(\alpha,\rho),\lambda)$

$$F[r,\lambda]=0$$

$$\begin{aligned}
&F[r,\ \lambda]\\
&\equiv \lambda r_\alpha(\alpha,\rho) - \frac{1}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}\frac{f_\rho(\rho')}{|x(\alpha,\rho)-x(\alpha',\rho')|}\cos(\alpha-\alpha')(r_\alpha(\alpha',\rho')-r_\alpha(\alpha,\rho))d\alpha' d\rho'\\
&+\frac{r_\alpha(\alpha,\rho)}{2\pi r(\alpha,\rho)}\int_0^\infty\int_{-\pi}^{\pi}\frac{f_\rho(\rho')}{|x(\alpha,\rho)-x(\alpha',\rho')|}\cos(\alpha-\alpha')(r(\alpha',\rho')-r(\alpha,\rho))d\alpha' d\rho'\\
&-\frac{1}{2\pi r(\alpha,\rho)}\int_0^\infty\int_{-\pi}^{\pi}\frac{f_\rho(\rho')}{|x(\alpha,\rho)-x(\alpha',\rho')|}\sin(\alpha-\alpha')(r(\alpha,\rho)r(\alpha',\rho')+r_\alpha(\alpha,\rho)r_\alpha(\alpha',\rho'))d\alpha' d\rho',
\end{aligned}$$

SQG as an equation for the rotating level sets of θ

We have to solve for the pair $(x(\alpha,\rho),\lambda)$

$$
\begin{aligned}
&-\lambda x(\alpha,\rho)\cdot x_\alpha(\alpha,\rho)\\
&+\frac{1}{2\pi}x_\alpha^\perp(\alpha,\rho)\cdot\int_0^\infty\int_{-\pi}^{\pi}\frac{f_\rho(\rho')}{|x(\alpha,\rho)-x(\alpha',\rho')|}x_\alpha(\alpha',\rho')d\alpha' d\rho'=0.
\end{aligned}
$$

We still have some freedom to choose the parametrization of the level sets. We choose radial coordinates:

$$x(\alpha,\rho)=r(\alpha,\rho)(\cos(\alpha),\sin(\alpha))$$

with $r:\mathbb{T}\times\mathbb{R}^+\to\mathbb{R}^+$. And we need to solve for the pair $(r(\alpha,\rho),\lambda)$

$$F[r,\lambda]=0$$

Important fact! $r(\alpha,\rho)=\rho$ is a solution for all λ

$$F[\rho,\lambda]=0\quad\forall\lambda$$

Linearization of $F[r, \lambda]$ around ρ

To apply bifurcation theory, we have to find an element, $\tilde{r}(\alpha, \rho)$ in the kernel of the linearization of the operator $F[r, \lambda]$ around $r(\alpha, \rho) = \rho$.

$$
\begin{aligned}
&(\partial_r F)[\rho, \lambda]\tilde{r}(\alpha, \rho) \\
=&\lambda \tilde{r}_\alpha(\alpha, \rho) - \frac{1}{2\pi}\int\int \frac{f_\rho(\rho')\cos(\alpha - \alpha')(\tilde{r}_\alpha(\alpha', \rho') - \tilde{r}_\alpha(\alpha, \rho))}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\alpha - \alpha')}} d\alpha' d\rho' \\
&+ \frac{\tilde{r}_\alpha(\alpha, \rho)}{2\pi}\int\int \frac{f_\rho(\rho')\cos(\alpha - \alpha')(\rho' - \rho)}{\rho\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\alpha - \alpha')}} d\alpha' d\rho' \\
&- \frac{1}{2\pi}\int\int \frac{f_\rho(\rho')(\rho - \rho'\cos(\alpha - \alpha'))\sin(\alpha - \alpha')(\rho\tilde{r}(\alpha', \rho') - \rho'\tilde{r}(\alpha, \rho))}{(\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\alpha - \alpha'))^{\frac{3}{2}}} d\alpha' d\rho'
\end{aligned}
$$

Therefore we have to solve for the pair $(\tilde{r}(\alpha, \rho), \lambda)$

$$(\partial_r F)[\rho, \lambda]\tilde{r}(\alpha, \rho) = 0.$$

The main equation

We introduce the ansatz

$$\tilde{r}(\alpha, \rho) = \rho B^m(\rho)\cos(m\alpha)$$

and we obtain the equation

$$\boxed{\tilde{I}(\rho)B^m(\rho) + \tilde{T}^m B^m(\rho) = \lambda B^m(\rho)},$$

with

$$\tilde{I}(\rho) = -\frac{1}{2\pi\rho}\int f_\rho(\rho')\,T^1\left(\frac{\rho}{\rho'}\right)d\rho'$$

$$\tilde{T}^m B(\rho) = \frac{1}{2\pi\rho}\int f_\rho(\rho')B(\rho')\frac{\rho'}{\rho}T^m\left(\frac{\rho}{\rho'}\right)d\rho'$$

Properties of $\tilde{I}$ and $\tilde{T}$.

$$T^m(s) = \int_{-\pi}^{\pi} \frac{\cos(mx)}{\sqrt{1+s^2-2s\cos(x)}}dx$$

- $T^m(s) > 0$
- $T^m(s) > T^{m+1}(s)$
- $T^m(s)$ is smooth but at $s = 1$ where $T^m(s) \sim -\log(|s-1|)$
- $\tilde{T}^m(s)$ is not self-adjoint

Then

- $\tilde{I}(\rho) \in C^3$ satisfies $I(\rho) > 0$
- $\tilde{T}^m$ is a compact operator mapping H^k to H^{k+1}, with $k = 0, 1, 2$, and

$$\|\tilde{T}^m\|_{L^2 \to L^2} \leq C \qquad \|\tilde{T}^m\|_{H^k \to H^{k+1}} \leq C(m)$$

- $\tilde{T}^m$ is "negative" ($T^m B(\rho) < 0$ if $B(\rho) > 0$)

Existence in L^2 for $m = 3$

We find $(B^3, \lambda_3) \in L^2 \times \mathbb{R}$ such that

$$\Theta^3 B^3(\rho) \equiv \tilde{I}(\rho) B^3(\rho) + \tilde{T} B^3(\rho) = \lambda_3 B^3(\rho)$$

We can construct an approximation B_{sj} for the symmetric problem

$$\Theta^3_S B_{sj} = \lambda^* B_{sj} + e, \quad \left(\Theta^3_S = \frac{1}{2}\left(\Theta^3 + \Theta^{3*}\right)\right)$$

with $\|e\|_{L^2}$ small and with an explicit bound (computer-assisted).
We look for a solution of the type

$$B^3 = B_{sj} + v, \qquad v \in B_{sj}^{\perp}$$

The equation for v reads

$$\Theta^3 v = (\lambda - \lambda^*) B_{sj} + \lambda v - e - \Theta^3_A B_{sj}$$
$$\Theta^3_A = \frac{1}{2}\left(\Theta^3 - \Theta^{3*}\right)$$

Existence in L^2 for $m = 3$

Taking scalar product with $u \in B_{sj}^{\perp}$ yields the equation

$$\langle \Theta^3 v - \lambda v, u \rangle = -\langle e + \Theta_A^3 B_{sj}, u \rangle \quad \forall u \in B_{sj}^{\perp},$$

Let's call

$$c^* = \inf_{\substack{v \in B_{sj}^{\perp} \\ \|v\|_{L^2}=1}} \langle \Theta_3 v, v \rangle$$

and let's assume that $c^* > -\infty$.Then for $\lambda < c^*$ the operator $\Theta^3 - \lambda \mathbb{I}$ is coercive in L^2. We can apply Lax-Milgram theorem to obtain the existence of $v^\lambda \in L^2$ satisfying

$$\langle \Theta^3 v^\lambda - \lambda v^\lambda, u \rangle = -\langle e + \Theta_A^3 B_{sj}, u \rangle \quad \forall u \in B_{sj}^{\perp},$$

Then there exists a function $d(\lambda)$ such that

$$\Theta^3 v^\lambda = (\lambda - \lambda^*) B_{sj} + \lambda v^\lambda - e - \Theta_A^3 B_{sj} + d(\lambda) B_{sj}$$

We just need to show that $d(\lambda)$ has a zero for some $\lambda < c^*$.

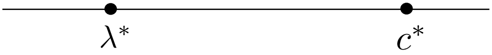
Existence in L^2 for $m = 3$

- $d(\lambda)$ is continuous
- $d(\lambda) > 0$ for $\lambda \to -\infty$
- $$d(\lambda) \leq \frac{\|\Theta_A^3 B_{sj} + e\|_{L^2}}{c^* - \lambda} \frac{\|e - \Theta_A^3 B_{sj}\|_{L^2}}{\|B_{sj}\|_{L^2}^2} + (\lambda^* - \lambda) + \frac{|\langle e, B_{sj}\rangle|}{\|B_{sj}\|_{L^2}^2}$$

We need to find $\lambda > \lambda^*$ but $\lambda < c^*$ to obtain $d(\lambda) < 0$ and therefore there exists λ_3 such that $d(\lambda_3) = 0$.
We have obtained that the pair $\left(B^3 = B_{sj} + v^{\lambda_3}, \lambda_3\right)$ satisfies

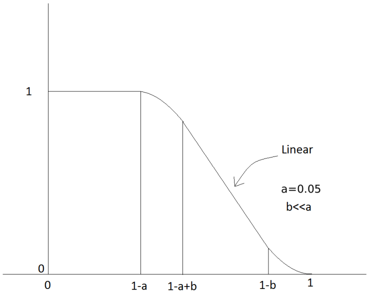
$$\Theta^3 B^3 = \lambda_3 B^3$$

Other way of looking at it



Other way of looking at it

Sharpening the profile of $f(\rho)$



By making $a, b \ll a$ small we make small the L^2-norm of $\Theta_A B_{sj}$. In our case, we take $b = \frac{1}{512}a$.

How do we use the computer to bound c^*?

We would like to find a bound from below for

$$c^* = \inf_{\substack{v \in B_{sj}^{\perp} \\ \|v\|_{L^2=1}}} \langle \Theta^3 v, v \rangle, \qquad \langle \tilde{I} v + \tilde{T} v, v \rangle \geq \min \tilde{I} + \langle \tilde{T} v, v \rangle$$

with $v \in B_{sj}^{\perp}$ and $\|v\|_{L^2} = 1$.
We recall that $\tilde{T}$ is compact and then we can approximate it by a finite dimensional operator by projecting onto a sufficiently large dimensional subspace. Let $\{v_i\}_{i=1}^{\infty}$ an orthogonal basis of $B_{sj}^{\perp}$, $v_0 = B_{sj}$ and

$$\tilde{T}_N = \sum_{i,j=0}^{N} \langle \tilde{T} v_i, v_j \rangle v_j \Pi_{v_i}$$

We bound

$$\min \tilde{I} + \langle \tilde{T} v, v \rangle \geq \min \tilde{I} + \underbrace{\langle \tilde{T}_N v, v \rangle}_{\text{finite dimensional}} - \underbrace{\left| \langle \left(\tilde{T} - \tilde{T}_N \right) v, v \rangle \right|}_{\text{generalized Young's inequality}}$$

In our case, $N = 24$. We take v_i to be Legendre polynomials adapted to our domain.

Computer-Assisted Proofs (finite dimensional part)

- We use interval arithmetics to give a rigorous enclosure of the result.

The finite dimensional part is bounded using Gershgorin's theorem (we get good bounds since our matrix is either diagonally dominant or small). We know that every eigenvalue of a matrix A is contained in the union of the disks:

$$D_i = \left\{ z \in \mathbb{C}, |z - A_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}} |A_{ij}| \right\}, \quad i = 1, \ldots, n.$$

Computer-Assisted Proofs (tails)

Using the generalized Young's inequality, we need to compute

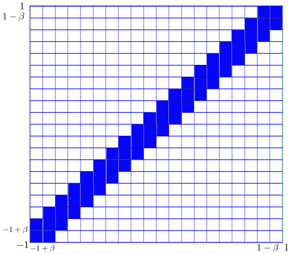
$$\left\| \int K(\rho,\rho')B(\rho')d\rho' \right\|_{L^2} \leq \max\{\|K(\rho,\rho')\|_{L^\infty_{\rho'}L^1_\rho}, \|K(\rho,\rho')\|_{L^\infty_\rho L^1_{\rho'}}\}\|B\|_{L^2}$$

Obstructions:

1. K is explicit but not easy to compute (it involves elliptic integrals).
2. K is singular along the diagonal $\rho = \rho'$.

Computer-Assisted Proofs (tails)

1. We perform (by hand) a series expansion around $\rho = \rho'$ up to errors of size $(\rho - \rho')^2 \log(|\rho - \rho'|)$ or smaller, with explicit bounds on the errors. Then, substitute K by its expansion and carry over the error bounds.
2. We split the integration region into two parts: singularity and rest.



The singularity is bounded integrating explicitly the logarithmic terms. The rest is calculated using an adaptive Gauss-Legendre quadrature of order 2.

H^3-Regularity

Once we have a solution $B^3 \in L^2$

$$\tilde{I}(\rho)B^3(\rho) - \lambda_3 B^3(\rho) = -\tilde{T}^3 B^3(\rho)$$

we can bootstrap because

1. $\tilde{I} \in C^3$
2. $\min \tilde{I} - \lambda_3 > 0$ (also computer-assisted)
3. $\tilde{T}$ maps H^k into H^{k+1} for $k = 0, 1, 2$.

Thank you!