Ergodicity of Markov processes: theory and computation (3)

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Last time

1. Coupling method and renewal theory
2. Exponential and power-law ergodicity

Recall from last time

1. Find a small set C.
2. Split the small set to get an atom α.
3. Independent coupling. Couple at time T (first simultaneous visit to the atom).
4. Random sum of random numbers episode 1: exponential/power-law tail of η_C gives exponential/power-law tail of η_α
5. Random sum of random numbers episode 2: exponential/power-law tail of η_α gives exponential/power-law tail of T

First passage time to the small set

General approach

1. Construct a Lyapunov function
2. Show that the “bottom” of the function landscape is a small set
3. Show that η_C has exponential/power-law tail

Adapted sequence

1. $\mathcal{F}_k = \sigma(\Phi_0, \Phi_1, \cdots, \Phi_k)$
2. Z_k is an adapted sequence such that Z_k is measurable on $\mathcal{F}_k$. $Z_k \geq 0$
3. $\tau^n = \min\{n, \tau, \inf\{k \geq 0 \,|\, Z_k \geq n\}$ for a stopping time τ.

Dynkin's formula

Theorem (Dynkin's formula)

$$\mathbb{E}_x[Z_{\tau^n}] = \mathbb{E}_x[Z_0] + \mathbb{E}_x\left[\sum_{i=1}^{\tau_n}(\mathbb{E}[Z_i \,|\, \mathcal{F}_{i-1}] - Z_{i-1})\right]$$

Recall: conditional expectation

1. Y: random variable, $\mathcal{F}$: sub sigma field
2. Conditional expectation $\mathbb{E}[Y|\mathcal{F}]$ is a $\mathcal{F}$- measurable random variable.
3. If Y is $\mathcal{F}$ measurable, then $\mathbb{E}[Y|\mathcal{F}] = Y$
4. $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]] = \mathbb{E}[Y]$

See proof on the whiteboard.

Dynkin's formula (2)

Proposition

Let f_k and s_k be two sequences of nonnegative functions.

If

$$\mathbb{E}[Z_{k+1} \mid \mathcal{F}_k] \leq Z_k - f_k(\Phi_k) + s_k(\Phi_k)$$

Then for any stopping time τ, we have

$$\mathbb{E}_x[\sum_{k=0}^{\tau-1} f_k(\Phi_k)] \leq Z_0(x) + \mathbb{E}_x[\sum_{k=0}^{\tau-1} s_k(\Phi_k)]$$

Proof on the whiteboard

Lyapunov function

Theorem

If there exists a function $V \geq 1$ such that

$$PV(x) - V(x) \leq -\beta V(x) + b\mathbf{1}_C(x)$$

for some $\beta > 0$, $b < \infty$, then for any $r \in (1, (1-\beta)^{-1})$, there exists $\epsilon > 0$ such that

$$V(x) \leq \mathbb{E}_x[\sum_{k=0}^{\eta_C - 1} V(\Phi_k) r^k] \leq \epsilon^{-1} r^{-1} V(x) + \epsilon^{-1} b\mathbf{1}_C(x)\,.$$

Proof on the whiteboard

Exponential tail of first passage time

1. $$\mathbb{E}_x[\sum_{k=0}^{\eta_C-1} r^k V(\Phi_k)] \geq \mathbb{E}_x[\sum_{k=0}^{\eta_C-1} r^k] = \frac{1}{r-1}\mathbb{E}_x[r^{\eta_C}-1] \geq c\mathbb{E}_x[r^{\eta_C}]$$

   for some constant c.
2. Hence $\mathbb{E}_x[r^{\eta_C}] < \infty$
3. $$\mathbb{P}[r^{\eta_C} \geq r^n] \leq \mathbb{E}_x[r^{\eta_C}]r^{-n}$$
4. $\mathbb{P}[\eta_C \geq n] \leq Cr^{-n}$ for some constant C.

Power-law tail of first passage time

Theorem

If there exists $m \geq 0$ such that for each $i = 1, \cdots, m$ and functions $V_0, V_1, \cdots, V_m$ such that

$$PV_{i-1} \leq V_{i-1} - c_i V_i + b_i \mathbf{1}_C \quad i = 1, \cdots, m$$

Then

$$\mathbb{E}_x \left[\sum_{k=0}^{\eta_C - 1} (k+1)^{i+1} V_i(\Phi_k) \right] \leq C_{i+1}(V_0(x) + 1)$$

for some $C_{i+1} \leq \infty$.

Reference: Jarner-Roberts 2002 AAP

Reduction to one Lyapunov function (1)

Lemma

If $V \geq 1$, $b, c \geq 0$, $\alpha < 1$ and

$$PV \leq V - cV^\alpha + b\mathbf{1}_C\,,$$

then for any $\eta > 0$ there exists some $b_1, c_1 < \infty$ such that

$$PV^\eta \leq V^\eta - c_1 V^{\alpha+\eta-1} + b_1\mathbf{1}_C\,.$$

Reduction to one Lyapunov function (2)

Theorem

If $V \geq 1$, $b, c \geq 0$, $\alpha < 1$ and

$$PV \leq V - cV^{\alpha} + b\mathbf{1}_C\,,$$

then for each $1 \leq \beta \leq (1-\alpha)^{-1}$, let $V_\beta(x) = V^{1-\beta(1-\alpha)}$, we have

$$\mathbb{E}_x\left[\sum_{k=0}^{\eta_C - 1}(n+1)^{\beta-1}V_\beta(\Phi_k)\right] \leq C_\beta(V(x)+1)$$

for some $C_\beta < \infty$

Reduction to one Lyapunov function (3)

Integer β only. Let $\gamma = 1 - \alpha$ and $m = \lceil \gamma^{-1} \rceil$. Let $V_0 = V$, $V_i = V^{1-i\gamma}$ for $i = 1, \cdots, m-1$.

Need to show that

$$PV_{i-1} \leq V_{i-1} - c_i V_i + b_i \mathbf{1}_C \text{ for each } i = 1, \cdots, m.$$

1. Case $i = 1$: $PV \leq V - cV^{1-\gamma} + b\mathbf{1}_C$, or $PV_0 \leq V_0 - cV_1 + b\mathbf{1}_C$.
2. Case $i > 1$: Let $\eta = 1 - (i-1)\gamma$, $V_{i-1} = V^{\eta}$. By the lemma, we have $PV^{\eta} \leq V^{\eta} - c_1 V^{\alpha+\eta-1} + b_1 \mathbf{1}_C$. Since $\alpha + \eta = \eta - \gamma = 1 - i\gamma$, we have $V^{\alpha+\eta-1} = V_i$, or $PV_{i-1} \leq V_{i-1} - c_i V_i + b_i \mathbf{1}_C$.
3. β is an integer that is less than m. Apply the theorem.

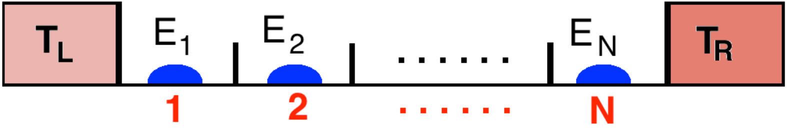
Lyapunov function method

Try to find a Lyapunov function $V(x)$

1. If $PV(x) - V(x) \leq -\beta V(x) + b\mathbf{1}_C(x)$, first passage time to C has exponential tail.
2. If $PV \leq V - cV^\alpha + b\mathbf{1}_C$ for some $\alpha < 1$, first passage time to C has power-law tail.

Finding a suitable Lyapunov function is the main difficulty.

Stochastic energy exchange model

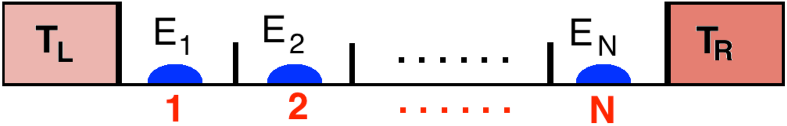


- A chain of N cells is connected to two heat baths.
- Cell i carries energy E_i.
- Exponential clock with rate $R(E_i, E_{i+1}) = \sqrt{\min\{E_i, E_{i+1}\}}$ is associated with each adjacent pair.
- When clock rings,

$$(E_i', E_{i+1}') = (E_i + E_{i+1})p, (E_i + E_{i+1})(1 - p)) .$$

p: uniform distribution on $(0, 1)$.

Stochastic energy exchange model (cont.)



- Bath temperatures T_L and T_R.
- Clocks between ends of chain and baths: $R(T_L, E_1)$ and $R(E_N, T_R)$
- Similar rule for an energy exchange involving heat baths.
- Heat bath energy $\sim \mathcal{E}(T_L)$ and $\sim \mathcal{E}(T_R)$ (exponential distribution).

Result

Theorem 1, Contraction of Markov operator, (Y. Li 2018 AAP)

For any $\gamma > 0$, there exists $\eta > 0$ such that for any μ, $\nu \in \mathcal{M}_\eta$,

$$\lim_{t\to\infty} t^{2-\gamma}\|\mu P^t - \nu P^t\|_{TV} = 0\,.$$

$\mathcal{M}_\eta$ is the measure class on which function

$$\sum_{m=1}^{N}\sum_{i=1}^{N-m+1}\Big(\sum_{j=0}^{m-1} E_{i+j}\Big)^{a_m\eta-1} + \sum_{i=1}^{N} E_i$$

is integrable, where $a_m = 1 - (2^{m-1}-1)/(2^N-1)$.

Result

Theorem 2, Properties of NESS (Y. Li, 2018 AAP)

There exists a unique invariant measure π that is absolutely continuous with respect to the Lebesgue measure. In addition, for any $\gamma > 0$, there exists $\eta > 0$ such that for any $\mu \in \mathcal{M}_\eta$,

$$\lim_{t \to \infty} t^{1-\gamma} \| \mu P^t - \pi \|_{TV} = 0$$

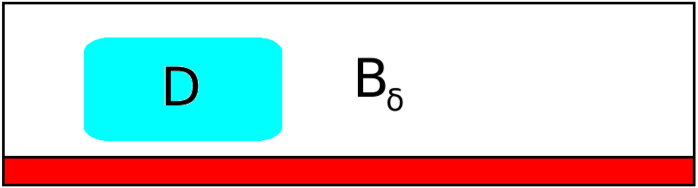
Theorem 3, Decay of Correlation (Y. Li, 2018 AAP)

For any $\gamma > 0$ there exists a $\eta > 0$ such that for any $\mu \in \mathcal{M}_\eta$, let functions ξ and $\varphi \in L^\infty(\mathbb{R}^N_+)$. Then

$$\left| \int_{\mathbb{R}^N_+} (P^t \zeta)(\mathbf{E}) \xi(\mathbf{E}) \mu(\mathrm{d}\mathbf{E}) - \int_{\mathbb{R}^N_+} (P^t \zeta)(\mathbf{E}) \mu(\mathrm{d}\mathbf{E}) \int_{\mathbb{R}^N_+} \xi(\mathbf{E}) \mu(\mathrm{d}\mathbf{E}) \right| = O(1) \cdot \left(\frac{1}{t^{2-\gamma}} \right)$$

as $t \to \infty$.

Strong Markov property



1. $B_\delta \subset \mathbb{R}_+^N$ is an "active set":

$$\inf\{E_i \,|\, \mathbf{E} = (E_1, \cdots, E_N) \in B_\delta\} \geq \delta\,.$$

2. $D \subset B_\delta$: uniform reference set.
3. $\mathbf{E}_n$: time-h sample chain

$$T_{n+1} = \inf_{k > T_n} \{\mathbf{E}_k \in B_\delta\}$$

$\hat{\mathbf{E}}_n = \mathbf{E}_{T_n}$: B_δ-induced chain.

Strong Markov property (2)

Induced Chain Lemma (Y. Li 2018 AAP)

Assume

- $$\mathbb{P}[T_{n+1} - T_n > n \,|\, E_{T_n}] \leq \xi(E_{T_n}) n^{-\alpha},$$
  where $\xi(\mathbf{E})$ is uniformly bounded in B_δ.
- $$\mathbb{P}_{\mathbf{E}_0}[\hat{\tau}_D > n] \leq \eta(\mathbf{E}_0) e^{-c\eta},$$

then for any small $\epsilon > 0$, there exists a constant c such that

$$\mathbb{P}_{\mathbf{E}_0}[\tau_D > n] \leq c(\eta(\mathbf{E}_0) + \xi(\mathbf{E}_0)) n^{-(\alpha - \epsilon)}$$

Tower construction of Lyapunov functions

- The most difficult part is to estimate

$$\mathbb{P}[T_{n+1} - T_n > n \mid E_{T_n}] .$$

- Need a Lyapunov function V such that

$$P^h V(\mathbf{E}) - V(\mathbf{E}) \leq -c_0 V^\alpha(\mathbf{E})$$

  for some $h > 0$.
- V take high value at boundary (small energy).

Tower construction of Lyapunov functions (2)

Let $a_i = 1 - \frac{2^{i-1}-1}{2^N-1}$ be a decreasing sequence.

- Natural Lyapunov function with respect to site i: $V_{1,i}(\mathbf{E}) = E_i^{a_1\eta-1}$, $\eta > 0$ is arbitrarily small.
- $P^h V_{1,i}$ decreases if $V_{1,i}$ is much bigger than its "neighbors".
- Question: how to build a global Lyapunov function from $V_{1,i}$?
- **Tower construction**:

$$V_k(\mathbf{E}) = \sum_{i=1}^{N-k+1} V_{k,i} = \sum_{i=1}^{N-k+1} \left(\sum_{j=0}^{k-1} E_{i+j} \right)^{a_k\eta-1}$$

  for $1 \leq k \leq N-1$.

- Global Lyapunov function

$$V(\mathbf{E}) = \sum_{i=1}^{N-1} V_i(\mathbf{E}) .$$

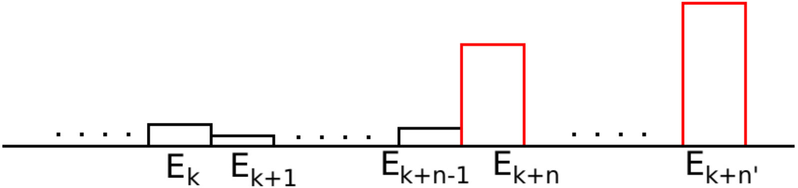
Tower construction of Lyapunov functions (3)

Main idea of the proof

Recall that

$$V_{n,k} = (E_k + \cdots + E_{k+n-1})^{a_n \eta - 1} .$$

- General rule: higher value on lower probability states
- Penalty for states that have consecutive low energy sites.
- If a $V_{n,k}$ is sufficiently large, then $E_k, \cdots, E_{k+n-1}$ are all small.
- If E_{k+n} is much larger, the expectation of $V_{n,k}$ decreases at the next energy exchange.



Tower construction of Lyapunov functions (4)

Main idea of the proof (cont.)

- Otherwise

$$(E_k + \cdots + E_{k+n-1} + E_{k+n})^{a_{n+1}\eta - 1} \gg (E_k + \cdots + E_{k+n-1})^{a_n\eta - 1}$$

- Easy to see the expected change of $V_{n,k}$ is dominated by that of $V_{n+1,k}$
- Boundary has temperature T_L, T_R. We can always find an $n' > n$ such that $E_{k+n'}$ is "much larger" that $E_k, \cdots, E_{k+n'-1}$.
- Same strategy on the left end.
- The expected increase of every $V_{n,k}$ can be bounded.
- When V is extremely large, the expected decrease dominates the expected increase.

Tower construction of Lyapunov functions (5)

- Idea of the tower construction: Dichotomy.
- For each $\mathbf{E}$, either $P^h V_{k,i}(\mathbf{E})$ decreases, or $V_{k,i}(\mathbf{E})$ is dominated by the "next level" $V_{k+1,i}$ (or $V_{k+1,i-1}$).
- $P^n V_{k,i}$ decreases if $V_{k,i}$ "touches" the boundary.

Theorem A (Y. Li 2018 AAP)

For any $\eta > 0$ and $h > 0$ small enough, there exist $c_0 > 0$, $M_0 > 1$ depending on η, N, and h, such that

$$(P^h)V(\mathbf{E}) - V(\mathbf{E}) \leq -c_0 V^{\alpha}(\mathbf{E})$$

for every $\mathbf{E} \in \{V > M_0\}$, where $\alpha = 1 - \frac{1}{2(1-\eta)}$.

$$B_\delta = \{V \leq M_0\}.$$

Thank you