Three notions of tropical rank for symmetric matrices

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The tropical semiring consists of the real numbers equipped with two operations

$$a \oplus b = \min(a, b) \qquad \text{and} \qquad a \odot b = a + b.$$

Example:

$$3 \oplus 4 = 3 \qquad \text{and} \qquad 3 \odot 4 = 7.$$

"Motivation"

$$\begin{aligned} (x^3 + \text{higher terms}) + (x^4 + \text{higher terms}) &= (x^3 + \text{higher terms}) \\ (x^3 + \text{higher terms}) \cdot (x^4 + \text{higher terms}) &= (x^7 + \text{higher terms}) \end{aligned}$$

We can do tropical linear algebra, for example

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \odot \begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 4 \end{bmatrix} \odot \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$$

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$$\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} = \begin{array}{c c} & \begin{array}{cc} 1 & 4 \end{array} \\ \begin{array}{c} 1 \\ 4 \end{array} & \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix} \end{array} \oplus \begin{array}{c c} & \begin{array}{cc} 4 & 1 \end{array} \\ \begin{array}{c} 4 \\ 1 \end{array} & \begin{pmatrix} 8 & 5 \\ 5 & 2 \end{pmatrix} \end{array}.$$

A symmetric matrix has symmetric rank k if it is the tropical sum of k symmetric rank 1 matrices, but no fewer.

Can we always find such a sum? How many rank 1 matrices are required?

linear space of symmetric matrices

Classically,

$$Secant^k(Segre) \cap L_{sym} = Secant^k(Segre \cap L_{sym}).$$

That is, a symmetric matrix of rank k can be written as a sum of k SYMMETRIC matrices of rank 1.

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For higher dimensional arrays, this is only conjecturally true:

Comon's Conjecture (2009): the rank of an order k, dimension n symmetric tensor over $\mathbb{C}$ equals its symmetric rank.

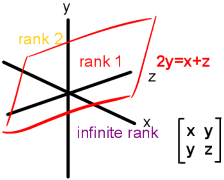
some cases proven by Comon-Golub-Lim-Mourrain (2008):

Symmetric tensor decomposition is important in signal processing, independent component analysis, ...

“Tropical Comon’s Conjecture:” rank equals symmetric rank, tropically?

In fact, symmetric rank may not even be finite

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} ? & -1 \\ -1 & ? \end{pmatrix} \oplus \cdots \quad \text{(infinite symmetric rank)}$$



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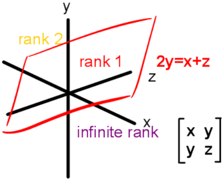
$$= \begin{pmatrix} 0 & -1 \\ 100 & 99 \end{pmatrix} \oplus \begin{pmatrix} 99 & 100 \\ -1 & 0 \end{pmatrix} \quad \text{(but finite rank)}$$



"Barvinok rank" (Develin, Santos, Sturmfels 2005)

What about when symmetric rank is finite? How large can it be? Surely it is bounded above by the dimension of the matrix?

n	1	2	3	4
maximum (finite) symmetric rank	1	2	3	4

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
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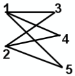

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CLIQUE COVER problem: express a given graph as a union of cliques. In each rank 1 summand, the off-diagonal zeroes form a clique in the *zero graph*, and these must cover the zero graph of the original matrix.

A graph on n nodes can require up to $\lfloor \frac{n^2}{4} \rfloor$ cliques to cover it; this bound is attained by $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ 

Theorem (Cartwright-C 2009) For $n \geq 4$, $\lfloor n^2/4 \rfloor$ is the maximum finite symmetric rank of an $n \times n$ matrix.

Similarly, the tropical Comon conjecture is false for higher dimensional symmetric tensors (graphs $\rightarrow$ hypergraphs).

What about the set of matrices of symmetric rank $\leq k$? It is a polyhedral fan (Develin 2006). What is its dimension?

Why is this even a good question?

Definition

The k^{th} tropical secant set of a subset $V \subseteq \mathbb{R}^n$ is the set

$$Sec^k(V) := \{v_1 \oplus \cdots \oplus v_k \ : \ v_i \in V\} \subseteq \mathbb{R}^n.$$

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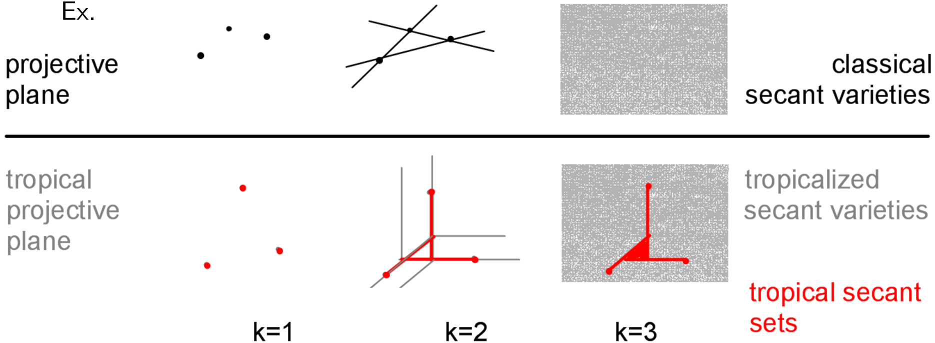
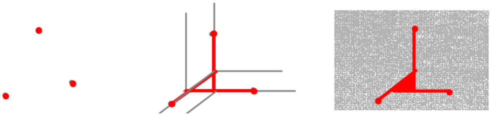
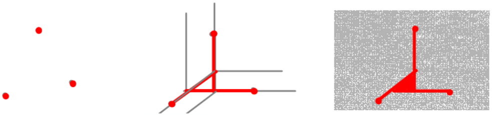
Ex.





For nice varieties, $Sec^k(Trop\ V) \subsetneq Trop(Sec^k\ V)$; the sets are generally far from equal. But are their dimensions equal?



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Examples:

	(symmetric matrices)	(diagonal-deleted symmetric Drton, Sturmfels, Sullivant 2007)	(tropicalization studied by Sturmfels & Speyer 2004)
irreducible variety	Veronese	factor analysis model	Grassmannian $(2, n)$
dim of k^{th} secant variety	$\binom{n+1}{2} - \binom{n-k+1}{2}$	$\min\{\binom{n}{2} - \binom{n-k}{2} + k, \binom{n}{2}\}.$	$\min\{k(2n-2k-1), \binom{n}{2}\}.$
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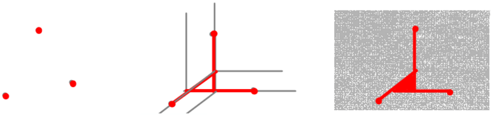

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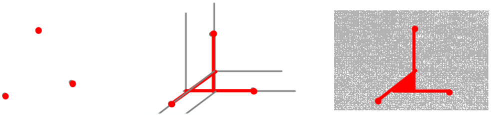

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Nonexamples: none known! (Draisma 2007 question/conjecture)

Moral: for irreducible varieties, tropical secant sets give lower bounds, and maybe even equalities, for the dimensions of classical secant varieties.

The tropical Grassmannian $Gr(2, n)$ is the set of $n \times n$ dissimilarity matrices satisfying the 3-term tropical Plücker relations: for all $i < j < k < l$,

$$\min\{p_{ij} + p_{kl}, p_{ik} + p_{jl}, p_{il} + p_{jk}\}$$

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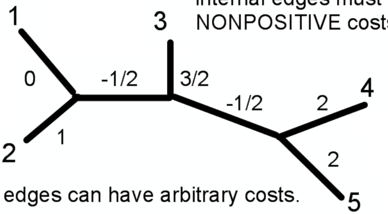
diagonal-deleted symmetric matrix

$$M = \begin{pmatrix} * & 1 & 1 & 1 & 1 \\ 1 & * & 2 & 2 & 2 \\ 1 & 2 & * & 3 & 3 \\ 1 & 2 & 3 & * & 4 \\ 1 & 2 & 3 & 4 & * \end{pmatrix}$$

internal edges must have NONPOSITIVE costs...



...leaf edges can have arbitrary costs.

Equivalently, it comes from pairwise cost-of-travel along a weighted tree on n nodes.

Tree mixtures studied in phylogenetics by Matsen-Mossel-Steel, Cueto

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$$\begin{aligned}
&\textcolor{blue}{x_{12}x_{13}x_{24}x_{35}x_{45} \oplus x_{12}x_{13}x_{25}x_{34}x_{45} \oplus x_{12}x_{14}x_{23}x_{35}x_{45} \oplus x_{12}x_{14}x_{25}x_{34}x_{35} \oplus x_{12}x_{15}x_{23}x_{34}x_{45} \oplus x_{12}x_{15}x_{24}x_{34}x_{35}} \\
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&\oplus x_{12}x_{13}x_{23}x_{45}^2 \oplus x_{12}x_{14}x_{24}x_{35}^2 \oplus x_{12}x_{15}x_{25}x_{34}^2 \oplus x_{13}x_{14}x_{34}x_{25}^2 \oplus x_{13}x_{15}x_{35}x_{24}^2 \oplus x_{14}x_{15}x_{45}x_{23}^2 \oplus x_{23}x_{24}x_{34}x_{15}^2 \\
&\oplus x_{23}x_{25}x_{35}x_{14}^2 \oplus x_{24}x_{25}x_{45}x_{13}^2 \oplus x_{34}x_{35}x_{45}x_{12}^2
\end{aligned}$$

$$\begin{pmatrix} * & 0 & 1 & 1 & 0 \\ 0 & * & 0 & 1 & 1 \\ 1 & 0 & * & 0 & 1 \\ 1 & 1 & 0 & * & 0 \\ 0 & 1 & 1 & 0 & * \end{pmatrix} = \begin{pmatrix} * & \textcolor{green}{0} & \textcolor{red}{\geq 1} & \textcolor{blue}{\geq 1} & \\ & * & \textcolor{blue}{\geq 0} & \textcolor{red}{\geq 1} & \\ & & * & \textcolor{green}{0} & \\ & & & * & \\ & & & & * \end{pmatrix} \oplus \cdots \oplus \cdots$$

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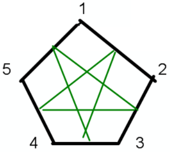
\end{aligned}
$$

$$
\begin{pmatrix}
* & 0 & 1 & 1 & 0 \\
0 & * & 0 & 1 & 1 \\
1 & 0 & * & 0 & 1 \\
1 & 1 & 0 & * & 0 \\
0 & 1 & 1 & 0 & *
\end{pmatrix}
=
\begin{pmatrix}
* & 0 & \geq 1 & \geq 1 & \\
 & * & \geq 0 & \geq 1 & \\
 & & * & 0 & \\
 & & & * & \\
 & & & & *
\end{pmatrix}
\oplus \cdots \oplus \cdots
$$



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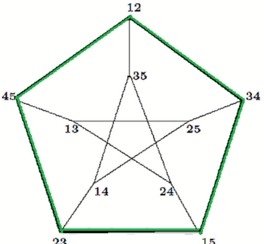
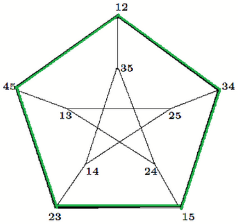


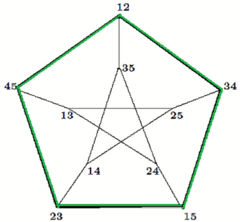


Theorem The chromatic number of the "conflict" hypergraph is a lower bound for rank.

Does this bound tell the truth?

- yes for tree rank on $n \leq 5$ taxa,
- no in general, but

Theorem In the case of a toric ideal and a universal Gröbner basis, the bound above is an equality.



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Thank you!

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