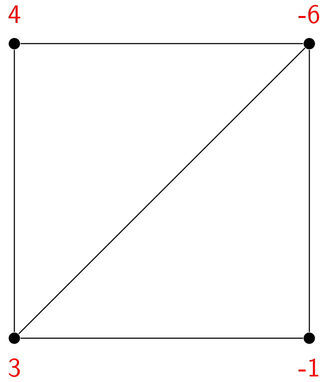
Chip-Firing on Representable Matroids

GOCC

Alex McDonough

5/6/20

Let's Fire Some Chips!



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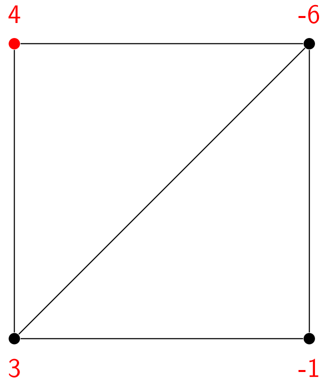

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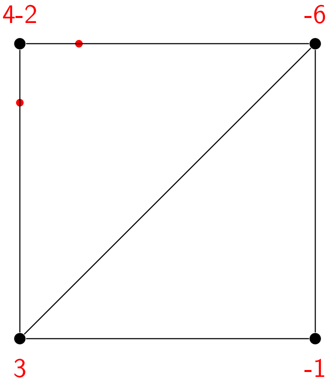

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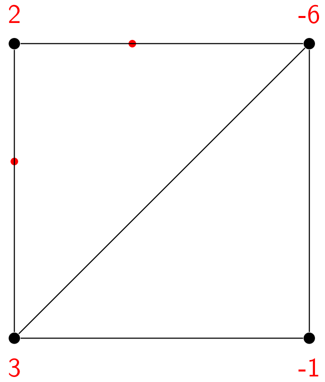

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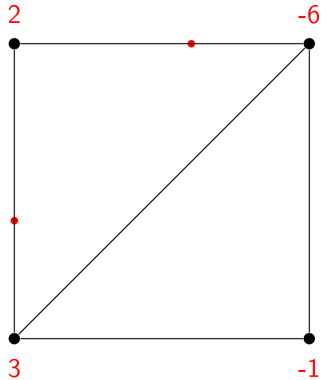

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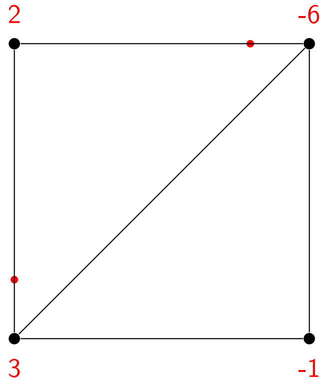

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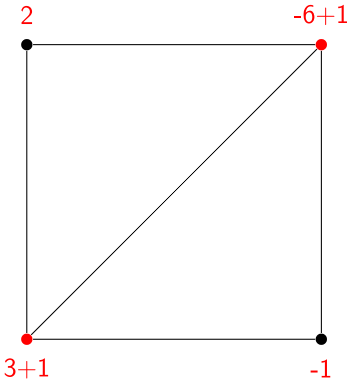

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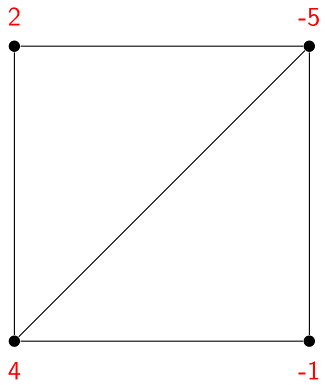

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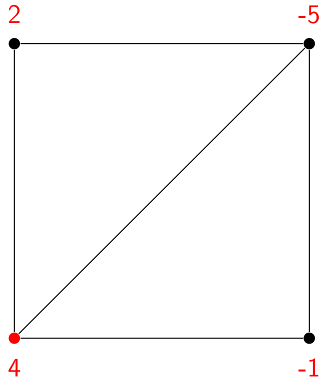

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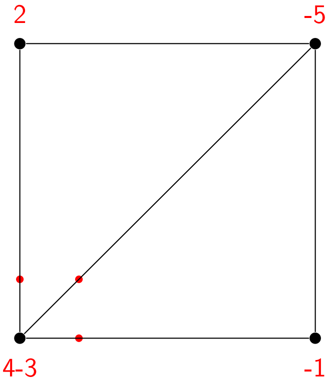

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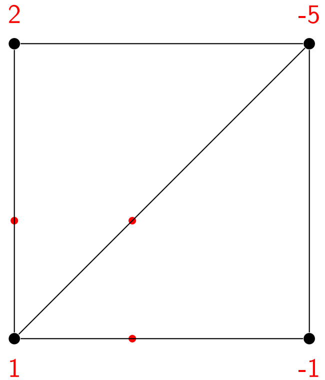

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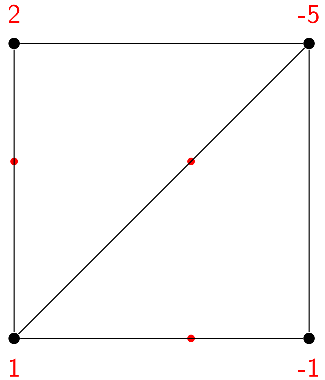

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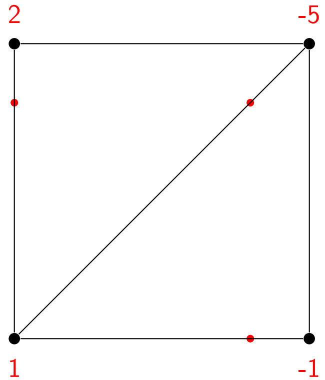

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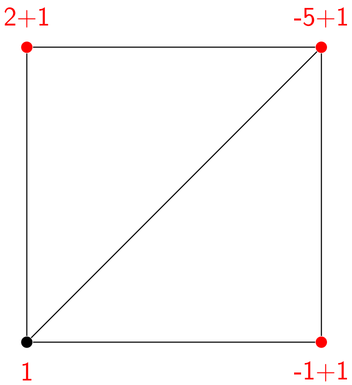

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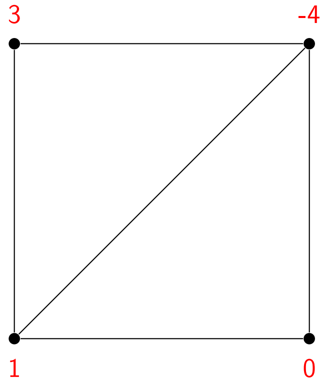

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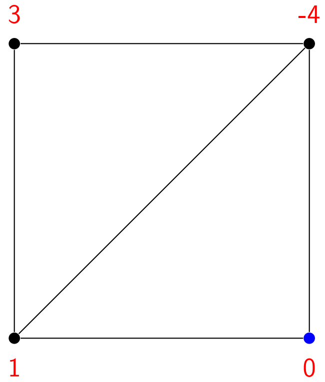

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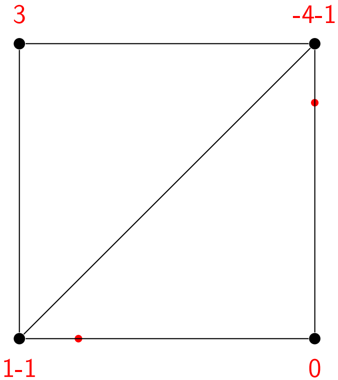

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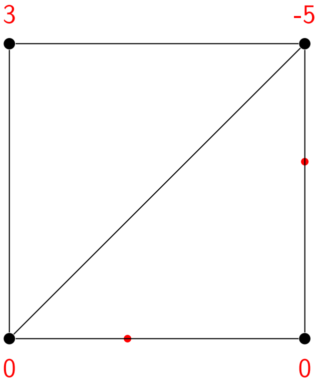

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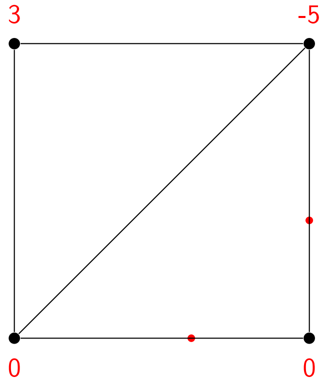

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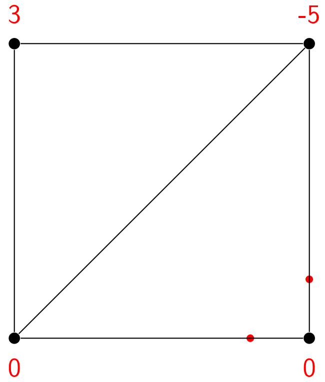

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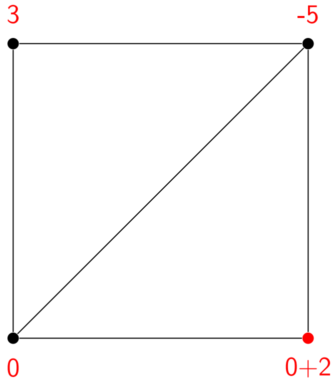

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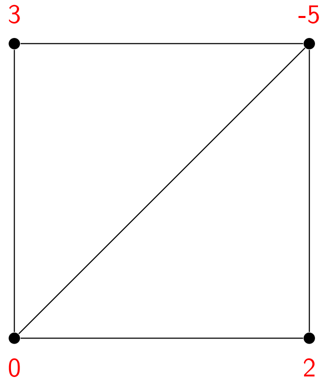

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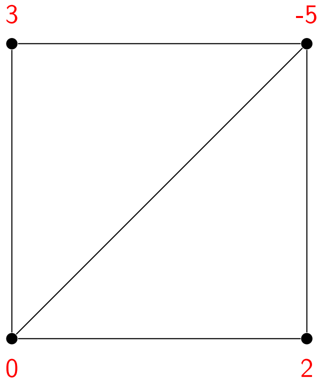

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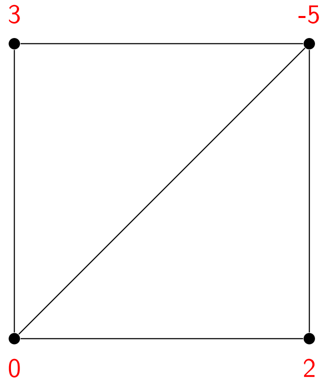

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No. For this reason, this is often called the "Abelian Sandpile Model".

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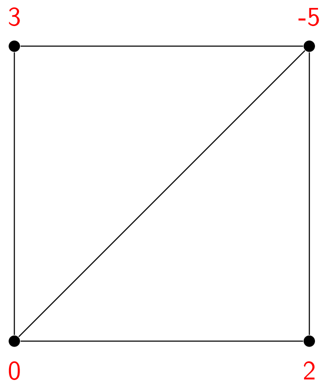

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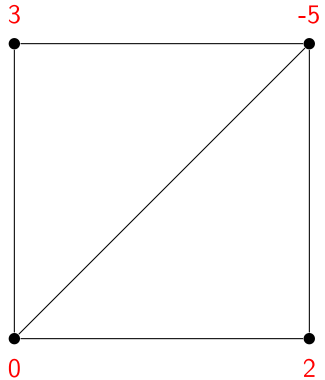

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It's not hard to show that firing equivalence is an equivalence relation.

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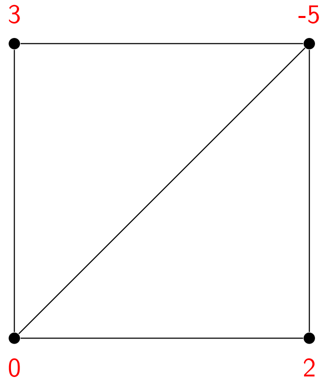

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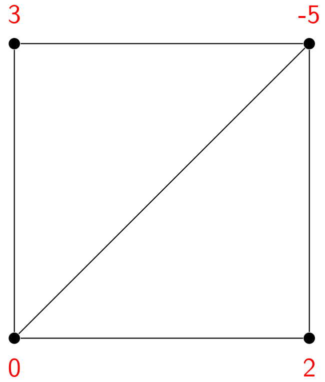

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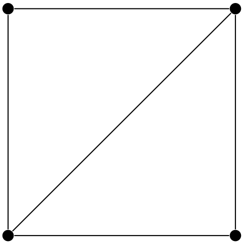
Theorem (Sandpile Matrix-Tree Theorem)

The size of $\mathcal{S}(G)$ is the number of spanning trees of G.

Edge-labeled Chip-Firing

- Instead of working with chips let's imagine some Frito-Lay transportation trucks driving from vertex to vertex.



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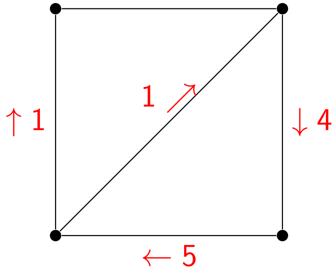

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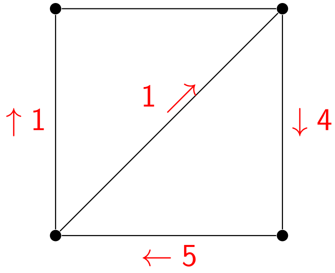

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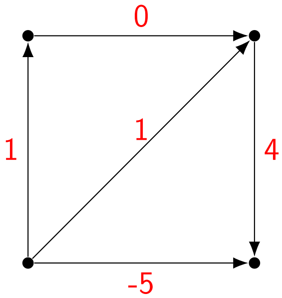

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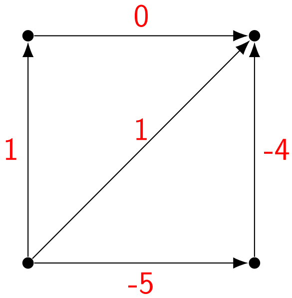

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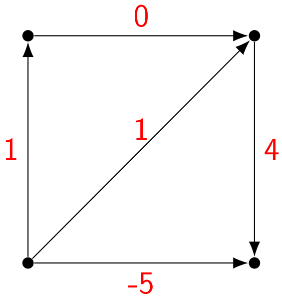

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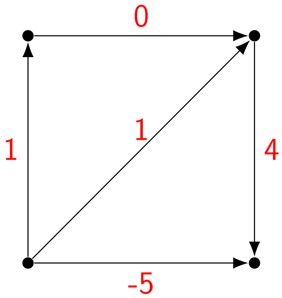

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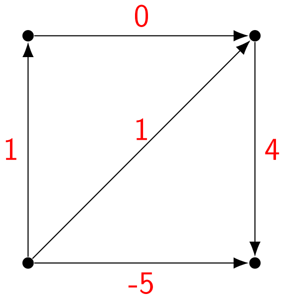

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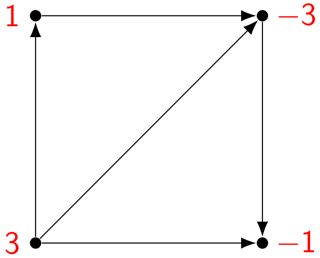

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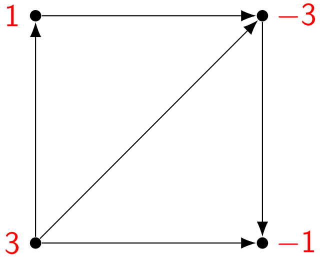

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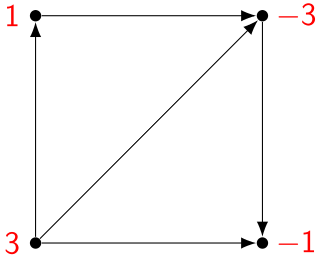

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Definition (Reduced Homology)

The group $\ker \epsilon / \operatorname{im} \partial$ is called the reduced 0^{th} homology group and is denoted $\tilde{H}_0(G)$.

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- Lets play around with $\mathcal{S}^e(G)$.

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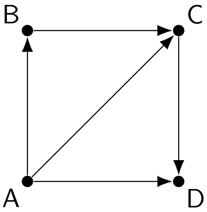

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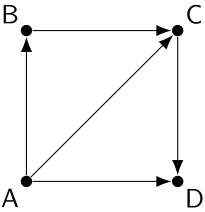

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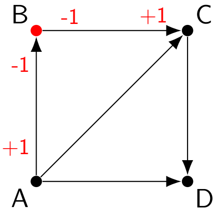

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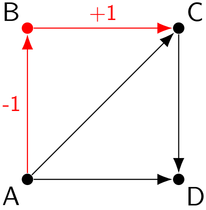

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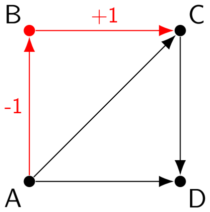

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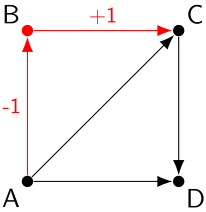

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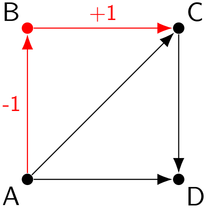

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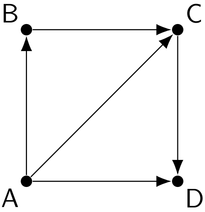

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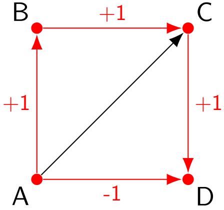

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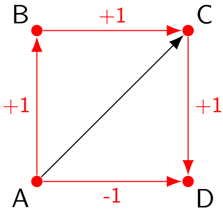

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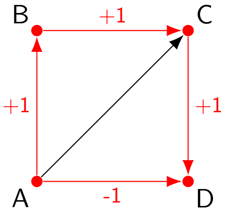

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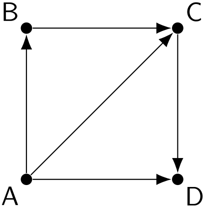

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- The matroid arising from this matrix has $E = \{A, B, C, D, E, F\}$ and $\mathcal{B} = \{\{A, B, D\}, \{A, C, D\}, \{A, D, F\}, \{B, C, D\}, \{B, D, F\}\}$.

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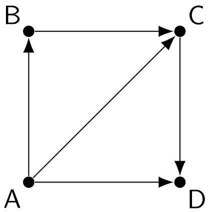

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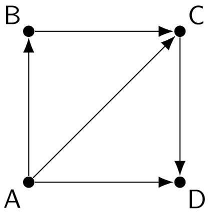

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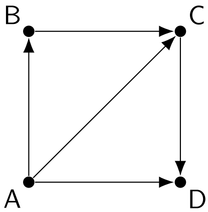

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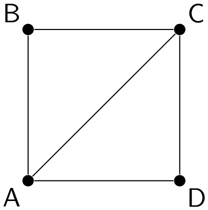
- The ground set corresponds to the edges of G and bases correspond to spanning trees of G.

Another Graph Representation

- We give a useful alternative method for producing a matrix D that represents the same matroid as ∂.



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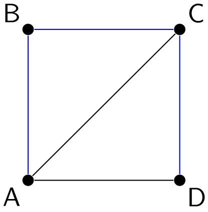

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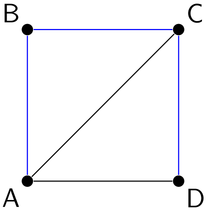

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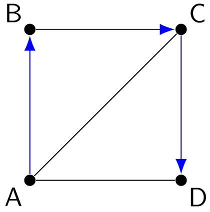

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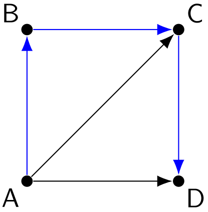

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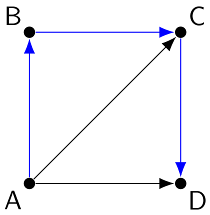
Back to $\mathcal{S}^e(G)$



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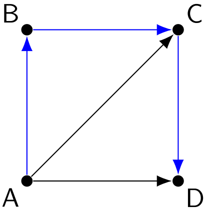

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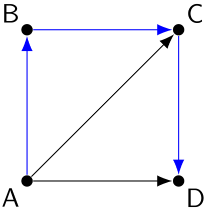

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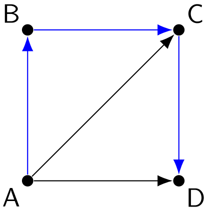

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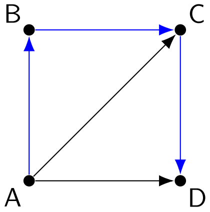

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Proposition

$\mathrm{im}_{\mathbb{Z}}(\hat{D}^T) = \ker_{\mathbb{Z}}(D)$.

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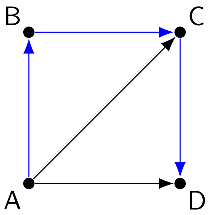

$$\mathcal{D} = \begin{array}{c} \begin{array}{ccccc} AB & BC & CD & AC & AD \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

- We saw before that c_1 and c_2 are edge firing equivalent if and only if they are equivalent modulo $\text{im}_{\mathbb{Z}}(\partial^T) \oplus \text{ker}_{\mathbb{Z}}(\partial)$.
- In fact, c_1 and c_2 are also edge firing equivalent if and only if they are equivalent modulo $\text{im}_{\mathbb{Z}}(D^T) \oplus \text{ker}_{\mathbb{Z}}(D)$.
- Let's add a few rows to D to get $\mathcal{D}$. Call the last 2 rows $\hat{D}$.

Proposition

$\text{im}_{\mathbb{Z}}(\hat{D}^T) = \text{ker}_{\mathbb{Z}}(D)$. *This implies that* $\text{im}_{\mathbb{Z}}(D^T) \oplus \text{ker}_{\mathbb{Z}}(D) = \text{im}_{\mathbb{Z}}(\mathcal{D}^T)$

The Matrix Sandpile Group

- Let D be an $n \times (n+m)$ matrix of the form: $\left(\begin{array}{c|c} I_n & M \end{array} \right)$ where M is any $n \times m$ integer matrix.

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The *sandpile lattice* of D, denoted $\mathcal{S}(D)$, is $\mathrm{im}_{\mathbb{Z}}(D^T) \oplus \ker_{\mathbb{Z}}(D)$. The *sandpile group* of D is $\mathbb{Z}^{n+m} / \left(\mathrm{im}_{\mathbb{Z}}(D^T) \oplus \ker_{\mathbb{Z}}(D)\right)$.

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$\mathrm{im}_{\mathbb{Z}}(D^T) \oplus \ker_{\mathbb{Z}}(D) = \mathrm{im}_{\mathbb{Z}}(\mathcal{D}^T)$. *Furthermore,* $|\mathcal{S}(D)| = |\mathcal{D}|$.

Finding $|\mathcal{D}|$

Lemma

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Proof.

$$\mathcal{D} = \left(\begin{array}{c|c} I_n & M \\ \hline -M^T & I_m \end{array}\right) \to \left(\begin{array}{c|c} M \cdot M^T + I_n & 0 \\ \hline -M^T & I_m \end{array}\right) = \left(\begin{array}{c|c} D \cdot D^T & 0 \\ \hline -M^T & I_m \end{array}\right) \to ☺$$

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Theorem (Consequence of Cauchy-Biney Formula)

$$|D \cdot D^T| = \sum_{s \subseteq [n+m], |s| = n} |D_s|^2$$

Cellular Matrix-Tree Theorem

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Proposition (Tutte, 1958)

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- This proves the Sandpile Matrix-Tree Theorem for graphs.

Theorem (Sandpile Matrix-Tree Theorem)

The size of $\mathcal{S}(G)$ is the number of spanning trees of G.

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- This works as long as Σ contains at least one basis with determinant 1 or -1. In particular, this works when Σ is the complete simplicial complex in any dimension and with any number of vertices.

Sandpile Torsors

Theorem (Duval-Klivans-Martin, 2009)

$$|\mathcal{S}(D)| = \sum_{s \in \mathcal{B}(D)} |D_s|^2$$

- Can we find a combinatorially interesting map f from $\mathcal{S}(D) \to \mathcal{B}(D)$ such that for each $s \in B(D)$, we have $f^{-1}(s) = |D_s|^2$?

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- The general result for any $D = \left(\begin{array}{c|c} I_n & M \end{array} \right)$ will appear in my upcoming paper.

Thanks For Listening!!!