A Higher-Dimensional Sandpile Map

Georgia Tech Algebra Seminar

Alex McDonough

Brown University

9/30/20

The Plan

- Define orientable arithmetic matroids

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- Define their sandpile group

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- Give a generalized Matrix-Tree Theorem
- Construct a periodic tiling of space
- Use it to construct “bijections”
- Show some pretty pictures

Representable Matroids

- In general, a *matroid* is a pair $(E, \mathcal{B})$, where $\mathcal{B} \subseteq \mathcal{P}(E)$, that satisfies some conditions. E is called the *ground set* while $\mathcal{B}$ is called the set of *bases*.

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- The size of each basis is the *rank* of the matroid, which is 2 for this example.

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- We call the triple $(E, \mathcal{B}, m)$ an *orientable arithmetic matroid* (oa-matroid).

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Technical Note (see Pagaria 2020 for Details)

Usually, m is defined on all subsets of E, not just $\mathcal{B}$. With our setup, we are actually working with oa-matroids that have the *strong GCD property*.

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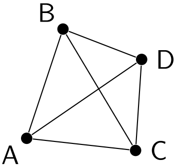
Proposition (Pagaria 2020)

If M generates a rank r matroid and $m(\{v_1, \dots, v_r\}) = 1$, then there is a unique(ish) standard representative matrix D that satisfies properties 1 and 2.

Cell Complexes and oa-Matroids

- Let Σ be a cell complex.



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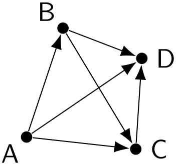

$$\begin{array}{c|cccc} & ABC & ABD & ACD & BCD \\ \hline AB & 1 & 1 & 0 & 0 \\ AC & -1 & 0 & 1 & 0 \\ AD & 0 & -1 & -1 & 0 \\ BC & 1 & 0 & 0 & 1 \\ BD & 0 & 1 & 0 & -1 \\ CD & 0 & 0 & 1 & 1 \end{array}$$

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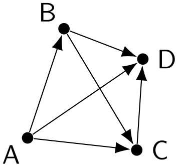

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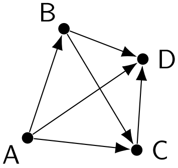

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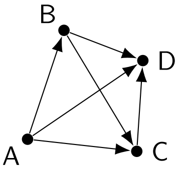

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- For now, let's not worry about oa-matroids without multiplicity 1 bases.

D and $\hat{D}$

- Let D be a standard representative matrix $\begin{bmatrix} I_r & N \end{bmatrix}$ where N is $r \times n$.

$$D = \begin{pmatrix} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \end{pmatrix}$$

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- $\hat{D}$ relates to D in several ways that we will explore on the next slide.

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- If we put D on top of $\hat{D}$, we get an invertible square matrix of the form:

$$\mathcal{D} = \begin{bmatrix} I_r & N \\ -N^T & I_n \end{bmatrix}.$$

Representable Matrix Sandpile Group

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- If D represents a regular matroid, $\mathcal{S}(D)$ is isomorphic to the usual regular matroid sandpile group.

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$$|\mathcal{S}(D)| = m(\{v_1, v_2\})^2 + m(\{v_1, v_3\})^2 + m(\{v_2, v_3\})^2 =$$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 + \det\left(\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\right)^2 + \det\left(\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}\right)^2 = 1^2 + 2^2 + 3^2 = 14$$

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- **This is the main result of this talk.**

Fundamental Parallelepipeds

Definition

The *fundamental parallelepiped* of a square matrix M with column vectors $v_1, \ldots, v_n$ is the set of points:

$$\left\{ \sum_{i=1}^{n} a_i v_i \mid 0 \leq a_i \leq 1 \right\}.$$

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- The polytope $\Pi_\bullet(M)$ is also called the *zonotope* or *minkowski sum* of the columns of M.
- In order to construct our maps, we associate each basis with the fundamental parallelepiped of a particular matrix.

Basis Parallelepipeds

Let $D = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$ which means that $\mathcal{D} = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ \hline -3 & -2 & 1 \end{pmatrix} \end{array}$.

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- For each basis $B \in \mathcal{B}$, we get a parallelepiped $P(B)$ by replacing the first r or last n entries of each column of $\mathcal{D}$ by 0 based on which columns make up B (see example).

$$P(\{v_1, v_2\}) = \Pi_\bullet \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \quad P(\{v_1, v_3\}) = \Pi_\bullet \left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \right)$$

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The Tile Associated with D

- We call $\bigcup_{B \in \mathcal{B}} P(B)$ the *tile associated with* D, denoted $T(D)$.

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$$T(D) = \Pi_\bullet\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\bigcup\Pi_\bullet\left(\begin{bmatrix}1 & 0 & 3\\0 & 0 & 2\\0 & -2 & 0\end{bmatrix}\right)\bigcup\Pi_\bullet\left(\begin{bmatrix}0 & 0 & 3\\0 & 1 & 2\\-3 & 0 & 0\end{bmatrix}\right)$$

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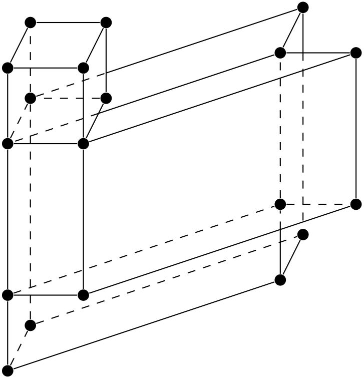

Some Fun With Blender

The Best Theorem I've Ever Proven

Theorem (M. 2020)

The parallelepipeds that make up $T(D)$ have non-overlapping interiors.

The Best Theorem I've Ever Proven

Theorem (M. 2020)

The parallelepipeds that make up $T(D)$ have non-overlapping interiors. Furthermore, the translates of $T(D)$ by integer linear combinations of rows of $\mathcal{D}$ form a non-overlapping tiling of $\mathbb{R}^{r+n}$.

A Tiling Demonstration (More Blender Fun)

A Proof that the $P(B_i)$ have Non-Overlapping Interiors

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$$\mathcal{D} = \left[\begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \\ \hline \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_4 & \hat{c}_5 \end{array}\right]$$

$$\mathcal{D} = \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 7 \\ 0 & 1 & -1 & 0 & -2 \\ \hline -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -7 & 2 & 0 & 0 & 1 \end{array}\right]$$

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- **Setup:** $\mathcal{D}$ is a nonsingular $(r+n) \times (r+n)$ integer matrix whose first r rows and last n rows are orthogonal. Label columns with c_i and $\hat{c}_i$ (see below).
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The Important Result: $d \cdot \hat{d} = 0$

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Since $B_1 \neq B_2$ we have a contradiction!

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- Here's a reminder of the second part of the theorem:

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- This theorem does not hold if we only assume orthogonality.

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Theorem (M. 2020)

For any choice of w *not in the span of a facet of* $P(B)$, $|f_w^{-1}(B)| = m(B)^2$

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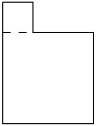

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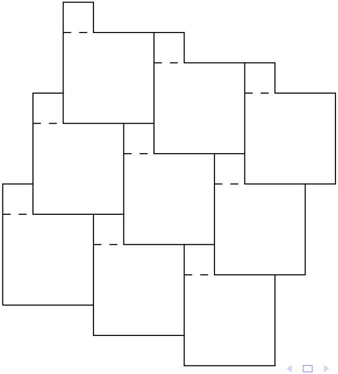

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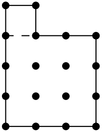

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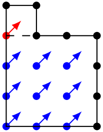

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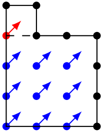

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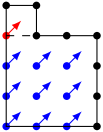

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- One of them is mapped to the basis of multiplicity 1. Nine of them are mapped to the basis of multiplicity 3.

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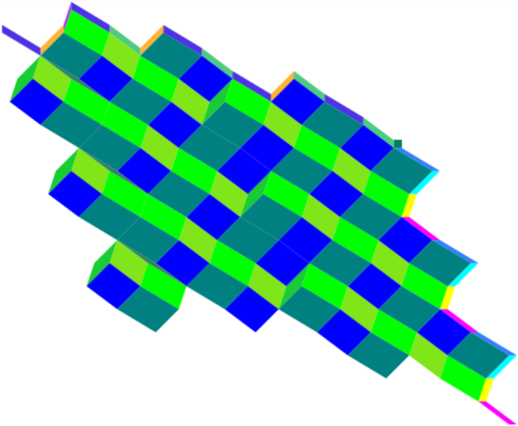

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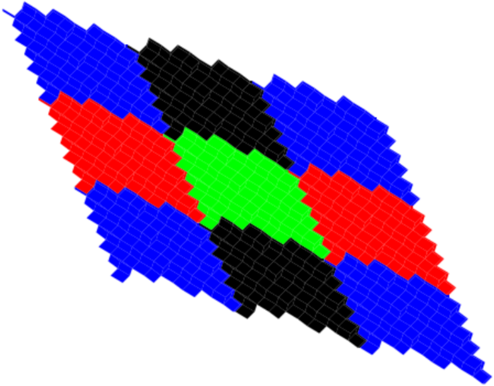

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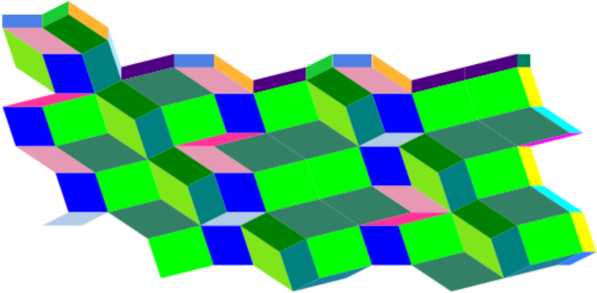

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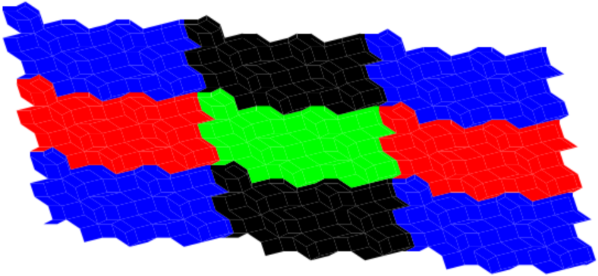

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- I'm really curious about the group $\mathcal{S}(\mathcal{D}^T)$.

Thanks For Listening!!!

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