Genus from a Sandpile Torsor Algorithm

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What is a Sandpile Torsor Algorithm?

Let $G = (V, E)$ with spanning trees $\mathcal{T}(G)$. Every vertex of G has an integral number of chips. A vertex can "fire", sending one chip across each incident edge.

Definition: Sandpile Group

The sandpile group $\mathrm{Pic}^0(G)$ is the set of chip configurations on G such that there are 0 total chips along with the equivalence relation given by firing vertices. (This is the cokernal of the reduced Laplacian). The group operation is pointwise addition.

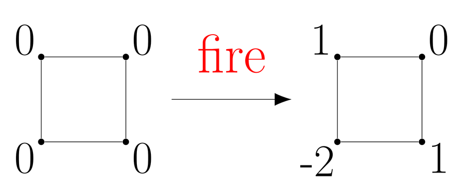

Figure 1: Equivalent elements of $\mathrm{Pic}^0(G)$

There is a very important relationship between G and $\mathrm{Pic}^0(G)$, whose exploration motivates the remainder of this poster.

Sandpile Matrix-Tree Theorem

The size of $\mathrm{Pic}^0(G)$ is the number of spanning trees of G.

Because these two sets are the same size, it seems plausible that we could find a canonical bijection between them. However, this is impossible without the following restrictions:

- We need to assign a cyclic order to the edges incident to each vertex. (With this added structure, G is called a "ribbon graph")
- Instead of looking for a bijection, we need to look for a "free transitive action" of the sandpile group on the set of spanning trees. In essence, this is a canonical bijection once we assign a spanning tree to the identity of $\mathrm{Pic}^0(G)$.
- We need to select a distinguished vertex of G.

Definition: Sandpile Torsor

A sandpile torsor for a ribbon graph G with distinguished vertex v is a free transitive action of $\mathrm{Pic}^0(G)$ on the spanning trees of G.

Definition: Sandpile Torsor Algorithm

A sandpile torsor algorithm assigns a sandpile torsor to every ribbon graph with every possible distinguished vertex. This assignment must be compatible with automorphisms of the ribbon graph.

Two known sandpile torsor algorithms are the Bernardi Process and the rotor routing process. See Figure 2, [HLM+08], and [BW17].

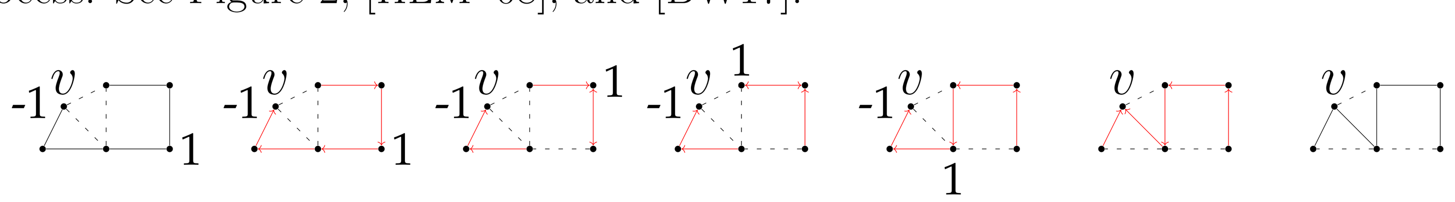

Figure 2: A demonstration of the rotor routing sandpile torsor algorithm.

Known Planarity Results

The *genus* of a ribbon graph G is the genus of the surface obtained after thickening the edges of G and then gluing disks to the boundary components. A ribbon graph is called *planar* if its genus is equal to 0. The inspiration for this paper comes from the following theorem.

Theorem(Chan-Church-Grochow, Baker-Wang)

The rotor routing and Bernardi processes are invariant to the choice of basepoint if and only if the ribbon graph is planar.

Exploring Higher Genus

Because it is possible to determine whether or not a ribbon graph is genus 0 from looking at a sandpile torsor algorithm, it is natural to ask whether sandpile torsor algorithms can be used to distinguish a genus m ribbon graph from a genus n ribbon graph for $m \neq n$ and $m, n \neq 0$. This question was first posed by Melody Chan.

First, we ask if the structure of the sandpile torsors is enough to determine genus.

Question 1

Let G and G' be two ribbon graphs with genus g and g' respectively and let α be a sandpile torsor algorithm. Assume that $V(G) = V(G')$, $\varphi : \mathcal{T}(G) \to \mathcal{T}(G')$ is a bijection, and $\psi : \mathrm{Pic}^0(G) \to \mathrm{Pic}^0(G')$ is an isomorphism such that for every vertex $v \in V(G)$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pic}^0(G) \times \mathcal{T}(G) \arrow[r, "\alpha_v(\mathrm{Pic}^0(G))"] \arrow[d, "\psi\times\varphi"'] & \mathcal{T}(G) \arrow[d, "\varphi"] \\ \mathrm{Pic}^0(G') \times \mathcal{T}(G') \arrow[r, "\alpha_v(\mathrm{Pic}^0(G'))"] & \mathcal{T}(G') \end{tikzcd}$$

Is it necessarily true that $g = g'$?

NO:

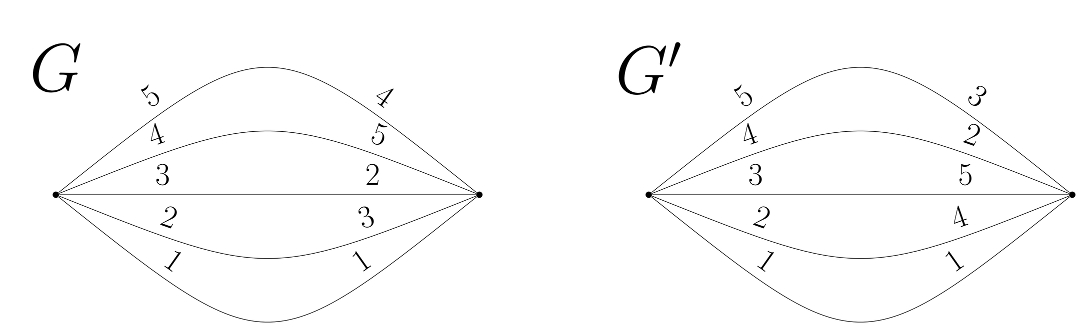

Figure 3: Two ribbon Graphs with the same sandpile torsor structure but different genus. The numbers give cyclic orders around each vertex.

Because structure is not enough, we next see if we can find genus from the precise output of a sandpile torsor algorithm.

Question 2

Consider the same conditions as Question 1, but where the isomorphism ψ is required to be the identity. Is it necessarily true that $g = g'$?

NO: (when α is the Bernardi or rotor routing process)

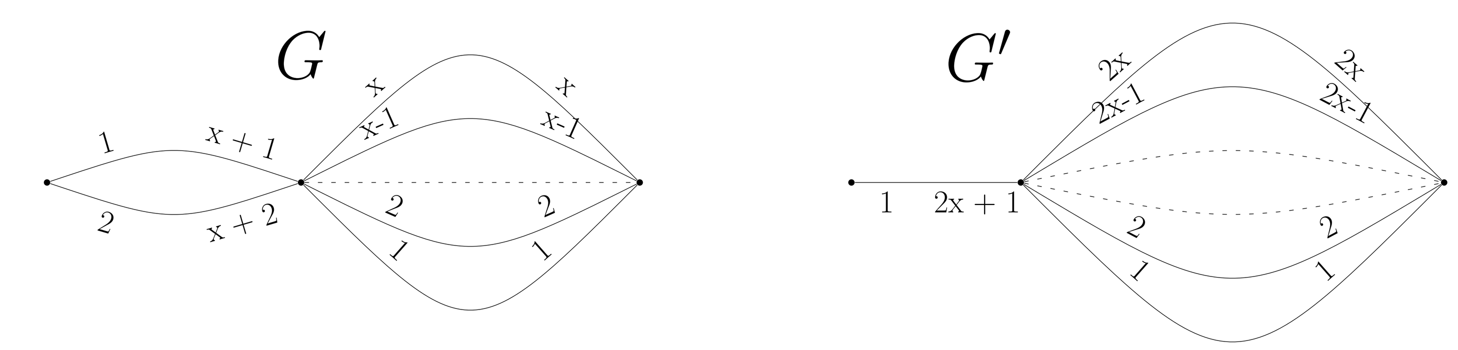

Figure 4: Two graphs with the same rotor routing/ Bernardi torsors but different genus.

Finally, we consider the case where edges are known and spanning trees are given as subsets of them.

Question 3

Let G be a ribbon graph. Suppose that we are given $V(G)$, $E(G)$, $\mathrm{Pic}^0(G)$, $\mathcal{T}(G)$ (with elements given as subsets of $E(G)$) and for every $v \in V(G)$, we are given the map

$$\mathrm{Pic}^0(G) \times \mathcal{T}(G) \xrightarrow{\alpha_v(\mathrm{Pic}^0(G))} \mathcal{T}(G)$$

where α_v is a sandpile torsor algorithm with basepoint v. Then, is it possible to determine the genus of G?

YES: (when α is the rotor routing process)

Our strategy is to find the cyclic order around each vertex using rotor routing with very specific sandpile elements and spanning trees. This process is most straightforward for cut-free graphs, where we can find exact cyclic orders around each vertex. For arbitrary ribbon graphs, we do not always have unique cyclic orders, but all solutions have the same genus.

[BW17] Matthew Baker and Yao Wang, *The bernardi process and torsor structures on spanning trees*, International Mathematics Research Notices (2017), rnx037.

[CCG14] Melody Chan, Thomas Church, and Joshua A Grochow, *Rotor-routing and spanning trees on planar graphs*, International Mathematics Research Notices **2015** (2014), no. 11, 3225–3244.

[HLM+08] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, and David B. Wilson, *Chip-firing and rotor-routing on directed graphs*, pp. 331–364, Birkhäuser Basel, Basel, 2008.