RATIONALITY OF COMPLETE INTERSECTIONS OF TWO QUADRICS OVER NONCLOSED FIELDS

BRENDAN HASSETT AND YURI TSCHINKEL, WITH AN APPENDIX BY JEAN-LOUIS COLLIOT-THÉLÈNE

Abstract. We study rationality problems for smooth complete intersections of two quadrics. We focus on the three-dimensional case, with a view toward understanding the invariants governing the rationality of a geometrically rational threefold over a non-closed field.

1. Introduction

An algebraic variety X defined over a field k is called *rational* if it is birational to projective space, i.e., its function field $k(X)$ is a purely transcendental extension of k. It is called *stably rational* over k if $X \times \mathbb{P}^n$ is rational over k, for some $n \in \mathbb{N}$. One speaks of *geometric rationality*, respectively, *geometric stable rationality*, if these properties hold over an algebraic closure $\bar{k}$ of k. In dimension one, there are easy effective criteria for rationality (and stable rationality): these depend only on the topology and the existence of a k-rational point. Geometric rationality in dimensions two and three has been one of the central questions of classical Italian algebraic geometry, with fundamental contributions by Enriques, Fano, and many others.

Rationality problems for geometrically rational surfaces X over *non-closed* fields k have been intensely studied and are well understood. Their minimal models over k are either conic bundles or del Pezzo surfaces. In addition to the existence of k-rational points $X(k)$, the essential information is encoded in the Picard group $\mathrm{Pic}(\bar{X})$, viewed as a Galois-module. Rationality can be detected by analyzing the Galois orbit structure on the set of exceptional curves: one of the main results here is that minimal surfaces X cannot be rational, with a few explicit exceptions. This allows us to completely settle the question of rationality of X. Stable rationality remains a challenging open problem.

In dimension 3, the minimal model program leads to conic bundles, del Pezzo fibrations, and Fano varieties. There is an extensive literature

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on rationality of these varieties over algebraically closed fields, see, e.g., [IP99]. Recently, there has been decisive progress on the stable rationality problem, essentially settling this problem in dimension 3, with the exception of varieties birational to a cubic [HKT16], [HT19b], [KO20]. These results relied on the specialization method introduced by Voisin [Voi15] and developed in [CTP16], see also [Voi16], [CT19]. However, little is known about rationality properties of geometrically rational threefolds over nonclosed fields such as finite fields or function fields of curves.

Here, we focus on the simplest geometrically rational, nontoric, example, the complete intersection of two quadrics

$$X \subset \mathbb{P}^5, \tag{1}$$

with a view toward understanding the invariants governing the rationality of a geometrically rational threefold over relatively simple ground fields. Indeed, rationality properties of toric varieties are controlled by the Picard group, as a Galois module, similar to the case of surfaces; the classification of nonrational tori can be found in [Kun87]. For X a smooth intersection of two quadrics, we have

$$\mathrm{Pic}(\bar{X}) = \mathbb{Z},$$

with trivial Galois action, and the Brauer group $\mathrm{Br}(X)$ is also trivial. As we will see, the intermediate Jacobian $\mathrm{IJ}(X)$ is the Jacobian of a curve of genus 2, over k. Similarly, there are no obstructions from the birational rigidity viewpoint. Are there any obstructions to rationality?

We find them in specializations. One of our main results, Theorem 29, is a proof of failure of stable rationality of general X as in (1). We use two different specializations of X. In Section 9 we use nonrational toric threefolds over k to obtain smooth examples over $k((\tau))$ that are nonrational but admit rational points. In Section 10, we work over $k = \mathbb{C}(t)$ and view X as a fourfold over $\mathbb{C}$, admitting a quadric surface bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. The failure of stable rationality of such X, over $\mathbb{C}$, can be proven using the techniques of [HPT18]; this implies that X, considered as a variety over $k = \mathbb{C}(t)$, is not stably rational over k.

On the other hand, natural rationality constructions interact in unexpected ways. In Section 2, we recall the geometry of the varieties of projective subspaces on higher-dimensional intersections of two quadrics. The most classical rationality construction – projection from lines – is discussed in Section 3. Another useful – and *a priori* distinct – rationality construction is presented in Section 4. Interestingly, when this construction applies, it forces the existence of a line over the ground field (Theorem 9). In Section 5 we study the connection to the intermediate Jacobian of X and its twisted forms; Question 11(ii) is a natural outgrowth of our examples. Section 6 presents a related rationality result (Theorem 14) asserting that odd-degree curves force the existence of a line and rationality. We then explore what happens as the corresponding cocycle associated with the intermediate Jacobian collapses. For example, X might admit a pair of skew lines over the ground field, addressed in Section 7. Given the importance of rational curves in all our constructions, we explore their geometry in Section 8. Real pencils of quadrics have been extensively studied; rationality in the three-dimensional case is discussed in Section 11. The Appendix, contributed after this manuscript was originally written, addresses several questions raised in the text and presents interesting examples and extensions.

Throughout, we work over a base field k that has characteristic not equal to two.

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2. Geometric background

In this section, $X \subset \mathbb{P}^n$ is a smooth complete intersection of two quadrics over k. We recall basic facts concerning varieties of linear subspaces of X.

2.1. **Varieties of linear subspaces.** Let $F_r(X) \subset \mathbb{G}(r,n)$ denote the variety of r-dimensional linear subspaces in X. It has expected dimension

$$e(r,n) = (r+1)(n-2r-2)$$

which is non-negative provided $n \geq 2r+2$.

Proposition 1. *If $n \geq 2r+2$ then $F_r(X)$ is smooth and nonempty of dimension $e(r,n)$. If $n > 2r+2$ it is also connected.*

The nonsingularity result is [Rei72, Th. 2.6] and nonemptyness is addressed in the introductions to Chapters 3 and 4 of [Rei72]. The connectedness assertion is [DM98, Th. 2.1].

The adjunction formula [DM98, p. 555] gives the dualizing sheaf

$$\omega_{F_r(X)} \simeq \mathcal{O}_{F_r(X)}(2r-n+3),$$

which is always non-negative provided $e(r,n) > 0$. It is negative unless $n = 2g+1$ and $r = g-1$ for some $g \in \mathbb{N}$.

2.2. **Connections with hyperelliptic curves and vector bundles.** Assume that $n = 2g+1$ and consider the fibration

$$\varrho : \mathrm{Bl}_X(\mathbb{P}^{2g+1}) \to \mathbb{P}^1$$

associated with the pencil of quadrics and the relative variety of maximal isotropic subspaces

$$F_g(\mathrm{Bl}_X(\mathbb{P}^{2g+1})/\mathbb{P}^1) \to \mathbb{P}^1$$

which factors

$$F_g(\mathrm{Bl}_X(\mathbb{P}^{2g+1})/\mathbb{P}^1) \xrightarrow{\varpi} C \xrightarrow{\gamma} \mathbb{P}^1,$$

where ϖ is smooth and γ is a double cover branched over $D \subset \mathbb{P}^1$ with $|D| = 2g+2$.

Over $\bar{k}$, X may be diagonalized

$$X = \{a_{00}x_0^2 + \cdots + a_{02g+1}x_{2g+1}^2 = a_{10}x_0^2 + \cdots + a_{12g+1}x_{2g+1}^2 = 0\},$$

and thus admits automorphisms by diagonal matrices with entries ± 1

$$(\mathbb{Z}/2\mathbb{Z})^{2g+1} \subset \mathrm{Aut}(\bar{X}).$$

As a Galois module, this may be represented as

$$H = \langle p_1, \ldots, p_{2g+2} \rangle \subset \mathrm{Pic}(\bar{C}) / \left\langle g_2^1 \right\rangle,$$

where the p_i are the branch points. This has the relations

$$2p_i \equiv 0, p_1 + \cdots + p_{2g+2} \equiv 0.$$

Desale-Ramanan [DR77] obtain:

Proposition 2. *Assume k is algebraically closed.*

— *$F_{g-1}(X)$ is a torsor over the Jacobian of C.*

— *$F_{g-2}(X)$ is the moduli space of rank-two vector bundles $\mathcal{E}$ with fixed odd determinant, i.e., an isomorphism*

$$\det(\mathcal{E}) \simeq \mathcal{L}.$$

The generic such bundle admits automorphisms by ± 1 and an action

$$\mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{M}$$

where $\mathcal{M}^{\otimes 2} \simeq \mathcal{O}_C$.

2.3. **Quadric pencils.** Consider the fibration

$$\varrho : P := \mathrm{Bl}_X(\mathbb{P}^n) \to \mathbb{P}^1$$

associated with the pencil of quadrics.

Proposition 3 (Brumer Theorem [Bru78])**.** *The variety X admits a rational point if and only if ϱ admits a section.*

Here we assume $n = 2g+1$ and write OGr_{g+1} for one of the two isomorphic connected components of the Grassmannian of maximal isotropic subspaces in a nondegenerate quadratic form in $2g+2$ variables. The relative variety of planes

$$F_g(P/\mathbb{P}^1) \to \mathbb{P}^1$$

factors

$$F_g(P/\mathbb{P}^1) \stackrel{\varpi}{\to} C \stackrel{\gamma}{\to} \mathbb{P}^1,$$

where ϖ is an étale OGr_g-bundle and γ is a double cover. The standard theory of quadratic forms yields a class $\alpha \in \mathrm{Br}(C)[2]$.

Now assume $n = 5$ and $g = 2$. Here OGr_3 is a Brauer-Severi threefold, geometrically isomorphic to $\mathbb{P}^3$. Thus

$$F_2(P/\mathbb{P}^1) \stackrel{\varpi}{\to} C$$

is an étale $\mathbb{P}^3$-bundle, hence the index of α divides four. Intersection with X gives an isomorphism

$$F_2(P/\mathbb{P}^1) \stackrel{\sim}{\to} R_2(X)$$

to the variety $R_2(X)$ of conics on X.

Remark 4. It follows that if X contains a conic over k then $C(k) \neq \emptyset$. The converse holds if $\alpha = 0$ or k is a C_1-field.

3. Rationality via lines

In this section, we recall a standard rationality construction: Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of two quadrics containing a line ℓ. Projecting from ℓ induces a birational morphism

$$\beta : \mathrm{Bl}_\ell(X) \to \mathbb{P}^{n-2}$$

that blows down all the lines in X incident to ℓ. Fix coordinates $x_0, \ldots, x_n$ so that $\ell = \{x_2 = \ldots = x_n = 0\}$ and

$$X = \{L_{00}x_0 + L_{01}x_1 + Q_0 = L_{10}x_0 + L_{11}x_1 + Q_1 = 0\},$$

where the L_{ij} are linear and the Q_i quadratic in $x_2, \dots, x_n$. The inverse mapping β^{-1} is given by a linear series of cubics associated with the 2×2 minors of a matrix

$$\begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \\ Q_0 & Q_1 \end{pmatrix}.$$

When $n = 5$ we recover the following data:

— the base locus of β^{-1}, a smooth quintic curve of genus two

$$C \subset \{L_{00}L_{11} - L_{01}L_{10} = 0\} \subset \mathbb{P}^3;$$

— a divisor on C of degree three, corresponding to one of the rulings of the quadric surface.

This is consistent with our previous notation as the space of conics $R_2(X)$ admits a natural morphism to C with rational fibers. Indeed, conics in X correspond under β to conics $R \subset \mathbb{P}^3$ incident to C in four points; we take the residual point in the intersection of $C \cap \operatorname{span}(R)$.

Remark 5. This yields a geometric explanation for the first assertion of Theorem 28 of [BGW17], in the genus two case.

We summarize this classical construction:

Construction 1. *(see* [CTSSD87a, Prop. 2.2]*) Let X be a smooth complete intersection of two quadrics. Suppose that $F_1(X)$ admits a k-rational point ℓ. Then X is rational over k.*

From this, we obtain

— over $k = \mathbb{C}$, X is rational provided $n \geq 4$;

— over $k = \mathbb{F}_q$, X is rational provided $n \geq 5$;

— over $k = \mathbb{C}(B)$, where B is a curve, X is rational provided $n \geq 6$.

Most of this is contained in [CTSSD87a, Th. 3.3,3.4], with the exception of the case of finite fields with $n = 5$. In this case, $F_1(X)$ is a principal homogeneous space over the Jacobian of a genus two curve (by Prop. 2 or [Rei72, Th. 4.8]) and thus admits k-rational points by Lang's Theorem [Lan55].

4. Rationality via points and quadric surface fibrations

In this section we present other rationality constructions.

4.1. **Double projection.** Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics.

Construction 2. [CTSSD87a, §3] *For each $x \in X(k)$, double projection from x induces a quadric surface bundle*

$$q : X' \to \mathbb{P}^1$$

with six degenerate geometrically integral fibers. The relative variety of lines factors

$$F_1(X'/\mathbb{P}^1) \xrightarrow{\phi} C_x \to \mathbb{P}^1$$

where ϕ is an étale $\mathbb{P}^1$ bundle, classified by $\alpha_x \in \mathrm{Br}(C_x)[2]$.

Proposition 6. *We have a natural diagram*

$$\begin{tikzcd}
F_1(X'/\mathbb{P}^1) \arrow[r, hook] \arrow[d] & F_2(P/\mathbb{P}^1) \arrow[d] \\
C_x \arrow[r, equal] \arrow[d] & C \arrow[d] \\
\mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1
\end{tikzcd}$$

giving a linear embedding of an étale $\mathbb{P}^1$-bundle into an étale $\mathbb{P}^3$-bundle.

The arrows are induced as follows:

— for the bottow row

$$\mathbb{P}^1 = \mathbb{P}(T_x\mathbb{P}^5/T_xX) = \mathbb{P}(\Gamma(\mathcal{I}_X(2)) = \mathbb{P}^1$$

by taking tangent spaces to the quadric hypersurfaces $\{Q_p\}$ in the pencil;

— for the top row, given $p \in \mathbb{P}^1$ the threefold

$$\mathbb{P}^4_p \cap Q_p$$

is a cone over the surface $q^{-1}(p)$ whence

$$F_1(q^{-1}(p)) \hookrightarrow F_2(\mathbb{P}^4 \cap Q_p) \hookrightarrow F_2(Q_p);$$

— the middle row is induced by the functoriality of Stein factorization.

A degree computation shows that the $\mathbb{P}^1$-fibers are linearly embedded in the $\mathbb{P}^3$-fibers.

Proposition 6 yields the compatibility of the pairs (C_x, α_x) with the pair (C, α) introduced in Section 2.3:

Corollary 7. *The curve C_x and Brauer class α_x are independent of x. The index of α equals two whenever X admits a rational point.*

Construction 3. *Retain the notation of Construction 2. Note that X' is rational when $\alpha = 0$.*

As a consequence, we obtain that X is rational [CTSSD87a, Th. 3.4]

(1) over k a C_i-field when $n \geq 2^{i+1}+2$;

(2) over k a p-adic field when $n \geq 11$.

4.2. **Geometric analysis.** We first elaborate the geometric implications of our rationality construction.

Let $X \subset \mathbb{P}^5$ denote a smooth complete intersection of two quadrics over k and suppose there exists an $x \in X(k)$ such that there exist distinct lines

$$x \in \ell_1, \ell_2, \ell_3, \ell_4 \subset \bar{X} = X_{\bar{k}}$$

with $N_{\ell_i/\bar{X}} \simeq \mathcal{O}_{\mathbb{P}^1}^2$. Double projection from x induces a rational map

$$X \dashrightarrow \mathbb{P}^1$$

defined away from the lines. We resolve indeterminacy

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X & & X' \arrow[d, "q"] \\ & & \mathbb{P}^1 \end{tikzcd}$$

where $\widetilde{X} \to X$ is obtained by blowing up x and the proper transforms of $\ell_1, \ldots, \ell_4$, and $\widetilde{X} \to X'$ blows down the exceptional divisors over the lines along the opposite rulings. The morphism $X' \to \mathbb{P}^1$ is a quadric surface bundle. Let $Y' \subset X'$ denote the proper transform of the exceptional divisor over x, a conic bundle over $\mathbb{P}^1$ with three degenerate fibers.

Suppose that the class $\alpha \in \mathrm{Br}(C)[2] = 0$ so that

$$\phi : F_1(X'/\mathbb{P}^1) \to C$$

admits a section. Then we may express

$$F_1(X'/\mathbb{P}^1) = \mathbb{P}(\mathcal{E})$$

where $\mathcal{E}$ is a vector bundle of rank two and odd determinant [DR77]. Note that it follows immediately that $[\mathrm{Pic}^1(C)] = 0$ as a principal homogeneous space under $A = \mathrm{Pic}^0(C)$. Thus we may normalize so that $\deg(\mathcal{E}) = 5$.

The structure of sections of ϕ is well-known: For each $\mathcal{L} \in \mathrm{Pic}^0(C)$, elements of

$$\mathbb{P}(\Gamma(C, \mathcal{E} \otimes \mathcal{L}))$$

yield sections of the projective bundle. Riemann-Roch shows these are parametrized by a Zariski $\mathbb{P}^2$-bundle over $\mathrm{Pic}^0(C)$. Their proper transforms in X are twisted cubic curves containing x; these are birational to a Zariski $\mathbb{P}^2$-bundle over $F_1(X)$ by:

Proposition 8. *Twisted cubic curves on X are residual to a line $\ell \subset X$ in the three-plane they span. Thus the twisted cubics are birational to a Zariski $\mathrm{Gr}(2,4)$-bundle over $F_1(X)$. Those passing through a fixed point form a Zariski $\mathbb{P}^2$-bundle.*

In particular, X admits a twisted cubic over k if and only if it admits a line over k.

Lines in X disjoint from the lines $\ell_1, \dots, \ell_4$ correspond to sections of

$$\Gamma(C, \mathcal{E} \otimes \mathcal{L}), \quad \mathcal{L} \in \mathrm{Pic}^{-1}(C);$$

Riemann-Roch shows there is a unique such section for generic $\mathcal{L}$.

We summarize this discussion as follows:

Theorem 9. *Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics. Suppose that*

— $X(k) \neq \emptyset$;

— the class $\alpha \in \mathrm{Br}(C)$ vanishes.

Then X is rational and contains a line.

Proof. We may assume that k is infinite – we have already seen that X always admits a line over finite fields. In this case, Remark 3.28.3 of [CTSSD87a] applies to show that X is unirational, and thus $X(k)$ is Zariski dense. Hence we may find a point x satisfying the genericity assumptions. This allows us to apply the construction above. $\square$

Remark 10. The presence of a line on X allows us to realize $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a bidegree $(2,3)$ curve (see Section 3). It is perhaps surprising that the triviality of α forces the existence of odd degree cycles on C.

5. Variety of lines as a cocycle

In this section, we explain the connection between the variety of lines on X and its Albanese variety.

Let $C \to \mathbb{P}^1$ denote the genus two curve associated with X. Note that $\mathrm{Pic}^0(C)$ and the Albanese of the variety of lines $F_1(X)$ are isomorphic by work of X. Wang [Wan18] – we call this abelian surface A. Moreover, we obtain identifications of principal homogeneous spaces over A:

$$2[F_1(X)] = \mathrm{Pic}^1(C).$$

The class $[F_1(X)]$, as a principal homogeneous space over A, has order dividing four by [Wan18, Th. 1.1].

The associated morphism

$$\kappa : \mathrm{Sym}^2(F_1(X)) \to \mathrm{Pic}^1(C)$$

has nodal Kummer surfaces as fibers. Over $C \subset \mathrm{Pic}^1(C)$, we obtain the reducible conics $R_{2,red}(X) \subset R_2(X)$. Thus we have a natural factorization

$$\begin{tikzcd} R_{2,red}(X) \arrow[r, hook] \arrow[dr] & F_2(X) \arrow[d] \\ & C \end{tikzcd}$$

realizing the Kummer fibers as 16-nodal quartic surfaces in the Brauer-Severi fibration associated with α.

We record some basic observations.

— If $[F_1(X)]$ is trivial then X admits a line over k and is rational by Construction 1.

— The Brauer class $\alpha \in \mathrm{Br}(C)[2]$ is determined geometrically by the class $[F_1(X)]$.

— The class $[F_1(X)]$ has index dividing four when $X(k) \neq \emptyset$; indeed, the lines incident to a point are defined over a quartic extension.

Thus the class $[F_1(X)]$ is the fundamental object for our purposes.

Question 11. [1]

i. If $[F_1(X)]$ has order four must X be irrational over k?

ii. Are there examples where X is rational and $[F_1(X)] \neq 0$?

The second question grew out of discussions with Colliot-Thélène.

Proposition 12. *Consider the following statements:*

(1) $[F_1(X)]$ *has order two;*

(2) $[F_1(X)]$ *has index two, i.e.,* $F_1(X)$ *admits a rational point over a quadratic extension of* k*;*

(3) $R_2(X) \neq \emptyset$*;*

(4) C *admits a rational point.*

Then we have

$$(3) \Rightarrow (4) \Rightarrow (1), (2) \Rightarrow (1).$$

Below we analyze the geometry of each case, with a view toward understanding whether these implications are strict.

1. Recent work [HT19a, BW19, KP20] finds that $[F_1(X)] = 0$ if and only if X is rational over k.

Remark 13. Suppose that k is a C_1-field and $C(k) \neq \emptyset$. Then X admits a conic $D \subset X$ defined over k. Projecting from D gives a fibration

$$\mathrm{Bl}_D(X) \to \mathbb{P}^2$$

in conics, with a quartic plane curve as degeneracy.

Using conic bundles for rationality constructions is problematic, as we cannot expect to find sections when the base is a surface. Such sections would yield extra divisor classes in the total space, which would have to be singular by the Lefschetz hyperplane theorem. (Our X is rational over $\bar{k}$ but not because of the conic fibration.)

6. Odd degree curves, secants, and rationality

In this section, we establish a connection between the existence of curves of odd degree on X and rationality. Our main result here is:

Theorem 14.[2] *Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics. Suppose that X admits a smooth geometrically connected curve $R \subset X$ of odd degree. Then X contains a line and is rational.*

6.1. **Background on secants.** Let $R \subset \mathbb{P}^5$ be a smooth projective geometrically connected curve of degree d and genus g. Let $\mathrm{Sec}(R)$ denote the closure of the secants of R. We have a diagram

$$\begin{tikzcd} B_2 \arrow[r, "\beta"] \arrow[d] & \mathrm{Sec}(R) \arrow[r, hook] & \mathbb{P}^5 \\ \mathrm{Sym}^2(R) & & \end{tikzcd}$$

where $\mathrm{Sym}^2(R)$ is the symmetric square of R, $B_2 \to \mathrm{Sym}^2(R)$ is $\mathbb{P}^1$-bundle parametrizing lines parametrized by pairs of points on R, and β is the canonical morphism.

Proposition 15. [Dal84, Th. 4.3] *Assume that the span of R has dimension at least four. Then β is birational onto its image and*

$$\deg(\mathrm{Sec}(R)) = \binom{d-1}{2} - g.$$

The formula has the following interpretation: Secants passing through a plane P correspond to nodes of the image of the projection

$$\pi_P : R \to \mathbb{P}^2;$$

the formula is the difference between the genus of R and the arithmetic genus of $\pi_P(R)$.

2. Th. A.5 of the Appendix offers a generalization.

Proposition 16. [Ber92, 1.4, 1.5] *Assume that length four subschemes of R impose independent conditions on linear forms. Then* $\mathrm{Sec}(R)$ *is smooth away from R and its projective tangent cone at $r \in R$ is obtained by the projection of R from the tangent line of R at r.*

Under our assumptions projection of R from the tangent line at r maps it isomorphically onto a degree $d-2$ curve in $\mathbb{P}^3$. Thus $\mathrm{Sec}(R)$ has multiplicity $d-2$ along R.

6.2. **Analysis in complete intersections of quadrics.** Let $X \subset \mathbb{P}^5$ denote a smooth complete intersection of quadrics. We denote its hyperplane class by h. Consider

$$X' = \{(x, \ell) : x \subset \ell\} \subset X \times F_1(X)$$

which has the following properties:
— $\pi_1 : X' \to X$ is generically finite of degree four;
— π_1 is branched over a hypersurface in X of degree eight.
It is clear that four lines pass through a generic point of X. For the second assertion, fix a generic line $\ell \in F_1(X)$ so the resulting

$$\ell' := X' \times_X \ell \to \ell$$

has two disjoint components ℓ and C, where $C \to \ell$ is a degree-three cover from a curve of genus two branched over eight points.

Proposition 17. *Let $R \subset X$ be smooth and geometrically connected of degree $d \geq 3$ and genus g. Assume that*
— $\mathrm{Sec}(R)$ *has isolated singularities of multiplicity $d-2$ at the generic point r of R, and*
— *finitely many secants to R are contained in X.*
Let Σ_R denote the sum of the secants to R in X counted with multiplicity. Then we have

$$(d-2)R + \Sigma_R \equiv \left(\binom{d-1}{2} - g\right) h^2$$

in the Chow group of 1*-cycles of X.*

This is similar to results of M. Shen [She14] for cubic threefolds.

Proof. We assume for the moment that R spans a subspace of dimension at least four. Since $[\mathrm{Sec}(R)] \equiv (\binom{d-1}{2} - g)h^2$ and $\mathrm{Sec}(R)$ intersects X in pure dimension one, we just need to interpret the terms of the intersection. The computation of tangent cones above means that $\mathrm{Sec}(R) \cap X$ has multiplicity $m \geq (d-2)$ along R.

Consider the induced morphism

$$R' := X' \times_X R \to F_1(X);$$

its arithmetic genus $p_a(R')$ satisfies

$$2p_a(R') - 2 = 4(2g-2) + 8d$$

whence $p_a(R') = 4g + 4d - 3$. We claim R' maps birationally onto its image D. Indeed, suppose that $[\ell] \in D$ has two points $r_1', r_2' \in R'$ lying over it. Let $r_1, r_2 \in R$ be their images in R; the line ℓ contains these points and is therefore a secant to R.

An intersection computation shows that the class $[D] \in N_1(F_1(X))$ (one-cycles up to numerical equivalence) is equal to $d\Theta$. Curves in this class have arithmetic genus d^2+1 and secants to R correspond to double points of the induced map $R' \to D$. There are

$$p_a(d\Theta) - p_a(R') = d^2 - 4d - 4g + 4$$

such points, whence

$$\deg(\Sigma_R) = d^2 - 4d - 4g + 4.$$

Comparing degrees

$$md + d^2 - 4d - 4g + 4 = 2(d-1)(d-2) - 4g$$

we conclude that $m = d-2$ and the formula follows.

Now assume that R spans a subspace of dimension three. If R is a twisted cubic then it is residual to a line ℓ in a complete intersection of linear linear forms; ℓ is the unique secant to R in X so that $\Sigma_R = \ell$. Then we obtain

$$R + \ell \equiv h^2$$

which is consistent with our general formula. If R is an elliptic quartic curve then

$$R \equiv h^2$$

which is also consistent with the formula. □

Proposition 18. *Let $R \subset X$ and assume that* $\mathrm{Sec}(R)$ *does not have isolated singularities of multiplicity $d-2$ at the generic point r of R. Then there are infinitely many secants to R contained in X.*

Proof. Under our assumptions, there exist points $p, q \in R \setminus \{r\}$ such that the secant $\ell(p,q)$ contains r. It follows that the line $\ell(p,q)$ meets X with multiplicity at least three and thus is contained in X. Varying over $r \in R$, we obtain a curve

$$B \to F_1(X)$$

such that R is contained in the ruled surface $\mathbb{P}(S^*|B) \subset X$ associated with B. Here S is the universal line sub-bundle on the Grassmannian $\mathrm{Gr}(2,6)$. □

Proposition 19. *Suppose that $R \subset X$ admits infinitely many secants on X. If R has odd degree then X admits a cycle of lines of odd degree.*

Proof. Let $B^\nu \to B \subset F_1(X)$ denote the normalization such that R factors through a ruled surface $\mathbb{F} := \mathbb{P}(S^*|B^\nu) \to X$. Writing

$$b = (\Theta \cdot B)_{F_1(X)},$$

we see that $[\mathbb{F}] = bh$ and, cutting by hyperplanes, we find that $\mathbb{F} \to B^\nu$ admits sections of class bh^2. These both have degree $4b$. If $\mathbb{F} \to B^\nu$ admits a multisection R then B^ν admits a zero-cycle of odd degree as well. Indeed, take the difference of R and a suitable multiple of the sections mentioned above. Thus $F_1(X)$ admits such a cycle as well. □

6.3. **Proof of Theorem 14.** If X contains a line then there is nothing to prove. Otherwise, Proposition 17 covers the 'generic' case. The number of secants with multiplicities is

$$\deg(\Sigma_R) = d^2 - 4d - 4g + 4,$$

which is odd whenever d is odd. However, $F_1(X)$ has a point whenever it admits a cycle of odd degree. When the genericity hypotheses fail to be satisfied, Proposition 18 puts us in the case where there are infinitely many secants to R on X. Here we apply Proposition 19 to conclude the result.

7. Two skew lines

The motivating question of this section is: Let $X \subset \mathbb{P}^5$ denote a smooth complete intersection of two quadrics over a field k. Suppose that X admits a line $\ell \subset X$ defined over a quadratic extension L/k, not containing a k-rational point of X. In other words, ℓ and its Galois conjugate ℓ' are disjoint. When does it follow that X is rational?

Example 20. Over $k = \mathbb{R}$ we are guaranteed lines over quadratic extensions. If $X(\mathbb{R}) = \emptyset$ then pairs of conjugate lines are necessarily skew but X is not rational. So the existence of two conjugate lines *per se* does not imply rationality. [3]

Construction 4. *(cf.* [CTSSD87a, §4,6]*) Suppose that X admits a pair of skew lines ℓ and ℓ', Galois conjugate over k. Express*

$$\mathrm{span}(\ell, \ell') \cap X$$

as a quadrilateral $\{\ell, m_1, \ell', m_2\}$. Then projecting from $\mathrm{span}(\ell, \ell')$ *gives a sextic del Pezzo fibration*

$$\varpi : Y \to \mathbb{P}^1,$$

3. Prop. A.6 of the Appendix addresses this further.

obtained by blowing down skew lines.

Our analysis will describe the degeneracy locus and associated Galois-theoretic data of ϖ.

7.1. **Details of the construction.** Let $m_1, m_2 \subset X$ be residual to ℓ and ℓ' in a codimension-two linear section of X

$$\operatorname{span}(\ell, \ell') \cap X = \ell \cup m_1 \cup \ell' \cup m_2.$$

Consider the hyperplane sections of X containing $\ell, \ell', m_1,$ and m_2, which are quartic del Pezzo surfaces with a fixed anti-canonical cycle of lines. The fiber-wise linear series consists of quadrics vanishing along ℓ and ℓ', which collapse m_1 and m_2.

Remark 21. In this situation, we have a natural rational map

$$\psi : X \dashrightarrow \mathbf{R}_{L/k}\mathbb{P}^3$$

induced by projection from ℓ and ℓ'. This factors as
— blowing up ℓ and ℓ';
— blowing down the lines m_1 and m_2 residual to ℓ and ℓ'.
Let Y denote the closed image of X. Geometrically, it is a complete intersection of three bidegree $(1,1)$ forms in $\mathbb{P}^3 \times \mathbb{P}^3$ with two ordinary threefold double points $y_1, y_2 \in Y$, the images of m_1 and m_2. Note that

$$\deg(Y) = \binom{6}{3} = 20, Y \subset \mathbb{P}^{12}. \tag{2}$$

The pencil of hyperplane sections of X containing the cycle of rational curves

$$\ell \cup m_1 \cup \ell' \cup m_2$$

gives a rational map

$$\varphi : X \dashrightarrow \mathbb{P}^1$$

that may be factored a follows.

Step 1. Let $X_1 \to X$ denote the blow up of ℓ and ℓ' with exceptional divisors $\hat{E}$ and $\hat{E}'$, each isomorphism to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that

$$(\hat{E})^3 = (\hat{E}')^3 = 0.$$

The proper transforms $\hat{m}_1$ and $\hat{m}_2$ have normal bundles $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Step 2. Let $X_2 \to X_1$ denote the blow up of $\hat{m}_1$ and $\hat{m}_2$ with exceptional divisors F_1 and F_2. We have

$$(F_1)^3 = (F_2)^3 = 2$$

from the normal bundle computation. Let $\tilde{E}$ and $\tilde{E}'$ denote the proper transforms of $\hat{E}$ and $\hat{E}'$.

Step 3. Let g denote the pull-back of the hyperplane class of X to X_2. We claim that

$$M = g - \tilde{E} - \tilde{E}' - F_1 - F_2$$

is basepoint free. We analyze it on each exceptional divisor.

We can write

$$\mathrm{Pic}(\tilde{E}) = \mathbb{Z}g + \mathbb{Z}e + \mathbb{Z}f_1 + \mathbb{Z}f_2,$$

where $e = -\tilde{E}|\tilde{E}$ and $f_i = F_i|\tilde{E}$, with nonzero intersections

$$g \cdot e = 1, f_1^2 = -1, f_2^2 = -1.$$

We have $\tilde{E}$ is a sextic del Pezzo surface and $M|\tilde{E}$ induces a conic bundle, which is basepoint free. We can write

$$\mathrm{Pic}(F_1) = \mathbb{Z}g + \mathbb{Z}\eta$$

with nonzero intersections $g\eta = 1$. We have

$$M|F_1 = g - 2g + (g + \eta) = \eta.$$

Thus F_1 and F_2 are collapsed to $\mathbb{P}^1$'s in the 'opposite' direction.

Note the further nonvanishing intersection numbers

$$g(\tilde{E})^2 = g(\tilde{E}')^2 = g(F_1)^2 = g(F_2)^2 = -1$$

and

$$\tilde{E}F_1^2 = \tilde{E}F_2^2 = \tilde{E}'F_1^2 = \tilde{E}'F_2^2 = -1.$$

Thus we have

$$(g - \tilde{E} - \tilde{E}' - F_1 - F_2)^3 = 4 - 3 - 3 - 3 - 3 + 3 + 3 + 3 + 3 - 2 - 2 = 0.$$

Step 4. We interpret this map as flopping $\tilde{m}_1$ and $\tilde{m}_2$, to get

$$\varphi' : X_1' \to \mathbb{P}^1.$$

The corresponding lines m_1 and m_2 in our pencil of hyperplane sections are contracted, yielding fibers that are sextic del Pezzo surfaces.

Step 5. Since

$$\chi(X_1') = \chi(X_1) = 4 - 4 + 2 + 2 = 4,$$

assuming φ' is degenerate along fibers with one ordinary double point, the degeneracy locus $\Delta \subset \mathbb{P}^1$ satisfies

$$4 = (2 - |\Delta|)6 + 5|\Delta| \text{ whence } |\Delta| = 8.$$

Our fibration thus has eight singular fibers.

Geometric description. The Galois representation associated with φ'

$$\rho : \mathrm{Gal}(k(\mathbb{P}^1)) \to \mathfrak{S}_2 \times \mathfrak{S}_3$$

may be interpreted as follows. The distinguished quadratic extension associated to the blow-up realizations over $\mathbb{P}^2$ is induced by L/k, the field of definition of ℓ and ℓ'.

We analyze conics in the fibers of φ'. Their proper transforms $Z \subset X$ are incident to ℓ and ℓ' but disjoint from m_1 and m_2. They are parametrized by a surface T fibered in conics over a genus-two curve C', which admits a morphism

$$C' \to C \hookrightarrow \mathrm{Pic}^1(C)$$

and a triple cover $C' \to \mathbb{P}^1$ with eight degenerate fibers. It follows that $C' \simeq C$ and the latter curve admits a degree-three morphism to $\mathbb{P}^1$.

7.2. **Application to rationality.** Retain the set-up of Construction 4, which yields a fibration $\varpi : Y \to \mathbb{P}^1$ in sextic del Pezzo surfaces where Y is birational to X over k. The generic fiber of ϖ is rational over $k(\mathbb{P}^1)$ if and only if ϖ admits a section. Indeed, a sextic del Pezzo surface over a field is rational whenever it admits a point [Man66, Cor. 1 to Thm. 3.10]. It follows that X is also rational over k.

However, such a section is the proper transform of a curve $R \subset X$ of odd degree – the hyperplanes in the pencil meet the section in one point outside $\{\ell, m_1, \ell', m_2\}$ and R meets these lines in pairs of conjugate points. Assuming that R is smooth, Theorem 14 implies that X actually contains a line (cf. Question 11).

8. Rational curves in higher degrees

In this section we discuss the geometry of spaces of rational curves. We have already described the structure of lines, conics, and twisted cubics on a smooth complete intersection $X \subset \mathbb{P}^5$ of two quadrics. We now describe the rational normal quartic curves

$$R \subset X_{\bar{k}}$$

by making reference to a rational parametrization

$$\rho : \mathbb{P}^3 \dashrightarrow X_{\bar{p}},$$

blowing up $C \subset \mathbb{P}^3$ realized as a $(2,3)$ divisor in a quadric surface $Q = \{F = 0\} \subset \mathbb{P}^3$ (see Section 3). The preimage $\rho^{-1}(R)$ is also a rational quartic curve R' with

$$R' \cap C = \{c_1, \dots, c_8\}.$$

Thus we obtain a rational map

$$M_0(X, 4) \dashrightarrow \mathrm{Sym}^8(C).$$

We claim this is generically finite of degree four. Indeed, we write

$$I_{c_1,\dots,c_8}(2) = \langle F, G \rangle$$

and we have an elliptic quartic curve

$$E = \{F = G = 0\}$$

and computing intersections in Q yields

$$(E \cap C)_Q = \{c_1, \dots, c_8, c_9, c_{10}\}.$$

Each R' arising as above sits in a member of the pencil

$$Q' \in \{sF + tG = 0\}$$

as a divisor of bidegree $(1,3)$. Rulings of quadrics in this pencil are parametrized by E, in which we have the equation

$$f_1 + 3f_2 | E = \mathcal{O}_E(c_1 + \dots + c_8)$$

or equivalently

$$2f_2 = \mathcal{O}_E(c_1 + \dots + c_8)(-1).$$

This has four solutions with the structure of a principal homogeneous space over $E[2]$.

The image of the proper transform of Q' is obtained in two steps

— blow up $c_1, \dots, c_{10}$;

— pinch along the E using the degree two morphism

$$E \to \mathbb{P}^1$$

induced by the ruling of Q contracted by ρ.

This is obtained by taking a quadric hypersurface section

$$\Sigma \subset X$$

containing R and double along ℓ – these form a linear system of dimension

$$21 - 2 - 9 - 3 - 6 = 1.$$

We are using the fact that $N_{\ell/X} = \mathcal{O}_{\mathbb{P}^1}^2$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ to compute the conditions imposed by insisting that the quadric is double along ℓ. Thus Σ is uniquely determined by ℓ and R.

9. Irrationality via specialization of birational types

9.1. The toric example. First, we work over an algebraically closed field.

Let X be toric, isomorphic to

$$x_0x_1 - x_2x_3 = x_2x_3 - x_4x_5 = 0,$$

with ordinary singularities at the coordinate axes. We may realize X as the image of the rational map

$$\begin{gathered} \mathbb{P}^3 \overset{j}{\dashrightarrow} X \subset \mathbb{P}^5 \\ [y_0, y_1, y_2, y_3] \mapsto [y_0y_1, y_2y_3, y_0y_2, y_1y_3, y_0y_3, y_1y_2] \end{gathered}$$

that blows up the coordinate points p_1, p_2, p_3, p_4 and collapses the lines $\ell_{12}, \dots, \ell_{34}$ joining them to singularities. Let $\beta : \widetilde{X} \to X$ denote the resolution obtained by blowing up the coordinate axes, or equivalently, blowing up $p_1, \dots, p_4$ and then the proper transforms of $\ell_{12}, \dots, \ell_{34}$.

Consider the the graph X' of the standard Cremona transformation

$$\begin{gathered} \iota : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \\ [y_0, y_1, y_2, y_3] \mapsto [1/y_0, 1/y_1, 1/y_2, 1/y_3], \end{gathered}$$

whose indeterminacy is given by the scheme

$$\{y_1y_2y_3 = y_0y_2y_3 = y_0y_1y_3 = y_0y_1y_2 = 0\}.$$

Letting z_0, z_1, z_2, z_3 be projective coordinates on the target projective space, we have

$$X' = \{y_0z_0 = y_1z_1 = y_2z_2 = y_3z_3\} \subset \mathbb{P}^3 \times \mathbb{P}^3.$$

It has ordinary singularities at 12 points

$$([1, 0, 0, 0], [0, 1, 0, 0]), ([0, 1, 0, 0], [1, 0, 0, 0]), \dots,$$

with three over each of the coordinate axes in the $\mathbb{P}^3$ factors.

We have a diagram

$$\begin{tikzcd}
 & X'' \arrow[dl] \arrow[ddr] & \\
\widehat{X} \arrow[d] \arrow[drr] & & \\
\widetilde{X} \arrow[dl] \arrow[dr] & & X' \\
\mathbb{P}^3 & X &
\end{tikzcd}$$

with arrows as follows:

— $\widehat{X} \to \widetilde{X}$ blows up the proper transforms of the lines $\ell_{12}, \dots, \ell_{34}$;

— $X'' \to X'$ blows up the 12 ordinary double points;

— $X'' \to \widehat{X}$ blows up the proper transforms of the 12 lines $L_{i;jk}$ in the exceptional divisors of $\widetilde{X} \to \mathbb{P}^3$ connecting the proper transforms of ℓ_{ij} and ℓ_{ik};

— $\widehat{X} \to X'$ contracts the proper transform of the $L_{i;jk}$.

In addition to the toric action, there is an action of a group

$$H \simeq \mathfrak{S}_4 \times \mathfrak{S}_2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3,$$

where the semidirect product is by permutation of the factors. The $\mathfrak{S}_4$ symmetry is the permutation of the coordinates on $\mathbb{P}^3$; the $\mathfrak{S}_2$ symmetry is induced by the standard Cremona transformation on $\mathbb{P}^3$. Thus the action of H on X' (and thus its resolution X'') is clear from the notation. The $\mathfrak{S}_4$ action is clearly regular on X as well; the regularity of the Cremona involution on X (and thus $\widetilde{X}$) reflects the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{P}^3 \arrow[r, dashed, "j"] \arrow[d, dashed, "\iota"'] & X \arrow[d, "\phi"] \\
\mathbb{P}^3 \arrow[r, dashed, "j"] & X
\end{tikzcd}$$

Here ι is the Cremona involution, j the birational map between $\mathbb{P}^3$ and X, and ϕ is the linear transformation

$$\phi(x_0, x_1, x_2, x_3, x_4, x_5) = (x_1, x_0, x_3, x_2, x_5, x_4)$$

that preserves X.

9.2. **The variety of lines.**

Proposition 22. *Consider the variety of lines $F_1(X)$.*

— $\deg(F_1(X)) = 32$ *;*

— *it has eight components isomorphic to* $\mathbb{P}^2$*:*

$$P_1, P_2, P_3, P_4 \quad \text{and} \quad P_{123}, P_{124}, P_{134}, P_{234};$$

— *it has four components isomorphic to the degree six del Pezzo surface:* $S_1, \ldots, S_4$ *;*
— S_1 *meets* $P_2, P_{123}, P_3, P_{134}, P_4, P_{124}$ *in a hexagonal anticanonical cycle;*
— P_1 *meets* S_2, S_3, S_4 *in a triangular anticanonical cycle;*
— P_{123} *meets* S_1, S_2, S_3 *in a triangular anticanonical cycle.*

Proof. First, note that X contains eight planes

$$\{x_0 = x_2 = x_4 = 0\}, \{x_0 = x_2 = x_5 = 0\}, \ldots, \{x_1 = x_3 = x_5 = 0\},$$

each containing three of the ordinary double points. The lines in these planes are the eight $\mathbb{P}^2$'s in $F_1(X)$. These may be interpreted via j as:
— P_i – corresponding to lines in the exceptional divisors over the p_i;
— P_{ijk} – corresponding to conics in $\mathbb{P}^3$ containing p_i, p_j, p_k.
A direct computation shows that F_1 is singular along those lines meeting the six singularities. These form 24 lines in $F_1(X)$, four associated with each singularity and three lying on each of the eight planes. We write
— S_i – corresponding to the lines in $\mathbb{P}^3$ through p_i.
These are parametrized by the blow-up of $\mathbb{P}^2$ at through noncollinear points, a sextic del Pezzo surface, cf. [AK77, Th. 1.10]. Note that the distinguished hexagon of lines in S_1 coincides with its intersections with the six planes indicated.

On the other hand, a computation shows that $F_1(X)$ is smooth of dimension two at all lines *not* meeting to the singularities. In particular, $F_1(X)$ has pure dimension two.

The statement on the degree of $F_1(X)$ can be obtained via Schubert calculus on the Grassmannian $\mathrm{Gr}(2,6)$. It reflects the fact that $\mathcal{O}_{\mathrm{Gr}(2,6)}(1)|F_1(X)$ is four times the principal polarization.

It remains to show that the enumerated lines cover all the lines on X. The sum of the degrees of the S_i equals 24; the sum of the degrees of the $\mathbb{P}^2$ components is 8. Thus we conclude $F_1(X)$ is the union of these 12 surfaces. □

Our notation is chosen compatibly with the action of $\mathfrak{S}_4 \times \mathfrak{S}_2$. The first factor permutes the indices. The second interchanges P_i and P_{jkl} where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

9.3. **Galois cohomology.** Rationality of a 3-dimensional torus T is governed by the Galois action on its lattice of characters $\mathfrak{X}^*(T)$; the action factors through a subgroup $G \subset H$, where

$$H \simeq \mathfrak{S}_4 \times \mathfrak{S}_2 \subset \mathrm{GL}_3(\mathbb{Z}) = \mathrm{Aut}(\mathfrak{X}^*(T)).$$

The main result of [Kun87] is:

— A 3-dimensional torus is nonrational if and only if G contains a subgroup $U_1 \simeq \mathfrak{S}_2 \times \mathfrak{S}_2$, see [Kun87, Section 3].

In particular, all T with $G \subset \mathfrak{S}_4$ are rational [Kun87, Lemma 1]. Explicitly, the generators of H are given as matrices

$$a := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, b := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

These satisfy the relations

$$a^4 = b^2 = (ab)^3 = 1, c^2 = 1,$$

where c is central. The generators of U_1 are given in [Kun87, Thm. 1], they are (modulo a change of basis)

$$(a^2bc)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad bc = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

A full list of nonrational three-dimensional tori is in [Kun87, Thm. 1].

Remark 23. Assume that the Galois group acts transitively on the set of 4 coordinate points $p_1, \ldots, p_4 \in \mathbb{P}^3$, giving rise to X as above. Then X is rational but does not contain lines defined over k, by the description in Proposition 22.

Remark 24. The group U_1 may be realized as a subgroup of H, e.g., by taking c as the generator of the second factor, $a = (1234)$ and $b = (12)$. This group does not fix any of the irreducible components of $F_1(X)$. On the other hand, the Cremona involution acts on the S_i by taking inverses and thus stabilizes points not on hexagon, i.e., lines on X not meeting singularities.

9.4. **Construction of nonrational examples.** Since the action of H is regular on the toric variety X, we can use it to obtain a twisted model of ${}^{\rho}X$ and the corresponding torus over a nonclosed field, provided there is a representation $\rho : \mathrm{Gal}(k) \to H$. For example, let K be a cubic extension over k and L and quadratic extension over K; assume that the Galois closures of K and L over k have Galois groups $\mathfrak{S}_3$ and H respectively. Let z be an indeterminate in K and $\theta \in K$ a primitive element; the equations

$$\mathrm{Tr}_{K/k}(z) = \mathrm{Tr}_{K/k}(\theta z) = 0$$

are independent. Let x be an indeterminate in L satisfying $\mathrm{Nm}_{L/K}(x) = z$. The equations

$$\mathrm{Tr}_{K/k}(\mathrm{Nm}_{L/K}(x)) = \mathrm{Tr}_{K/k}(\theta\,\mathrm{Nm}_{L/K}(x)) = 0$$

are homogeneous and define a locus on $\mathbb{P}(\mathbf{R}_{L/k}\mathbb{A}^1) \simeq \mathbb{P}^5$ geometrically isomorphic to X.

The resulting variety is one of the minimal nonrational toric threefolds considered in [Vos01, Ex. 6.2.1].

9.5. **Specialization of birational types.** We will use a special case of [KT19, Th. 16]:

Theorem 25. *Let k be a field, $K = k((\tau))$, $B = \mathrm{Spec}(k[[\tau]])$, and $\mathcal{X} \to B$ a flat projective morphism from a scheme smooth over k with closed fiber $\mathcal{X}_0$. We assume that*

- *the generic fiber $\mathcal{X}_\tau$ is smooth and rational (resp. stably rational) over K;*
- *$\mathcal{X}_0$ is geometrically irreducible and reduced;*
- *$\mathcal{X}_0$ is singular along a subscheme Y that is smooth over k;*
- *the blowup $\beta : \mathrm{Bl}_Y(\mathcal{X}_0) \to \mathcal{X}_0$ resolves the singularities of $\mathcal{X}_0$;*
- *the exceptional divisor D of β is smooth over Y and rational over $k(Y)$.*

Then $\mathcal{X}_0$ is rational (resp. stably rational) over k.

Our assumptions mean that the proper transform of $\mathcal{X}_0$ meets the exceptional divisor of $\mathrm{Bl}_Y(\mathcal{X})$ transversally.

Proof. It suffices to verify that the pair $(\mathcal{X}, \mathcal{X}_0)$ has B-rational singularities. Our argument is an extension of [KT19, Ex. 13].

Let $\mathcal{X}' = \mathrm{Bl}_Y(\mathcal{X})$ with exceptional divisor $E \simeq \mathbb{P}(N_{Y/\mathcal{X}})$; $\mathcal{X}'_0$ is a normal crossings divisor and write D for the intersection of E with the proper transform of $\mathcal{X}'_0$, i.e., the exceptional locus of β. Using [KT19, §4], we write

$$\partial_{\mathcal{X}'_0}(\mathcal{X}') = [E \to \mathcal{X}'_0] + [\mathrm{Bl}_Y(\mathcal{X}_0) \to \mathcal{X}'_0] - [D \times \mathbb{A}^1 \to \mathcal{X}'_0] \tag{3}$$

$$= [\mathrm{Bl}_Y(\mathcal{X}_0) \to \mathcal{X}'_0], \tag{4}$$

where the cancellation comes from definition of the Burnside group and the fact that E and D are both rational over $k(Y)$. We conclude that

$$\partial_{\mathcal{X}_0}(\mathcal{X}) = [\mathcal{X}_0^{\mathrm{smooth}} \hookrightarrow \mathcal{X}_0],$$

the desired condition on the singularities. □

We apply this to the examples of Section 9.4:

Theorem 26. *Let k be a function field of a complex curve, $\mathcal{X}_0$ a nonrational toric complete intersection of two quadrics over k, and $\mathcal{X} \to B$ a deformation of $\mathcal{X}_0$ with smooth total space. Then the generic fiber of $\mathcal{X}_\tau$ is not (stably) rational over $k((\tau))$.*

Note that we can easily choose the deformation so that $\mathcal{X} \to B$ admits sections, e.g., by choosing the deformed quadratic equations to vanish at a smooth point of $\mathcal{X}_0$. This construction yields examples over function fields of complex surfaces that are irrational yet admit rational points.

Corollary 27. *There exist examples of smooth complete intersections of two quadrics in $\mathbb{P}^5$ over $\mathbb{C}(t,\tau)$ that are not rational (or stably rational) but admit rational points.*

Proof. Here $\mathcal{X}_0$ is the twisted form of our toric complete intersection of two quadrics. We take Y to be its six ordinary double points. A deformation that is versal for these singularities has smooth total space. Thus $D \to Y$ is a smooth quadric surface fibration over a complex curve. The Tsen-Lang theorem shows that any such fibration is rational, so the hypotheses of Theorem 25 are satisfied. □

Remark 28. Toric degenerations of Fano threefolds have attracted attention in connection with mirror symmetry and the theory of Landau-Ginzburg models. We expect that the technique presented here will permit the construction of numerous nonrational but geometrically rational smooth Fano threefolds.

10. Irrationality via decomposition of the diagonal

Here we establish the following result, which answers a question of Colliot-Thélène from 2005:

Theorem 29. *There exist smooth complete intersections of two quadrics $X \subset \mathbb{P}^5$ over the field $k = \mathbb{C}(t)$ that fail to be stably rational over k.*

The remainder of this section is devoted to the proof. The idea is to view the quadric surface bundle over $\mathbb{P}^1$, over $k = \mathbb{C}(t)$, as a quadric surface bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, over $\mathbb{C}$. If X were rational over k then the associated fourfold would be rational over $\mathbb{C}$. The quadric surface bundle degenerates along a curve of bidegree $(6,6)$. To this we will apply the specialization method of [Voi15] or [CTP16], as in [HPT18]. This entails two steps:

- — exhibit a specialization with nontrivial unramified Brauer group and
- — show that the singularities of the specialization are mild.

Throughout, we make reference to the geometric analysis in Section 4.2.

10.1. **The case of function fields.** We now assume that $k = \mathbb{C}(t)$. We have $X(k) \neq \emptyset$ by the Tsen-Lang Theorem; indeed, rational points are Zariski dense as X is rationally connected over the function field of a curve. (See also the unirationality results in [CTSSD87a, Prop. 2.3].)

We write down representative models for projective equations of models over $\mathbb{P}^1$. The complete intersection of two quadrics sits

$$\mathcal{X} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}),$$

where the first summand arises from the section associated with $x \in X(k)$ and the first four summands arise from the tangent space T_xX. Fix $\{x_0\}$, $\{x_1, x_2, x_3\}$, and $\{x_4, x_5\}$ to be weighted variables corresponding to the summands. We write equations $F = G = 0$ where

$$F = 2x_0x_4 + c_{11}x_1^2 + \cdots + 2l_{14}x_1x_4 + \cdots + q_{44}x_4^2 + 2q_{45}x_4x_5 + q_{55}x_5^2$$

and

$$G = 2x_0x_5 + d_{11}x_1^2 + \cdots + 2m_{14}x_1x_4 + \cdots + r_{44}x_4^2 + 2r_{45}x_4x_5 + r_{55}x_5^2$$

with the c_{ij} and d_{ij} of degree d, the l_{ij} and m_{ij} of degree $d+1$ and the q_{ij} and r_{ij} of degree $d+2$. The projection from T_xX is given by $[x_4, x_5]$; write $x_5 = tx_4$ which we'll take as the second grading. The elimination is obtained by taking $tF - G$, substituting $x_5 = tx_4$, and then re-homogenizing the t variable. Thus we obtain a quadric surface bundle

$$\mathcal{X}' \subset \mathbb{P}(\mathcal{O}^3_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)) \to \mathbb{P}^1 \times \mathbb{P}^1$$

with equation

$$L_{11}x_1^2 + 2L_{12}x_1x_2 + \cdots + 2Q_{14}x_1x_4 + \cdots + C_{44}x_4^2 = 0$$

associated with the symmetric matrix

$$A = \begin{pmatrix} L_{11} & L_{12} & L_{13} & Q_{14} \\ L_{12} & L_{22} & L_{23} & Q_{24} \\ L_{13} & L_{23} & L_{33} & Q_{34} \\ Q_{14} & Q_{24} & Q_{34} & C_{44} \end{pmatrix}$$

where the L_{ij} are bidegree $(d, 1)$, the Q_{ij} bidegree $(d+1, 2)$ and C_{44} bidegree $(d+2, 3)$. The degeneracy locus $D \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(4d+2, 6)$.

Conversely, given a symmetric matrix A of forms with the prescribed bidegrees, we may reverse the process to recover $\mathcal{X}$ from $\mathcal{X}'$. The construction depends on

$$6 \cdot 2(d+1) + 3 \cdot 3(d+2) + 4(d+3) - (9 + 1 + 3 \cdot 4) - 6 = 25d + 14$$

parameters. However, a generic form of bidegree $(4d+2,6)$ depends on

$$7(4d+3)-(1+6)=28d+14$$

parameters.

The model of C

$$\mathcal{C}\to\mathbb{P}^1\times\mathbb{P}^1$$

is a double cover branched over a degree-six multisection. The unramified element $\alpha\in\mathrm{Br}(\mathcal{C})[2]$ governs the rationality of the generic fiber.

If $d=0$ then $\mathcal{C}\to\mathbb{P}^1\times\mathbb{P}^1$ is a double cover branched over a curve of bidegree $(2,6)$ which has trivial Brauer group; here the fibration q must have a section. The case $d=1$ yields a bidegree-$(6,6)$ degeneracy locus – we focus on this. However, note that the determinantal condition gives a codimension **three** locus in the parameter space of $(6,6)$ forms.

10.2. **Application to Theorem 29.** Specialize to

$$2E\cup F_1\cup F_1'\cup F_2\cup F_2'$$

where E is bidegree $(2,2)$, F_1 and F_1' are of bidegree $(1,0)$, F_2 and F_2' are of bidegree, and all the fibers are tangent to E. Such configurations depend on **two** parameters.

We put these in the prescribed determinantal form. First write

$$F_1=\{y_1=0\},F_1'=\{z_1=0\},F_2=\{y_2=0\},F_2'=\{z_2=0\}$$

and set

$$E=\{g(y_1,z_1;y_2,z_2)=0\},\quad g\in\mathbb{C}[y_1,z_1;y_2,z_2]_{(2,2)},$$

with g chosen such that E has the tangencies specified above. We set

$$A=\begin{pmatrix} y_1z_1 & 0 & 0 & 0\\ 0 & y_1z_2 & 0 & 0\\ 0 & 0 & y_2z_1 & 0\\ 0 & 0 & 0 & y_2z_2g\end{pmatrix}.$$

Let $\mathcal{C}_0\to\mathbb{P}^1\times\mathbb{P}^1$ denote the quadric surface bundle associated with this quadratic form.

Pirutka's technique [Pir18, Th. 3.17] shows that $\mathcal{C}_0$ has unramified cohomology, arising from the pull-back of the class of $\mathrm{Br}(\mathbb{C}(\mathbb{P}^1\times\mathbb{P}^1))$ ramified along

$$F_1\cup F_1'\cup F_2\cup F_2'$$

with higher-order ramification at the four points of intersection. While $\mathcal{C}_0$ is singular, it admits a universally CH_0-trivial resolution of singularities following the procedure in [HPT18, §5]. Indeed, the configuration of singularities here is étale-locally equivalent at each point to a stratum of the configuration considered in [HPT18]. There we had a cycle of three rational curves, each with multiplicity two and simply tangent to the smooth (multiplicity-one) component of the degeneracy locus at a point; here we have a cycle of four rational curves with the same tangency to E.

Now suppose

$$\mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1$$

is branched over a *very general* $(6,6)$ curve. An application of [Voi15] or [CTP16], as in [HPT18], shows that for such $\mathcal{C}$ the corresponding fourfold lacks an integral decomposition of the diagonal and thus fails to be stably rational.

11. The real case

In this section, we settle the rationality problem over the real numbers.

11.1. **Normal forms for pencils of quadrics.** Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of two quadrics over $\mathbb{R}$.

Write $X = \{Q_0 = Q_1 = 0\}$ with associated pencil

$$P = \{s_0 Q_0 + s_1 Q_1\} \subset \mathbb{P}^n \times \mathbb{P}^1$$

and binary form of degree $n+1$

$$F(s_0, s_1) = \det(s_0 Q_0 + s_1 Q_1) \in \mathbb{R}[s_0, s_1].$$

Note that $F(s_0, s_1)$ has no multiple roots because X is smooth. We write a normal form under linear changes of coordinates, following [Tho91, Th. 2], expressed as an orthogonal sum of matrix blocks:

— associated with nonreal roots $a + bi$ of $F(1, \rho)$ the block

$$\begin{pmatrix} b & a-\rho \\ a-\rho & -b \end{pmatrix}; \tag{5}$$

— associated with real roots a of $F(1, \rho)$ the block

$$\begin{pmatrix} \pm(a-\rho) \end{pmatrix}.$$

Provided that $s_0 \nmid F(s_0, s_1)$, i.e., Q_1 is nondegenerate, these are the only blocks that may arise.

11.2. **Isotopy classification.** We follow [Kra18, §1].

Consider the degree-two covering $S^1 \to \mathbb{P}^1$ obtained by realizing

$$S^1 = \{(s_0, s_1) : s_0^2 + s_1^2 = 1\}.$$

Let

$$\widetilde{P} := P \times_{\mathbb{P}^1} S^1 \to P$$

denote the associated covering. The advantage of passing to S^1 is that the equation $s_0 Q_0 + s_1 Q_1$ lifts to a well-defined family of quadratic forms over S^1. (Over $\mathbb{P}^1$ the forms are defined up to sign.)

The *positive index function* $I^+ : S^1 \to \mathbb{Z}$ is defined as the number of positive eigenvalues of the associated form $s_0 Q_0 + s_1 Q_1$. It satisfies the following:

— I^+ is piecewise constant with $2k \leq 2n+2$ jumps of height ± 1;

— $I^+(-s_0, -s_1) = n + 1 - I^+(s_0, s_1)$ provided (s_0, s_1) is not one of the points of discontinuty.

We are using the fact that X is smooth which means $F(s_0, s_1)$ has no multiple roots so a quadric drops rank at $(s_0, s_1) \in S^1$ by at most one.

We extract a combinatorial invariant: A point of discontinuity for I^+ is *positive* if I^+ increases as we cross, moving counter-clockwise. Each positive point of discontinuity matches with its antipodal negative point of discontinuity. Observe that

— $k = 0$ only when n is odd and $F(s_0, s_1)$ has no real nontrivial roots, i.e., our pencil is a sum of $\frac{n+1}{2}$ two-dimensional real blocks (5);

— when $k \neq 0$ we have $k \in \{1, \ldots, n+1\}$ and $k \equiv n + 1 \pmod 2$.

This reflects the fact that the number of real roots of a polynomial $p(s) \in \mathbb{R}[s]$ has the same parity as $\deg(p)$.

Decompose

$$k = k_1 + \cdots + k_{2s+1}, k_i \in \mathbb{N},$$

where each k_i represents that number of consecutive positive points of discontinuity, with the indices increasing as we move counter-clockwise around S^1. The length of the decomposition is odd because the sign of the quadratic form is reversed under the antipodal involution of S^1. We impose an equivalence relation, identifying decompositions related by cyclic permutations or reversing all the terms.

Theorem 30. [Kra18] *Isotopy classes of smooth complete intersections of two quadrics in $\mathbb{P}^n$ correspond to equivalence classes of odd decompositions*

$$k_1 + \cdots + k_{2s+1} = k \leq n + 1$$

where k is a non-negative integer with parity equal to $n + 1$ (allowing $k = 0$ when n is odd).

The isotopy class corresponding to the trivial decomposition of $k = n+1$ is the case where $X(\mathbb{R}) = \emptyset$.

Example 31.

($n = 2$) There are three isotopy classes corresponding to
$$(1), (3), (1, 1, 1).$$

($n = 3$) There are four isotopy classes corresponding to
$$(0), (2), (2, 1, 1), (4).$$

($n = 4$) There are seven isotopy classes corresponding to
$$(1), (3), (1, 1, 1), (5), (3, 1, 1), (2, 2, 1), (1, 1, 1, 1, 1).$$

($n = 5$) There are nine isotopy classes corresponding to
$$(0), (2), (4), (2, 1, 1), (6), (4, 1, 1), (3, 2, 1), (2, 2, 2), (2, 1, 1, 1, 1).$$

11.3. **Existence of maximal linear subspaces.** Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of two quadrics and write
$$\dim(X) = n - 2 = 2m + 1 \text{ or } 2m,$$
depending on the parity of n. There exist linear subspaces in $X_{\mathbb{C}}$ of dimension $\leq m$. The Lefschetz hyperplane theorem shows that larger dimensional subspaces are not possible.

Theorem 32. [Kra18, §2] *The variety X contains a real linear subspace of maximal dimension m provided that*
$$m + 1 \leq I^{+}(s_0, s_1) \leq m + 3$$
for each $(s_0, s_1) \in S^1$.

Example 33. For threefolds $X \subset \mathbb{P}^5$ we have the variety of lines $F_1(X)$ admits real points for the decompositions
$$(0), (2), (2, 1, 1), (2, 2, 2), (2, 1, 1, 1, 1).$$
These correspond to sequences of nondegenerate signatures on S^1:
$$(3,3)$$
$$(2, 4)(3, 3)(4, 2)(3, 3)$$
$$(2, 4)(3, 3)(4, 2)(3, 3)(4, 2)(3, 3)(2, 4)(3, 3)$$
$$(2, 4)(3, 3)(4, 2)(3, 3)(2, 4)(3, 3)(4, 2)(3, 3)(2, 4)(3, 3)(4, 2)(3, 3)$$
$$(2, 4)(3, 3)(4, 2)(3, 3)(4, 2)(3, 3)(4, 2)(3, 3)(2, 4)(3, 3)(2, 4)(3, 3)$$
Here we omit the points of discontinuity.

In particular, decompositions (4), $(4, 1, 1)$ and $(3, 2, 1)$ admit real points but no real lines.

11.4. **Topological types in the threefold case.** See [Kra18, Th. 5.4] for topological types of complete intersections of two quadrics $X \subset \mathbb{P}^5$ over $\mathbb{R}$. We list the types that admit real points but not real lines:

— (4): $X(\mathbb{R})$ is diffeomorphic to the sphere S^3;

— $(4,1,1)$: $X(\mathbb{R})$ is diffeomorphic to the disjoint union $S^3 \sqcup S^3$;

— $(3,2,1)$: $X(\mathbb{R})$ is diffeomorphic to the product of spheres $S^1 \times S^2$.

In particular, $X(\mathbb{R})$ is disconnected in case $(4,1,1)$ thus irrational over $\mathbb{R}$. It is not *a priori* clear whether cases (4) or $(3,2,1)$ are rational; we explore this in Theorem 36. Certainly there is no topological obstruction to realizing S^3 with a rational threefold

$$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2\} \subset \mathbb{P}^4.$$

And there is no topological obstruction for $(3,2,1)$ – case (2) yields examples that are rational but diffeomorphic to $S^1 \times S^2$. See [Kol98] for more discussion.

11.5. **A refinement of the intermediate Jacobian criterion.** Let Y be a smooth projective geometrically rational threefold over $\mathbb{R}$. If Y is rational over $\mathbb{R}$ then its intermediate Jacobian is isomorphic to the Jacobian of a smooth projective (not necessarily irreducible) curve D over $\mathbb{R}$ [BW20, Cor. 2.8]

$$\mathrm{IJ}(Y) \simeq \mathrm{J}(D).$$

Fix a family of codimension-two algebraic cycles

$$\mathcal{Z} \subset Y \times B$$

flat over the base B. Assume that:

— given $b_0 \in B(\mathbb{C})$, the morphism

$$\begin{array}{rcl} B_{\mathbb{C}} & \to & \mathrm{IJ}(Y_{\mathbb{C}}) \\ b & \mapsto & [\mathcal{Z}_b - \mathcal{Z}_{b_0}] \end{array}$$

is an isomorphism;

— the Albanese $\mathrm{Alb}(B) \simeq \mathrm{J}(D)$ over $\mathbb{R}$.

Under the first assumption, B carries the structure of a principal homogeneous space over an abelian variety, which is isomorphic to $\mathrm{J}(D)$ by the second assumption.

Proposition 34. *Retain the assumptions above. Assume that*

— *Y is rational over $\mathbb{R}$;*

— $\mathrm{IJ}(Y)$ *admits no factors as a principally polarized abelian variety that are elliptic or the product of two complex conjugate elliptic curves.*

Then there exist principally polarized factors J_i of $\mathrm{IJ}(Y)$*, isoomorphisms $\eta_i : \mathrm{J}(D_i) \to J_i$, and degrees d_i such that*

$$[B] = \sum_i \eta_{i*}[\mathrm{CH}^1(D_i)_{d_i}].$$

Elliptic factors make the bookkeeping more complex: There exist nonisomorphic genus-one curves with isomorphic Jacobians. By 'complex conjugate elliptic curves' we mean that complex conjugation interchanges the two factors/components.

Lemma 35. *Each irreducible factor of* $\mathrm{IJ}(Y)$ *is elliptic, a product of two complex conjugate elliptic curves, or of the form $J_i \simeq \mathrm{J}(D_i)$, where D_i is an irreducible smooth projective curve over $\mathbb{R}$ of arithmetic genus at least two. In the last case, D_i is uniquely determined by its Jacobian.*

This follows from the Torelli Theorem over nonclosed fields [Lau01].

Proof. The birational map $\mathbb{P}^3 \dashrightarrow Y$ admits a factorization as iterated blow-ups and blow-downs along smooth centers [AKMW02]:

$$\begin{tikzcd}
Z_0 \arrow[ddd] \arrow[dr] & & Z_1 \arrow[dl] & \cdots & Z_{n-1} \arrow[dr] & & Z_n \arrow[dl] \arrow[ddd] \\
& Z_{0,1} & \cdots & \cdots & \cdots & Z_{n-1,n} & \\
& & & & & & \\
\mathbb{P}^3 \arrow[rrrrrr, dashed] & & & & & & Y
\end{tikzcd}$$

For notational simplicity we will assume that $n = 0$ and write $Z = Z_0 = Z_n$. There is no harm in doing this as the general case follows by iterating the argument below.

We apply the blow-up formula for a smooth curve A in a smooth threefold W [Ful98, §6.7]

$$\mathrm{CH}^2(\mathrm{Bl}_A(W)) \simeq \mathrm{CH}^1(A) \oplus \mathrm{CH}^2(W).$$

This is compatible with algebraic families: A family of codimension-two cycles on $\mathrm{Bl}_A(W)$ induces a morphism from the base of the family to $\mathrm{CH}^1(A)$.

Thus the intermediate Jacobian of Z in our factorization is a direct product of the Jacobians of the irreducible curves $A_1, \dots, A_N$ that were blown up at various stages and

$$\mathrm{CH}^2(Z) = \mathbb{Z}^m \oplus (\oplus_{i=1}^N \mathrm{CH}^1(A_i)).$$

We organize the centers based on which survive in Y. An essential center is a connected curve of positive genus whose Jacobian contributes as a factor of $\mathrm{IJ}(Y)$. There may be inessential centers, e.g., positive genus curves which are blown-up and then blown-down at a subsequent step in the factorization. However, these fail to contribute to the Chow group of Y.

Decompose our principally polarized abelian variety

$$\mathrm{J}(D) \simeq \mathrm{IJ}(Y) = \prod_{j=1}^{r} \mathrm{J}(D_j)^{n_j} \times \mathrm{J}(E)$$

where the D_j are distinct and of genus at least two, and E includes the genus one components. Under our assumptions, the last factor vanishes. A principal homogeneous space over such a product is a product of principal homogenous spaces over the factors. Consider the corresponding factors in $\mathrm{IJ}(Z)$

$$\mathrm{IJ}(Z) \simeq \prod_{j=1}^{r} \mathrm{J}(D_j)^{N_j} \times J$$

where J corresponds to the genus-one factors and the higher-genus factors not among $\{D_1, \dots, D_r\}$.

The blowup $Z \to Y$ implies that each $\mathrm{J}(D_j)^{n_j}$ sits in $\mathrm{J}(D_j)^{N_j}$ with projector

$$\Pi_j : \mathrm{J}(D_j)^{N_j} \to \mathrm{J}(D_j)^{n_j}.$$

The associated contribution to $\mathrm{CH}^2(Y)$ is given by applying Π_j to $\mathrm{CH}^1(D_j)^{N_j}$, regarded as a sum of compatible principal homogeneous spaces over $\mathrm{J}(D_j)^{N_j}$. Interpret Π_j as a matrix with entries endomorphisms of $\mathrm{J}(D_j)$. Since this respects principal polarizations, it takes the shape of a projection onto n_j of the factors, up to isomorphisms of those factors. Reindex with $i = 1, \dots, n_1 + \cdots + n_r$ so that each irreducible factor $\mathrm{J}(D_i) \subset \mathrm{IJ}(Y)$ gets its own index; the D_i need not be distinct. Then the summand of $\mathrm{CH}^2(Y)$ associated with D_i is obtained by applying these isomorphisms to cocycles of the form $\mathrm{CH}^1(D_i)_{d_i}$ for suitable degrees d_i.

Our family of cycles over B gives a morphism of principal homogeneous spaces

$$B \to \prod \eta_{i*}(\mathrm{CH}^1(D_i)_{d_i})$$

that becomes an isomorphism over $\mathbb{C}$. Hence it is an isomorphism over $\mathbb{R}$. □

11.6. **Application to complete intersections of quadrics.** We return to assuming that $X \subset \mathbb{P}^5$ is a smooth complete intersection of two quadrics over $\mathbb{R}$. We focus on the case where rationality remains open, i.e., $X(\mathbb{R}) \neq \emptyset$ but X does not admit a real line. Note that X

automatically admits a conic over $\mathbb{R}$, as the pencil of quadric hypersurfaces containing it admits members of signature $(3,3)$ that contain isotropic planes. Recall from Section 2.3 that the space of such conics is an étale $\mathbb{P}^3$-bundle over the genus two curve C associated with the pencil, which thus admits real points.

We apply Proposition 34 to $B = F_1(X)$ and $\mathcal{Z}$ the universal family of lines on X. The work of Wang [Wan18] (see also [Rei72, Th. 4.8]) shows that $F_1(X)$ is a principal homogeneous space over $\mathrm{J}(C)$, satisfying

$$2[F_1(X)] = [\mathrm{CH}^1(C) = \mathrm{Pic}^1(C)].$$

Since C is smooth of genus two, $\mathrm{J}(C)$ is simple as a principally polarized abelian surface – it cannot be a product of real or complex conjugate elliptic factors, which are associated with nodal stable curves of genus two. Since $C(\mathbb{R}) \neq \emptyset$ we have $2[F_1(X)] = 0$. However, then we would have $F_1(X) \simeq \eta_* \mathrm{CH}^1(D_i)_d$ for some d and some endomorphism $\eta : \mathrm{J}(D_i) \to \mathrm{J}(D_i)$; here D_i is an essential center of $\mathbb{P}^3 \dashrightarrow X$. Again, the Torelli Theorem [Lau01] guarantees that $D_i \simeq C$. We derive a contradiction as

$$F_1(X)(\mathbb{R}) = \emptyset \quad C(\mathbb{R}) \neq \emptyset.$$

We summarize this as follows:

Theorem 36. *Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics over $\mathbb{R}$ not containing a real line. Then X is not rational over $\mathbb{R}$.*

Remark 37. Colliot-Thélène points out that this fails for complete intersections of two quadrics $X \subset \mathbb{P}^4$ over $\mathbb{R}$. For instance, this happens when X is the blowup of two pairs of complex conjugate points on a quadric surface that contains no real lines.

Appendice, par J.-L. Colliot-Thélène

Dans toute cette note, k désigne un corps de caractéristique différente de 2.

A.1. **Quadriques avec une sous-variété de degré impair.** On a le lemme bien connu de T. A. Springer (voir [Lam73, Chap. VII, Thm. 2.3]) :

Lemme A.1. *Si une forme quadratique sur un corps k possède un zéro non trivial sur une extension impaire du corps de base, alors elle possède un zéro non trivial sur k.*

La proposition suivante est aussi bien connue [EKM08, Prop. 68.1].

Proposition A.2. *Soit k un corps. Soit $Q \subset \mathbb{P}^n$, $n \geq 2$, une quadrique lisse déployée, c'est-à-dire définie par une forme quadratique déployée.*

(i) La dimension maximale d'un sous-espace linéaire de Q est $[(n-1)/2]$.

(ii) Considérons l'application image directe $CH_r(Q) \to CH_r(\mathbb{P}^n) = \mathbb{Z}$ entre les groupes de Chow de cycles de dimension r, avec $0 \leq r \leq n-1$. Pour $r \leq [(n-1)/2]$, cette application est surjective. Pour $r > [(n-1)/2]$, son image est $2.\mathbb{Z}$: toute sous-variété intègre de Q de dimension $r > [(n-1)/2]$ est de degré pair.

Il serait surprenant que l'énoncé suivant n'ait pas été déjà établi.

Théorème A.3. *Soit k un corps. Soit $Q \subset \mathbb{P}^n_k$, $n \geq 1$, une quadrique lisse. S'il existe une sous-k-variété géométriquement intègre $W \subset Q \subset \mathbb{P}^n_k$ de dimension r et de degré impair dans $\mathbb{P}^n_k$, alors Q contient un espace linéaire $\mathbb{P}^r_k$.*

Démonstration. Supposons la quadrique donnée par l'annulation d'une forme quadratique $q(x_0, \ldots, x_n)$. Si la forme q est de rang pair et hyperbolique, alors $n+1 = 2d$ et $q = 0$ contient un espace linéaire $\mathbb{P}^d_k$. Si q est de rang $n+1$, avec $n = 2d$ et s'écrit comme somme orthogonale d'une forme quadratique hyperbolique de rang $2d$ et d'une forme de rang 1, alors $q = 0$ contient un espace linéaire $\mathbb{P}^{d-1}_k$. Dans ces deux cas, d'après la proposition A.2, la démonstration est achevée.

Toute k-variété $W \subset \mathbb{P}^n_k$ de degré impair contient des points fermés P de degré $[k(P) : k]$ impair. Si de plus W est géométriquement intègre, alors ses points fermés de degré impair sont denses pour la topologie de Zariski de W. D'après le lemme A.1, l'hypothèse exclut donc que la forme quadratique q soit anisotrope.

On peut donc supposer que $q(x_0, \ldots, x_n)$ s'écrit sous la forme

$$q(x_0, \ldots, x_n) = x_0x_1 + \cdots + x_{2s}x_{2s+1} + g(x_{2s+2}, \ldots, x_n)$$

avec $s \geq 0$ et avec g une forme quadratique anisotrope en au moins 2 variables. Considérons l'application rationnelle de $\mathbb{P}^n_k$ vers $\mathbb{P}^{n-2s-2}$ envoyant $(x_0, \ldots, x_n)$ sur $(x_{2s+2}, \ldots, x_n)$. Cette application est définie hors du fermé défini par

$$(x_{2s+2}, \ldots, x_n) = (0, \ldots, 0).$$

Sa restriction à $W \subset Q$ est donc définie hors du fermé $F \subset W$ défini par ces mêmes équations, donc en dehors du fermé de W défini par

$$(x_{2s+2}, \ldots, x_n) = (0, \ldots, 0)$$

et

$$x_0x_1 + \cdots + x_{2s}x_{2s+1} = 0.$$

Si $F \neq W$, on a alors une application rationnelle de W dans la quadrique anisotrope de $\mathbb{P}^{n-2s-2}$ définie par $g(x_{2s+2}, \ldots, x_n) = 0$. Comme les points fermés de degré impair sont Zariski denses dans W, et que la quadrique anisotrope ci-dessus ne possède pas de point fermé de degré impair par le lemme A.1, ceci est impossible.

On a donc $F = W$. La variété W de dimension r est contenue dans le fermé de $\mathbb{P}^n_k$ défini par $(x_{2s+2}, \ldots, x_n) = (0, \ldots, 0)$, qui est un sous-espace projectif $\mathbb{P}^{2s+1}_k$, et dans la quadrique de cet espace projectif définie par $x_0x_1 + \cdots + x_{2s}x_{2s+1} = 0$. Comme W est de degré impair, d'après la proposition A.2, ceci force $r \leq s$. Ainsi $q = 0$ contient un espace linéaire $\mathbb{P}^r_k$. $\square$

A.2. **Intersection de deux quadriques avec une sous-variété de degré impair.** Le théorème suivant est dû à M. Amer [Ame76]. Le cas $d = 1$ fut établi indépendamment par A. Brumer. Le théorème général est établi de nouveau, en toute caractéristique, dans un tapuscrit de D. Leep [Lee].

Théorème A.4. *Soient f et g deux formes quadratiques en $n+1$ variables sur le corps k. La forme quadratique $f + tg$ s'annule sur un sous-espace linéaire de dimension d de $k(t)^{n+1}$ si et seulement si $f = g = 0$ s'annule sur un espace linéaire de dimension d de k^{n+1}.*

Le cas $n = 5, r = 1$ du théorème suivant est établi, par une autre méthode, dans [HT19c, Thm. 14].

Théorème A.5. *Soit $X \subset \mathbb{P}^n_k$, $n \geq 3$, une intersection complète lisse de deux quadriques.*

(i) S'il existe une sous-k-variété géométriquement intègre $W \subset X \subset \mathbb{P}^n_k$ de dimension r et de degré impair dans $\mathbb{P}^n_k$, alors X contient un espace linéaire $\mathbb{P}^r_k$.

(ii) Si de plus $r \geq 1$, alors X est k-birationnelle à $\mathbb{P}^{n-2}_k$.

Démonstration. Soit $X \subset \mathbb{P}^n_k$ définie par l'annulation de deux formes quadratiques $f = g = 0$. La quadrique lisse Q sur le corps $K = k(t)$ définie par $f + tg = 0$ contient la sous-K-variété géométriquement intègre $W \times_k K$, qui est de degré impair et de dimension r. D'après le théorème A.3, elle contient un espace linéaire $\mathbb{P}^r_K$. D'après le théorème A.4, la K-variété X contient un espace linéaire $\mathbb{P}^r_k$. Ceci établit (i), et (ii) en résulte d'après [CTSSD87a, Prop. 2.2]. $\square$

A.3. **Intersection de deux quadriques qui contiennent une paire rationnelle de droites gauches.** On répond ici négativement à la question du §7 de [HT19c], et on donne simultanément des exemples un peu plus simples que ceux du Corollaire 27 du paragraphe 9 de [HT19c].

Commençons par des exemples sur le corps des réels, variantes de [CTS80, §2, p. 128] et [CT18, §1, Prop. 1.3].

Proposition A.6. *Soient $n \geq 5$ un entier et $X \subset \mathbb{P}^n_{\mathbb{R}}$, une intersection complète lisse de deux quadriques donnée par un système d'équations homogènes :*

$$f(x_2, \dots, x_n) - x_0x_1 = 0 = g(x_2, \dots, x_n) - (x_0 - x_1)(x_0 - 2x_1),$$

avec $f(x_2, \dots, x_n)$ et $g(x_2, \dots, x_n)$ deux formes quadratiques à coefficients réels définies positives.

Alors :

(i) L'espace topologique $X(\mathbb{R})$ a deux composantes connexes.

(ii) La $\mathbb{R}$-variété X n'est pas stablement rationnelle, ni même rétractilement rationnelle.

(iii) La $\mathbb{C}$-variété $X_{\mathbb{C}}$ contient un espace linéaire $\mathbb{P}^m_{\mathbb{C}}$ avec $m = [(n-3)/2]$ qui ne rencontre pas son congugué complexe.

Démonstration. Il n'y a pas de point de $X(\mathbb{R})$ avec $(x_0, x_1) = (0, 0)$. On dispose donc de l'application continue $X(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ définie par (x_0, x_1). Son image est la réunion des intervalles $[0, 1]$ et $[2, \infty]$. L'espace $X(\mathbb{R})$ a donc au moins deux composantes connexes, et c'est le maximum possible pour une intersection lisse de deux quadriques. Pour les conséquences de la non connexité de $X(\mathbb{R})$ sur la non rationalité d'une $\mathbb{R}$-variété projective et lisse $X/\mathbb{R}$, je renvoie aux références données dans [CT18, Théorème 1.1]. Pour l'énoncé (iii), il suffit d'observer que la section Y de X par $x_0 + x_1 = 0$ est une intersection de deux quadriques dans $\mathbb{P}^{n-1}_{\mathbb{R}}$ qui satisfait $Y(\mathbb{R}) = \emptyset$. On sait que toute intersection de deux quadriques dans $\mathbb{P}^{n-1}_{\mathbb{C}}$ contient un espace linéaire de dimension $m = [(n-3)/2]$; ceci résulte par exemple de la combinaison du théorème A.4 et du théorème de Tsen. Comme on a $Y(\mathbb{R}) = \emptyset$, un tel sous-espace linéaire de $Y_{\mathbb{C}}$ ne saurait rencontrer son conjugué dans $Y_{\mathbb{C}}$. $\square$

La proposition suivante utilise la méthode de spécialisation sur une variété pas trop singulière possédant des invariants non ramifiés non triviaux [CTP16].

Proposition A.7. *Soit $p \neq 2$ un nombre premier. Soit $\mathbb{F}$ un corps fini de caractéristique p assez gros. Sur tout corps K avec $\mathbb{F}(x) \subset K \subset \mathbb{F}((x))$, sur tout corps K avec $\mathbb{C}(x)(y) \subset K \subset \mathbb{C}((x))((y))$, sur tout corps de nombres K, et sur tout corps p-adique K, il existe une intersection lisse de deux quadriques $X \subset \mathbb{P}^5_K$ qui contient un K-point, qui* *possède une paire de droites gauches définies sur une extension quadratique de K, et qui n'est pas rétractilement rationnelle, et en particulier n'est pas stablement K-rationnelle.*

Démonstration. On utilise les exemples donnés avec Coray et Sansuc dans [CTCS80] ; voir aussi la liste d'exemples de [CTSSD87b, §15].

Soit k un corps de caractéristique différente de 2 et $a \in k$ non carré. Soient (x, y, z, t, u, v) des coordonnées homogènes de $\mathbb{P}^5_k$.

Soit $\alpha = \sqrt{a}$. Soit $X \subset \mathbb{P}^5_k$ définie par le système

$$q_1 = x^2 - ay^2 - uv = 0$$

$$q_2 = z^2 - at^2 - (u - cv)(u - dv) = 0,$$

avec $uv(u - cv)(u - dv)$ sans facteur multiple.

Elle contient le point rationnel lisse M de coordonnées $(1, 0, 1, 0, 1, 0)$.

La classe de l'algèbre de quaternions $((u - cv)/v, a) \in \mathrm{Br}(k(X))$ est non ramifiée sut tout modèle projectif et lisse de X. Comme a n'est pas un carré dans k, cette classe ne provient pas de $\mathrm{Br}(k)$. Ces deux énoncés sont établis dans [CTCS80, Prop. 6.1 (iii)].

Le lieu non lisse de X est formé des deux points fermés R et S définis l'un par $u = v = 0 = z = t = 0$ et donc $x^2 - ay^2 = 0$, l'autre par $u = v = x = y = 0$ et $z^2 - at^2 = 0$. On dispose d'une résolution des singularités $f : \tilde{X} \to X$ qui est un isomorphisme au-dessus du complémentaires de R et S et telle que les fibres $f^{-1}(R)$ et $f^{-1}(S)$ sont des quadriques lisses de dimension 2 sur le corps $k(\sqrt{a})$ possédant un $k(\sqrt{a})$-point : les $k(\sqrt{a})$-variétés $f^{-1}(R)$ et $f^{-1}(S)$ sont donc universellement CH_0-triviales.

La variété X contient deux droites gauches conjuguées

$$x - \alpha y = z - \alpha t = u = v = 0$$

et

$$x + \alpha y = z + \alpha t = u = v = 0.$$

On déforme maintenant X en une intersection lisse de deux quadriques contenant deux droites conjuguées et contenant le point M. Il suffit pour cela de prendre une intersection lisse $f_1 = f_2 = 0$ de deux quadriques dans $\mathbb{P}^5_k$ contenant les deux droites gauches ci-dessus et contenant le point M (voir [CTSSD87a, §4 et §1] ; c'est ici que l'on suppose le corps fini assez gros). On considère alors l'intersection complète lisse de deux quadriques X_λ sur le corps $K = k(\lambda)$ donnée par

$$q_1 + \lambda f_1 = 0,$$

$$q_2 + \lambda f_2 = 0.$$

La K-variété X_λ possède un point K-rationnel et contient deux droites conjuguées. Le théorème de spécialisation sous la forme [CTP16, Thm. 1.12] montre que X_λ n'est pas K-rétractilement rationnelle.

On peut prendre pour k tout corps assez gros de caractéristique différente de 2 pour lequel $k \neq k^2$. Par exemple un corps fini $\mathbb{F}$ de caractéristique différente de 2 assez gros, ou $k = \mathbb{C}((T))$ ou $k = \mathbb{C}(T)$.

L'argument donne ainsi des exemples de $X \subset \mathbb{P}^5_K$ non rétractilement rationnels sur $K = \mathbb{C}((T_1))((T_2))$, sur $K = \mathbb{C}(T_1, T_2)$, sur $K = \mathbb{F}((T))$, sur $K = \mathbb{F}(T)$. On peut aussi faire une déformation en inégale caractéristique et faire des exemples sur tout corps p-adique (de corps résiduel assez gros et non dyadique), et de là sur tout corps de nombres. □

On trouvera un exemple analogue pour les hypersurfaces cubiques de $\mathbb{P}^4_{\mathbb{Q}_p}$ dans [CTP16, Thm. 1.21].

A.4. **Intersection de deux quadriques non rationnelles en dimension quelconque sur des corps non ordonnables.** On a déjà donné de tels exemples sur les réels (Prop. A.6). On peut donner des exemples sur des corps non ordonnables. L'argument donné ici a déjà été développé dans [CT18, Thm. 4.1] pour les hypersurfaces cubiques diagonales. On renvoie à [CT18] pour plus de détails sur les outils employés (cohomologie galoisienne, cohomologie non ramifiée).

Soit k un corps de caractéristique différente de 2, contenant $a \in k^* \setminus k^{*2}$. Soit $K_n = k(s_1, \dots, s_n)$ le corps des fonctions rationnelles en $n \geq 0$ variables. Soient $b_1, \dots, b_n \in k$ et $X_n \subset \mathbb{P}^{n+4}_{K_n}$ l'intersection complète de deux quadriques donnée par le système d'équations :

$$\phi = x^2 - ay^2 - uv + \sum_{i=1}^{n} s_i y_i^2 = 0$$

$$\psi = 2(x^2 - az^2) - (u+v)(2u - v) + \sum_{i=1}^{n} b_i s_i y_i^2 = 0$$

en les variables homogènes $(x, y, z, u, v, y_1, \dots, y_n)$. La K_n-variété X_n possède le K_n-point $(x, y, z, u, v, y_1, \dots, y_n) = (1, 0, 0, 1, 1, 0, \dots, 0)$. On supposera X lisse. C'est le cas si le polynôme homogène $det(\lambda\varphi + \mu\psi)$ est séparable. Si k est infini ou fini avec assez d'éléments, il existe des éléments $b_1, \dots, b_n \in k$ qui satisfont ces conditions.

Proposition A.8. *Soit $K_n(X_n)$ le corps des fonctions de la K_n-variété X_n. Le cup-produit*

$$\alpha_n := ((u+v)/v, a, s_1, \dots, s_n) \in H^{n+2}(K_n(X_n), \mathbb{Z}/2)$$

des classes de $((u+v)/v, a, s_1, \dots, s_n)$ *dans*

$$K_n(X_n)^*/K_n(X_n)^{*2} = H^1(K_n(X_n), \mathbb{Z}/2)$$

est non ramifié sur la K_n*-variété* X_n*, et n'appartient pas à l'image de* $H^{n+2}(K_n, \mathbb{Z}/2)$*. Ainsi la* K_n*-variété* X_n *n'est pas* CH_0*-triviale et en particulier n'est pas rétractilement rationnelle.*

Démonstration. Pour $n = 0$, la classe de quaternions $((u+v)/v, a)$ est un élément non constant de la 2-torsion du groupe de Brauer de la surface $X_0 \subset \mathbb{P}^4_k$, voir [BSD75, §4]. Cela définit donc une classe dans $H^2_{nr}(k(X_0)/k, \mathbb{Z}/2)$ qui ne vient pas de $H^2(k, \mathbb{Z}/2)$. On notera que cela exclut la présence d'un couple de droites gauches conjuguées sur X_0.

Soit $n \geq 1$. Supposons l'énoncé démontré pour $n-1$. Sur la K_n-variété lisse X_n, la classe α_n est non ramifiée en dehors des diviseurs définis par $v = 0$ et par $u + v = 0$. Le diviseur Δ défini par $v = 0$ est intègre, donné par le système

$$x^2 - ay^2 - \sum_{i=1}^{n} s_i y_i^2 = 0,$$

$$2(x^2 - az^2) - 2u^2 - \sum_{i=1}^{n} b_i s_i y_i^2 = 0.$$

Le résidu de α_n en Δ est le cup-produit

$$\beta_n := (a, s_1, \dots, s_n) \in H^{n+1}(K_n(\Delta), \mathbb{Z}/2).$$

L'identité $x^2 - ay^2 - \sum_{i=1}^{n} s_i y_i^2 = 0$ sur Δ implique que β_n est nul. L'argument sur le diviseur intègre défini par $u + v = 0$ est identique. La classe α_n est donc non ramifiée sur la K_n-variété X_n.

On considère par ailleurs le modèle propre sur $K_{n-1}[s_n]$ défini par le même système d'équations que X_n. La fibre au-dessus de $s_n = 0$ n'est autre que le cône sur la K_{n-1}-variété X_{n-1} définie par le système d'équations :

$$x^2 - ay^2 - \sum_{i=1}^{n-1} s_i y_i^2 = 0,$$

$$2(x^2 - az^2) - (u+v)(2u - v) - \sum_{i=1}^{n-1} b_i s_i y_i^2 = 0.$$

Le résidu de α_n au point générique de cette variété est la classe

$$\alpha_{n-1} = ((u+v)/v, a, s_1, \dots, s_{n-1}),$$

qui par hypothèse de récurrence n'est pas dans l'image de $H^{n+1}(K_{n-1}, \mathbb{Z}/2)$. Comme la fibre $s_n = 0$ a multiplicité 1, la comparaison des résidus en $s_n = 0$ montre que α_n n'est pas dans l'image de $H^{n+2}(K_n, \mathbb{Z}/2)$. $\square$

L'argument ci-dessus peut s'adapter en inégale caractéristique et donne sur tout corps p-adique K avec $p \neq 2$ des exemples d'intersection lisse de deux quadriques $X \subset \mathbb{P}^5_K$ non rétractilement rationnelle.

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