

# REDUCTION OF BRAUER CLASSES ON K3 SURFACES, RATIONALITY AND DERIVED EQUIVALENCE

SARAH FREI, BRENDAN HASSETT, AND ANTHONY VÁRILLY-ALVARADO

ABSTRACT. Given a smooth projective variety over a number field and an element of its Brauer group, we consider the specialization of the Brauer class at a place of good reduction for the variety and the class. We are interested in the case of K3 surfaces. We show that a Brauer class on a very general polarized K3 surface over a number field becomes trivial after specialization at a set of places of positive natural density. We deduce that there exist cubic fourfolds over number fields that are conjecturally irrational, with rational reduction at a positive proportion of places. We also deduce that there are twisted derived equivalent K3 surfaces which become derived equivalent after reduction at a positive proportion of places.

## 1. INTRODUCTION

Suppose that  $X$  is a smooth projective surface over a number field  $k$ ; write  $\mathrm{Br}(X) := H_{\mathrm{et}}^2(X, \mathbb{G}_m)_{\mathrm{tors}}$  for its Brauer group. For a place  $\mathfrak{p}$  of  $k$  of good reduction for  $X$ , let  $X_{\mathfrak{p}}$  denote the reduction modulo  $\mathfrak{p}$ . Similarly, for  $\alpha \in \mathrm{Br}(X)$  and a place  $\mathfrak{p}$  where  $\alpha$  is unramified, we let  $\alpha_{\mathfrak{p}}$  be the image of  $\alpha$  under the reduction map  $\mathrm{Br}(X) \rightarrow \mathrm{Br}(X_{\mathfrak{p}})$ . What can we say about the density of the set

$$\mathcal{S}(X, \alpha) := \{\mathfrak{p} : \alpha_{\mathfrak{p}} = 0 \in \mathrm{Br}(X_{\mathfrak{p}})\}?$$

We might also ask for algebraicity, i.e.,  $\alpha_{\mathfrak{p}} = 0 \in \mathrm{Br}(\overline{X}_{\mathfrak{p}})$  on passing to an algebraic closure.

Rationality questions for fourfolds give an impetus for considering problems of this kind [AIM]. Several papers [Has99, Kuz10, HPT18, AHTVA19] illustrate how the rationality of complex fourfolds may be controlled by Brauer classes on surfaces: For certain smooth projective fourfolds  $Y$ , there exist a surface  $X$  and a Brauer class  $\alpha$  on  $X$  such that  $Y$  is rational whenever  $\alpha = 0$ . When  $Y$  is defined over a number field, we may also consider

$$\mathcal{R}(Y) := \{\mathfrak{p} : Y_{\mathfrak{p}} \text{ is smooth and rational}\}.$$

Totaro's specialization technique [Tot16], applied where rationality is not a deformation invariant, gives examples where  $\mathcal{R}(Y) \neq \emptyset$  with  $\overline{Y}$  irrational [Fre]. Can  $\mathcal{R}(Y)$  be infinite when  $\overline{Y}$  is not rational? Can it have positive natural density?

**1.1. K3 surfaces.** Now let  $X$  be a K3 surface. Let  $T(X)$  be the transcendental cohomology of  $X$ , i.e., the orthogonal complement of the Néron-Severi group  $\mathrm{NS}(X_{\mathbb{C}}) \subset H^2(X(\mathbb{C}), \mathbb{Z})$ .

---

*Date:* July 18, 2022, corrected June 29, 2023.

1991 *Mathematics Subject Classification.* Primary 14J28, 14F22; Secondary 14E08, 14F08.

S. F. was partially supported by NSF grant DMS-1745670.

B. H. was partially supported by NSF grant DMS-1701659 and Simons Foundation Award 546235.

A. V.-A. was partially supported by NSF grants DMS-1352291 and DMS-1902274.

Since  $T(X)_{\mathbb{Q}} := T(X) \otimes \mathbb{Q}$  is a rational Hodge structure of K3 type, the endomorphism algebra  $E := \text{End}_{\text{Hdg}}(T(X)_{\mathbb{Q}})$  is a totally real or a CM field [Zar83, Theorem 1.5.1].

**Theorem 1.1.** *Let  $X$  be a K3 surface over a number field  $k$ . Assume that  $E$  is totally real, and that  $\dim_E(T(X)_{\mathbb{Q}})$  is odd. Let  $\alpha \in \text{Br}(X)$ . Then the set  $\mathcal{S}(X, \alpha)$  of places  $\mathfrak{p}$  such that  $\alpha_{\mathfrak{p}} \in \text{Br}(X_{\mathfrak{p}})$  vanishes contains a set of positive natural density.*

**Remark 1.2.** For a very general polarized K3 surface  $X$ , we have  $\text{NS}(X_{\mathbb{C}}) \simeq \mathbb{Z}$ , in which case  $\dim_E(T(X)_{\mathbb{Q}})$  is odd and  $E$  is totally real [Huy16, Remark 3.3.14(ii)]. Moreover, there exist K3 surfaces over number fields such that  $\text{NS}(X_{\mathbb{C}}) \simeq \mathbb{Z}$  [Ter85, Ell04, vL07] and  $E = \mathbb{Q}$  [Noo95, Mas96].

Without the assumptions that  $E$  is totally real and that  $\dim_E(T(X)_{\mathbb{Q}})$  is odd, Theorem 1.1 is false. By [Cha14, Theorem 1(2)], if  $E$  is a CM field or  $\dim_E(T(X)_{\mathbb{Q}})$  is even, then after a finite field extension, there is a set of places  $\mathcal{S}$  of natural density one for which  $\text{rk NS}(\bar{X}) = \text{rk NS}(\bar{X}_{\mathfrak{p}})$ . On the other hand, a jump in the Picard rank upon reduction is required for a transcendental Brauer class to become algebraic (or vanish). For a prime  $\ell$  and a fixed embedding  $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$ , it is well known that  $\text{NS}(\bar{X}) \otimes \mathbb{Z}_{\ell} \hookrightarrow \text{NS}(\bar{X}_{\mathfrak{p}}) \otimes \mathbb{Z}_{\ell}$ , which by the Kummer sequence forces a nontrivial kernel for the map  $\text{Br}(\bar{X})[\ell^{\infty}] \rightarrow \text{Br}(\bar{X}_{\mathfrak{p}})[\ell^{\infty}]$ , so when the Picard rank jumps upon reduction, the Brauer group must shrink. Thus, there is necessarily a trade-off between Brauer classes modulo  $\mathfrak{p}$  and algebraic classes in  $\text{NS}(\bar{X}_{\mathfrak{p}})$ .

Costa and Tschinkel [CT14] found experimentally that the Picard rank may jump over a positive-density set of places for *some* K3 surfaces of Picard rank two: jumping occurs at half of all primes. This was explained by Costa, Elsenhans, and Jahnel [CEJ20] using the concept of the *jump character*, which encodes the discriminant of the Galois representation on transcendental cohomology. The finite extension stipulated in [Cha14, Theorem 1(2)] trivializes this character. This discriminant was studied earlier by de Jong and Katz in the case of even-dimensional hypersurfaces [dJK00].

**Example 1.3.** Let  $X'$  be the double cover of  $\mathbb{P}_{\mathbb{Q}}^2$  cut out by

$$w^2 = xyz(2x + 4y - 3z)(x - 5y - 3z)(x + 3y + 3z),$$

whose minimal desingularization  $X$  is a K3 surface over  $\mathbb{Q}$ . In [EJ21, Example 5.5] Elsenhans and Jahnel show that  $\text{rk NS}(X) = \text{rk NS}(\bar{X}) = 16$  and  $E = \mathbb{Q}$ , so that  $\dim_E(T(X)_{\mathbb{Q}})$  is even. They further show, using [EJ21, Corollary 4.7] and computations for the jump character, that the finite field extension of [Cha14, Theorem 1(2)] is trivial, so that there is a set of places  $\mathcal{S}$  of  $\mathbb{Q}$  of natural density one for which the Picard rank does not jump upon reduction, and hence any nontrivial Brauer class remains nontrivial.

**Remark 1.4.** In light of Example 1.3, it would be very interesting to show that, if one relaxes the requirement that  $\mathcal{S}(X, \alpha)$  contain a set of positive natural density to the weaker statement that it is *infinite*, then Theorem 1.1 holds when  $E$  is a CM field or  $\dim_E(T(X)_{\mathbb{Q}})$  is even. Recent work of Shankar, Shankar, Tang and Tayou [SSTT19] suggests that such a statement may be within reach.

It would also be interesting to extend Theorem 1.1 to cases where the jump character explains jumping of the Picard rank.

**1.2. Application: cubic fourfolds.** Special cubic fourfolds of certain discriminants comprise natural classes of fourfolds  $Y \subset \mathbb{P}_k^5$  to which one can associate a pair  $(X, \alpha)$ , where  $X$  is a K3 surface and  $\alpha \in \text{Br}(X)$ . It is expected that a very general such  $Y$  is irrational [Kuz10]; however, if  $\alpha = 0$  then in some cases it is possible to show that  $Y$  is rational [HPT18, AHTVA19].

Let  $h$  be the restriction to  $Y$  of a hyperplane class in  $\mathbb{P}^5$ . We consider  $Y$  for which there exists a saturated lattice  $K_d \subset H^2(Y, \Omega_Y^2) \subset H^4(Y, \mathbb{Z})$  of rank 2 equal to

$$K_8 = \langle h^2, P \rangle \simeq \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{or} \quad K_{18} = \langle h^2, T \rangle \simeq \begin{pmatrix} 3 & 6 \\ 6 & 18 \end{pmatrix}.$$

Write  $\mathcal{C}_{K_8}$  and  $\mathcal{C}_{K_{18}}$  for the respective moduli spaces of pairs  $(Y, K_d)$ .

**Theorem 1.5.** *Let  $Y \subset \mathbb{P}_k^5$  be a cubic fourfold over a number field  $k$ . Assume that  $Y$  is a very general fourfold in  $\mathcal{C}_{K_8}$  or  $\mathcal{C}_{K_{18}}$ . Then there exists a set of places  $\mathcal{S}$  of  $k$  of positive natural density for which the reduction  $Y_{\mathfrak{p}}$  is rational for every  $\mathfrak{p} \in \mathcal{S}$ .*

**1.3. Application: derived equivalences.** Let  $X$  be a K3 surface with a polarization  $h$ , and let  $v = (r, c, s) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}$  be a primitive Mukai vector. The moduli space  $M$  of Gieseker  $h$ -semi-stable sheaves on  $X$  of rank  $r$ , first Chern class  $c$  and Euler characteristic  $r + s$  is itself a K3 surface if  $c^2 - 2rs = 0$  and  $h$  is  $v$ -generic; see §6 [Muk84]. The space  $M$  need not be fine: there is a natural Brauer class  $\alpha \in \text{Br}(M)$  that can obstruct the existence of a universal sheaf on  $X \times M$ . However, there is a  $k$ -linear derived equivalence

$$D^b(M, \alpha) \simeq D^b(X),$$

first observed by Căldăraru [Că102] in the case  $k = \mathbb{C}$ . We call the pair  $(M, \alpha)$  a **twisted K3 surface** associated to  $X$ .

**Theorem 1.6.** *Let  $X$  be a very general K3 surface of degree  $2d$  over a number field  $k$ , and let  $(M, \alpha)$  be an associated twisted K3 surface parametrizing geometrically stable sheaves on  $X$ . Then there exists a set of places  $\mathcal{S}$  of  $k$  of positive natural density such that for  $\mathfrak{p} \in \mathcal{S}$ , the reduction  $M_{\mathfrak{p}}$  is a fine moduli space, and there is an  $\mathbb{F}_{\mathfrak{p}}$ -linear derived equivalence  $D^b(X_{\mathfrak{p}}) \cong D^b(M_{\mathfrak{p}})$ .*

**1.4. Outline of the paper.** In §§2–4 we present the proof of Theorem 1.1. It relies on extracting information on the Picard groups of reductions modulo places from the action of Frobenius on finite Galois modules. To draw such conclusions, we need information about the Mumford-Tate groups. Other technical inputs include the integral Tate conjecture for K3 surfaces over finite fields and open image theorems for K3 surfaces over number fields. We present the two applications above in §§5–6. In §5 we address specialization of rationality for cubic fourfolds; unfortunately, no smooth complex cubic fourfolds are known to be irrational. In §6 we illustrate how twisted derived equivalences of K3 surfaces specialize to finite fields.

**Notation.** For a field  $k$ , we write  $\bar{k}$  for a fixed algebraic closure of  $k$ . For a  $k$ -variety  $X$ , we let  $\bar{X} := X \times_k \bar{k}$ . When  $k$  is a number field, we write  $k_{\mathfrak{p}}$  for the completion of  $k$  with respect to the prime ideal  $\mathfrak{p} \subset \mathcal{O}_k$ , and  $\mathbb{F}_{\mathfrak{p}}$  for the residue field.

For a number field  $k$ , we say that a place  $\mathfrak{p}$  is a place of good reduction for a smooth proper  $k$ -variety  $X$  there is a smooth proper morphism  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_{k_{\mathfrak{p}}}$  such that  $\mathcal{X}_{k_{\mathfrak{p}}} \simeq X_{k_{\mathfrak{p}}}$ . In this case, we write  $X_{\mathfrak{p}}$  for the closed fiber over  $\mathbb{F}_{\mathfrak{p}}$ . We say that a place is finite if its residue field is finite.

For an endomorphism  $f: M \rightarrow M$  of a free module  $M$  over a ring  $R$ , we write  $\mathfrak{E}(f, \lambda)$  for the eigenspace of  $\lambda \in R$ , where  $\lambda$  is not a zero divisor.

**Acknowledgments.** We thank Ravi Vakil for asking the third named author whether a statement like Theorem 1.5 could be true at the 2015 Arizona Winter School. We thank Nicolas Addington, Martin Bright, Jean-Louis Colliot-Thélène, Edgar Costa, Ofer Gabber, Daniel Huybrechts, Evis Ieronymou, Daniel Loughran, and Yuri Tschinkel for valuable mathematical comments and discussions, and Isabel Vogt for pointing out the reference [AIM]. We thank an anonymous referees for valuable comments and on earlier versions of this paper.

## 2. INGREDIENTS FOR THE PROOF

**2.1. Mumford-Tate groups.** Let

$$\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}}$$

be the Deligne torus, and write  $w: \mathbb{G}_{m, \mathbb{R}} \rightarrow \mathbb{S}$  for the weight cocharacter, which is given on  $\mathbb{R}$ -points by the natural inclusion  $\mathbb{R}^{\times} = \mathbb{G}_{m, \mathbb{R}}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ . Given a finite-dimensional  $\mathbb{Q}$ -vector space  $V$ , a  $\mathbb{Q}$ -Hodge structure of weight  $m$  on  $V$  determines and is determined by a representation  $h: \mathbb{S} \rightarrow \text{GL}(V_{\mathbb{R}})$  such that  $h \circ w$  is given on  $\mathbb{R}$ -points by  $a \mapsto a^{-m} \cdot \text{Id}_V$ . A  $\mathbb{Z}$ -Hodge structure is defined analogously, starting with a free  $\mathbb{Z}$ -module  $V$  of finite rank. We refer to a  $\mathbb{Q}$ - or a  $\mathbb{Z}$ -Hodge structure as simply a Hodge structure to avoid clutter.

**Example 2.1.** For a smooth projective complex variety  $X$ , the singular cohomology  $V := H^m(X(\mathbb{C}), \mathbb{Q})$  gives rise to a  $\mathbb{Q}$ -Hodge structure of weight  $m$ . The intersection pairing, appropriately modified by a sign, defines a polarization on  $H^{\dim X}(X(\mathbb{C}), \mathbb{Q})$ . When  $m = 2n$  is even, applying an  $n$ -fold Tate twist, we obtain a Hodge structure  $V := H^{2n}(X(\mathbb{C}), \mathbb{Q}(n))$  of weight 0.

**Example 2.2.** For a complex K3 surface  $X$  the  $\mathbb{Q}$ -Hodge structure  $H = H^2(X(\mathbb{C}), \mathbb{Q}(1))$  of weight 0 arising from singular cohomology splits as a direct sum  $\text{NS}(X)_{\mathbb{Q}}(1) \oplus T(X)_{\mathbb{Q}}(1)$ . The vector space  $T(X)_{\mathbb{Q}}$  itself carries a polarized Hodge structure, polarized by the restriction  $\phi$  to  $T(X)_{\mathbb{Q}}$  of the cup product on  $H^2(X(\mathbb{C}), \mathbb{Q})$ . The ring  $\mathcal{O} := \text{End}_{\text{Hdg}}(T(X))$  of integral Hodge endomorphisms is an order of the endomorphism algebra  $E := \text{End}_{\text{Hdg}}(T(X)_{\mathbb{Q}})$ . In [Zar83, Theorem 1.5.1], Zarhin shows that  $E$  is either a totally real or a CM field.

**Definition 2.3.** For a  $\mathbb{Q}$ -Hodge structure  $V$ , the Mumford-Tate group  $\mathrm{MT}(V)$  of  $V$  is the smallest algebraic subgroup of  $\mathrm{GL}(V)$  over  $\mathbb{Q}$  such that  $h(\mathbb{S}(\mathbb{R})) \subset \mathrm{MT}(V)(\mathbb{R})$ . For a  $\mathbb{Z}$ -Hodge structure  $V$ , the integral Mumford-Tate group  $\mathcal{MT}(V)$  of  $V$  is the group subscheme of  $\mathrm{GL}(V)$  over  $\mathbb{Z}$  constructed as the Zariski closure of  $\mathrm{MT}(V_{\mathbb{Q}})$  in  $\mathrm{GL}(V)$ .

When  $V$  is polarizable,  $\mathrm{MT}(V)$  is a reductive connected linear algebraic group over  $\mathbb{Q}$  [Sch11, Proposition 2]. When  $X$  is a complex K3 surface,  $\mathrm{MT}(\mathrm{H}^2(X(\mathbb{C}), \mathbb{Q}))$  admits a representation into an orthogonal group of dimension 22, and  $\mathrm{MT}(\mathrm{H}^2(X(\mathbb{C}), \mathbb{Q})) \cong \mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}})$ .

**Theorem 2.4** ([Zar83]). *Let  $X$  be a complex K3 surface such that the endomorphism algebra  $E$  is a totally real field. Then  $\mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}})$  is isomorphic to the centralizer of  $E$  in the group of orthogonal similitudes  $\mathrm{GO}(\mathrm{T}(X)_{\mathbb{Q}}, \phi)$ <sup>1</sup>.*

*Proof.* Recall that the Hodge group  $\mathrm{Hdg}(\mathrm{T}(X)_{\mathbb{Q}}) \subset \mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}})$  is the smallest algebraic subgroup defined over  $\mathbb{Q}$  containing  $h(\mathbb{U}(\mathbb{R}))$ , where  $\mathbb{U} \subset \mathbb{S}$  is the unit circle [Zar83, p. 196]. In [Zar83, §2.1, Theorem 2.2.1], Zarhin shows that there is a unique  $E$ -bilinear form

$$\Phi: \mathrm{T}(X)_{\mathbb{Q}} \times \mathrm{T}(X)_{\mathbb{Q}} \rightarrow E$$

such that  $\phi = \mathrm{Tr}_{E/\mathbb{Q}}(\Phi)$ , and that  $\mathrm{Hdg}(\mathrm{T}(X)_{\mathbb{Q}})$  is isomorphic to the Weil restriction of scalars  $\mathrm{Res}_{E/\mathbb{Q}}(\mathrm{SO}_E(\mathrm{T}(X)_{\mathbb{Q}}, \Phi))$ . This is clearly contained in the centralizer of  $E$  in  $\mathrm{SO}(\mathrm{T}(X)_{\mathbb{Q}}, \phi)$ ; Zarhin's proof shows the reverse inclusion. By [Spr09, Theorem 6.4.7(i)], the centralizer of  $E$  is connected. Thus the assertion for  $\mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}})$  follows.  $\square$

**Corollary 2.5.** *If  $E$  is totally real, then the Mumford-Tate group  $\mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}}(1))$  is isomorphic to the centralizer of  $E$  in the special orthogonal group  $\mathrm{SO}(\mathrm{T}(X)_{\mathbb{Q}}, \phi_{\mathbb{Q}})$ .*

*Proof.* Since  $\mathrm{T}(X)_{\mathbb{Q}}(1)$  is a Hodge structure of weight 0, we know by [Sch11, Proposition 2(i)] that  $\mathrm{MT}(\mathrm{T}(X)_{\mathbb{Q}}(1)) \subset \mathrm{SL}(\mathrm{T}(X)_{\mathbb{Q}})$ . Now the result follows from Theorem 2.4.  $\square$

**2.2.  $\ell$ -adic representations and the Mumford-Tate conjecture.** Let  $X$  be a smooth projective variety defined over a number field  $k$ . Fix a prime  $\ell$ , and let

$$\rho_{\ell}: \mathrm{Gal}(\bar{k}/k) \rightarrow \mathrm{GL}(\mathrm{H}_{\mathrm{et}}^{2i}(\bar{X}, \mathbb{Z}_{\ell}(i))) \quad (2.1)$$

be the  $\ell$ -adic Galois representation arising from the action of the absolute Galois group  $\mathrm{Gal}(\bar{k}/k)$  on the cohomology group  $\mathrm{H}_{\mathrm{et}}^{2i}(\bar{X}, \mathbb{Z}_{\ell}(i))$ . The Zariski closure of  $\mathrm{im}(\rho_{\ell})$  is the  $\ell$ -adic algebraic monodromy group, which we denote by  $\mathcal{G}_{\ell}$ . Let  $G_{\ell}$  denote the generic fiber of  $\mathcal{G}_{\ell}$ , which is the Zariski closure of the image of the Galois representation for  $\mathrm{H}_{\mathrm{et}}^{2i}(\bar{X}, \mathbb{Q}_{\ell}(i))$ . The Mumford-Tate Conjecture predicts a connection between the Mumford-Tate group of  $\mathrm{H} := \mathrm{H}^{2i}(X(\mathbb{C}), \mathbb{Q}(i))$ , whose formulation is Hodge-theoretic, and the  $\ell$ -adic algebraic monodromy group, defined arithmetically.

**Conjecture 2.6** (Mumford-Tate Conjecture, [Ser86]). *Under the comparison isomorphism  $\mathrm{H}_{\mathrm{et}}^{2i}(\bar{X}, \mathbb{Q}_{\ell}(i)) \cong \mathrm{H} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$ , the Mumford-Tate group  $\mathrm{MT}(\mathrm{H}) \times_{\mathbb{Q}} \mathbb{Q}_{\ell}$  is isomorphic (as an algebraic group) to the identity component  $G_{\ell}^{\circ}$  of the  $\ell$ -adic algebraic monodromy group.*

<sup>1</sup>Please see the Correction below.

The conjecture has been proved for K3 surfaces over number fields:

**Theorem 2.7** ([Tan90], [Tan95], and independently [And96]). *The Mumford-Tate conjecture holds for K3 surfaces over number fields: that is, for a K3 surface  $X$  defined over a number field, we have  $G_\ell^\circ \simeq \text{MT}(\mathbb{H}) \times_{\mathbb{Q}} \mathbb{Q}_\ell$  for  $\mathbb{H} = \mathbb{H}^2(X(\mathbb{C}), \mathbb{Q}(1))$ .*  $\square$

**2.3. The Integral Mumford-Tate conjecture.** For a smooth projective variety  $X$  defined over a number field  $k$ , the Mumford-Tate conjecture can also be stated integrally:  $\mathcal{MT}(\mathbb{H}^{2i}(X(\mathbb{C}), \mathbb{Z}(i)) \times_{\mathbb{Z}} \mathbb{Z}_\ell)$  is isomorphic to  $\mathcal{G}_\ell^\circ$ . This version of the conjecture is equivalent to statement above: given the isomorphism of  $\mathbb{Q}_\ell$ -group schemes, taking their Zariski closures gives the  $\mathbb{Z}_\ell$ -isomorphism, and given the isomorphism of  $\mathbb{Z}_\ell$ -group schemes, take the generic fibers. For a thorough and illuminating discussion of this version of the Mumford-Tate conjecture, as well as other variants, see [CM20].

**2.4. Open image theorems.** From the integral version of the Mumford-Tate conjecture for K3 surfaces over number fields, we would like to make a conclusion about how a component of the image of  $\rho_\ell$  sits inside  $\mathcal{G}_\ell^\circ \cong \mathcal{MT}(\mathbb{H}^2(X(\mathbb{C}), \mathbb{Z}(1))) \times_{\mathbb{Z}} \mathbb{Z}_\ell$ .

First, as a consequence of the Hodge-Tate decomposition [Fal88], the representation (2.1) is of Hodge-Tate type. By [Bog80, Theorem 1], it follows that the image of the  $\text{Gal}(\bar{k}/k)$ -representation  $\mathbb{H}_{\text{et}}^{2i}(\bar{X}, \mathbb{Q}_\ell(i))$  is open in  $G_\ell(\mathbb{Q}_\ell)$ . Since there is a finite field extension  $k^c/k$  for which after base changing to  $k^c$ , the image of the Galois representation is connected, we know that  $G_\ell$  has finitely many connected components. Thus the image of the  $\text{Gal}(\bar{k}/k)$ -representation  $\mathbb{H}_{\text{et}}^{2i}(\bar{X}, \mathbb{Q}_\ell(i))$  has a finite index subgroup that is open in  $G_\ell^\circ(\mathbb{Q}_\ell)$ . By construction  $G_\ell^\circ(\mathbb{Q}_\ell)$  is open in  $\mathcal{G}_\ell^\circ(\mathbb{Z}_\ell)$ , so the integral version of the Mumford-Tate conjecture for K3 surfaces implies:

**Corollary 2.8.** *If  $X$  is a K3 surface over a number field, then  $\text{im}(\rho_\ell)$  has a finite index subgroup that is isomorphic to an open subgroup of  $(\mathcal{MT}(\mathbb{H}^2(X(\mathbb{C}), \mathbb{Z}(1))) \times_{\mathbb{Z}} \mathbb{Z}_\ell) (\mathbb{Z}_\ell)$ .*  $\square$

Let  $X$  be a K3 surface over a number field. Setting  $i = 1$ , the representation  $\rho_\ell$  introduced in §2.2 is the inverse limit of the finite-level representations

$$\rho_{\ell,n}: \text{Gal}(\bar{k}/k) \rightarrow \text{GL}(\mathbb{H}_{\text{et}}^2(\bar{X}, \mu_{\ell^n}))$$

Letting  $\pi_n^\ell: \mathcal{G}(\mathbb{Z}_\ell) \rightarrow \mathcal{G}(\mathbb{Z}/\ell^n\mathbb{Z})$  denote the projections of this inverse system, the following diagram

$$\begin{array}{ccc} \text{Gal}(\bar{k}/k) & \xrightarrow{\rho_\ell} & \mathcal{G}(\mathbb{Z}_\ell) \\ & \searrow \rho_{\ell,n} & \downarrow \pi_n^\ell \\ & & \mathcal{G}(\mathbb{Z}/\ell^n\mathbb{Z}) \end{array}$$

commutes for all  $n$ . For  $n' \geq n$  we denote by  $\pi_{n',n}^\ell: \mathcal{G}(\mathbb{Z}/\ell^{n'}\mathbb{Z}) \rightarrow \mathcal{G}(\mathbb{Z}/\ell^n\mathbb{Z})$  the intermediate projection of the inverse system. In this context, the openness of  $\text{im} \rho_\ell$  in  $\mathcal{G}_\ell(\mathbb{Z}_\ell)$  implies the following more explicit statement.

**Corollary 2.9** ( $\ell$ -adic open image theorem). *There exists an integer  $n_0 > 0$  such that  $\text{im}(\rho_\ell) = (\pi_{n_0}^\ell)^{-1}(\text{im}(\rho_{\ell,n_0}))$ . In particular,  $\text{im}(\rho_{\ell,n}) = (\pi_{n,n_0}^\ell)^{-1}(\text{im}(\rho_{\ell,n_0}))$  for all  $n \geq n_0$ .*  $\square$

To prove Theorem 1.1 in the case where the order of the Brauer class  $\alpha$  is a composite integer  $m$ , we require an  $m$ -adic open image theorem. While it is possible to prove such a result by studying the independence of the images of the  $\rho_\ell$  for the primes  $\ell$  that divide  $m$  [Ser13], the theorem is already an immediate consequence of the much deeper adélic open image theorem of Cadoret and Moonen:

**Theorem 2.10** ([CM20, Theorem 6.6]). *The image of the Galois representation*

$$\rho_{\hat{\mathbb{Z}}}: \text{Gal}(\bar{k}/k) \rightarrow \text{GL}(\text{H}_{\text{et}}^2(\bar{X}, \hat{\mathbb{Z}})),$$

*has a finite index subgroup isomorphic to an open subgroup of  $(\mathcal{MT}(\text{H}^2(X(\mathbb{C}), \mathbb{Z}(1))) \times_{\mathbb{Z}} \hat{\mathbb{Z}})$  ( $\hat{\mathbb{Z}}$ ).*

We will make use of the resulting  $m$ -adic open image theorem in the following way. Write  $m = \ell_1^{e_1} \cdots \ell_r^{e_r}$  with  $\ell_i$  distinct primes,  $1 \leq i \leq r$ .

**Corollary 2.11.** *There exists an integer  $m_0 > 0$  depending on  $m$  such that  $\text{im}(\rho_{\hat{\mathbb{Z}}})$  contains the kernel of the reduction modulo  $m^{m_0}$ -map  $\mathcal{G}(\hat{\mathbb{Z}}) \rightarrow \mathcal{G}(\mathbb{Z}/m^{m_0}\mathbb{Z})$ . In particular, for  $n \geq m_0$ , an element  $(\gamma_1, \dots, \gamma_r) \in \mathcal{G}(\mathbb{Z}/\ell_1^{e_1 n}\mathbb{Z}) \times \cdots \times \mathcal{G}(\mathbb{Z}/\ell_r^{e_r n}\mathbb{Z})$  which reduces to the identity modulo  $m^{m_0}$  is contained in the image of  $\rho_{m,n}: \text{Gal}(\bar{k}/k) \rightarrow \text{GL}(\text{H}_{\text{et}}^2(\bar{X}, \mu_{m^n}))$ .  $\square$*

**2.5. Frobenius conjugacy classes.** Let  $X$  be a smooth and proper scheme over a number field  $k$ , and fix a finite place  $\mathfrak{p}$  of good reduction for  $X$ . For any choice of inclusion  $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$ , the image of  $\text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) \hookrightarrow \text{Gal}(\bar{k}/k)$  is a decomposition group  $D_{\mathfrak{p}}$ . Different choices for this embedding give rise to conjugate decomposition groups. The kernel of the natural, continuous surjective map  $\text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) \twoheadrightarrow \text{Gal}(\bar{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$  is the inertia group  $I_{\mathfrak{p}}$ . The group  $\text{Gal}(\bar{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$  is topologically generated by the Frobenius endomorphism, which we call  $\text{Frob}$ . Via the surjection above, we may pick a lift of this generator to  $D_{\mathfrak{p}} \cong \text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$ , which we call  $\text{Frob}_{\mathfrak{p}} \in \text{Gal}(\bar{k}/k)$ , a lift of Frobenius to characteristic zero. By [SGAIV-3, Exp. XVI, Corollaire 2.2] for  $\ell \neq \text{char } \mathbb{F}_{\mathfrak{p}}$ , the  $\text{Gal}(\bar{k}/k)$ -representation  $\text{H}_{\text{et}}^2(\bar{X}, \mathbb{Z}_{\ell}(1))$  is unramified at  $\mathfrak{p}$ , and so while  $\text{Frob}_{\mathfrak{p}}$  is only defined up multiplication by elements in  $I_{\mathfrak{p}}$ , the resulting action on  $\text{H}_{\text{et}}^2(\bar{X}, \mathbb{Z}_{\ell}(1))$  does not depend on this choice. However, it *does* depend on the choice of embedding  $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$ , which is discussed more below.

The embedding  $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$  induces an isomorphism  $\text{H}_{\text{et}}^{2i}(\bar{X}, \mathbb{Z}_{\ell}(i)) \cong \text{H}_{\text{et}}^{2i}(X_{\bar{k}_{\mathfrak{p}}}, \mathbb{Z}_{\ell}(i))$  [SGAIV-3, Exp. XII, Corollaire 5.4] for which the action of  $D_{\mathfrak{p}}$  on the left-hand object agrees via the isomorphism with the action of  $\text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$  on the right-hand object. There is also an isomorphism  $\text{H}_{\text{et}}^{2i}(X_{\bar{k}_{\mathfrak{p}}}, \mathbb{Z}_{\ell}(i)) \cong \text{H}_{\text{et}}^{2i}(\bar{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell}(i))$  [SGAIV-3, Exp. XVI, Corollaire 2.2] for which the action of  $D_{\mathfrak{p}}/I_{\mathfrak{p}} \cong \text{Gal}(k_{\mathfrak{p}}^{\text{nr}}/k_{\mathfrak{p}})$  is compatible with that of  $\text{Gal}(\bar{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$ . Thus we see that via these isomorphisms, the action of  $\text{Frob}_{\mathfrak{p}}$  on  $\text{H}_{\text{et}}^{2i}(\bar{X}, \mathbb{Z}_{\ell}(i))$  and of  $\text{Frob}$  on  $\text{H}_{\text{et}}^{2i}(\bar{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell}(i))$  agree.

We are interested in properties of an element  $\text{Frob}_{\mathfrak{p}}$  that may be extracted from its action on  $\text{H}_{\text{et}}^{2i}(\bar{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell}(i))$ . If we care only about, e.g., the characteristic polynomial of  $\rho_{\ell}(\text{Frob}_{\mathfrak{p}})$  then we can read this off from any element conjugate to  $\rho_{\ell}(\text{Frob}_{\mathfrak{p}})$  by a linear automorphism of the cohomology defined over  $\bar{\mathbb{Q}}_{\ell}$ .

However, we might ask for more refined data such as the *position* of an eigenspace of  $\rho_{\ell}(\text{Frob}_{\mathfrak{p}})$  in  $\text{H}_{\text{et}}^{2i}(\bar{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell}(i))$  associated with a given root of the characteristic polynomial,

and wish to read that off from the eigenspaces for  $\rho_\ell(\text{Frob}_{\mathfrak{p}})$ . Understanding how much this data depends on the choice of element  $\text{Frob}_{\mathfrak{p}}$  requires understanding  $\rho_\ell(\text{Frob}_{\mathfrak{p}})$  up to finer equivalence relations, e.g., conjugation by linear automorphisms of  $H_{\text{et}}^{2i}(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_\ell(i))$ , by the image of  $\text{Gal}(\overline{k}/k)$  in this linear group, or by a suitable congruence subgroup for the integral Mumford-Tate group contained in this image. The open image theorem (Cor. 2.9) permits this reduction in our situation. There is an extensive literature on classifying elements of matrix groups over various rings up to conjugacy by prescribed subgroups e.g. [GS80, AO83].

By considering the Galois action on  $H_{\text{et}}^2(\overline{X}, \mu_{\ell^n})$  with  $n$  large, the representation  $\rho_{\ell, n}$  factors through  $\text{Gal}(K/k)$  for  $K$  some finite Galois extension of  $k$ . We will express our desired properties so that we may extract the needed information from these finite representations. This is crucial to our arguments, as it enables the use of Chebotarev density theorem in the proof of Theorem 1.1.

We make use of this circle of ideas to identify eigenspaces of  $\text{Frob}_{\mathfrak{p}}$  with root-of-unity eigenvalues. By the Tate Conjecture for K3 surfaces over finite fields, which has been proved by the combined works of [Cha13], [LMS14], [MP15], and [KMP16], and the Integral Tate Conjecture [Tat68, Theorem 5.2], these eigenspaces will correspond to subspaces in  $H_{\text{et}}^2(\overline{X}, \mathbb{Z}_\ell(1))$  which become algebraic upon reduction modulo  $\mathfrak{p}$ .

### 3. PRELIMINARIES

We let  $X$  be a K3 surface over a number field  $k$ . Let  $H = H^2(X(\mathbb{C}), \mathbb{Z}(1))$ , which is a  $\mathbb{Z}$ -Hodge structure of weight 0. Its corresponding integral Mumford-Tate group  $\mathcal{MT}(H) \subset \text{GL}(H)$  is the Zariski closure in  $\text{GL}(H)$  of the Mumford-Tate group  $\text{MT}(H_{\mathbb{Q}}) \subset \text{GL}(H_{\mathbb{Q}})$ , as in Definition 2.3. Since  $T(X)_{\mathbb{Q}}(1)$  and  $\text{NS}(X_{\mathbb{C}})_{\mathbb{Q}}(1)$  are orthogonal direct summands of  $H_{\mathbb{Q}}$  and Mumford-Tate groups act trivially on weight-zero Hodge classes, we have an isomorphism

$$\text{MT}(H_{\mathbb{Q}}) \simeq \text{MT}(T(X)_{\mathbb{Q}}(1)).$$

Over  $\mathbb{Z}$  we lack direct sum decompositions but still obtain a homomorphism of group schemes

$$\mathcal{MT}(H) \rightarrow \mathcal{MT}(T(X)(1)). \quad (3.1)$$

**Proposition 3.1.** *Assume that the Hodge endomorphism algebra  $E$  of  $X$  is totally real, and that  $\dim_E T(X)_{\mathbb{Q}}$  is odd. Let  $U \subseteq \mathcal{MT}(H)$  be the set of elements  $\psi$  such that:*

- (1) *the action of  $\psi$  on  $H^2(X(\mathbb{C}), \mathbb{Q}(1))/(\text{NS}(X_{\mathbb{C}}) \otimes \mathbb{Q}(1))$  has a  $(+1)$ -eigenspace of dimension 1 as a vector space over  $E$ ;*
- (2) *the only root of unity that is an eigenvalue for  $\psi$  is 1.*

*Then  $U$  is a Zariski dense open subset of  $\mathcal{MT}(H)$ .*

We remark that this proposition is a slight generalization of [Cha14, Proposition 15(2)].

*Proof.* Let  $U_{\circ} \supset U$  denote the locus obtained by weakening the first condition, so that  $(+1)$  has algebraic multiplicity  $\leq 1$  for  $\psi$  over  $E$ . We show that  $U_{\circ}$  is Zariski open. Prescribing an eigenvalue, or imposing a lower bound on its multiplicity as a root of the characteristic

polynomial, cuts out a Zariski-closed subset, so avoiding certain eigenvalues or imposing upper bounds on their algebraic multiplicity is a Zariski-open condition.

Next, we show that  $U_\circ = U$ , i.e., the action of any  $\psi \in U_\circ$  on  $T(X)_\mathbb{Q}(1)$  has a nonzero  $(+1)$ -eigenspace over  $E$ . Let  $T = T(X)$ . There is a unique  $E$ -bilinear form  $\Phi: T_\mathbb{Q} \times T_\mathbb{Q} \rightarrow E$ , compatible with the pairing  $\phi: T_\mathbb{Q} \times T_\mathbb{Q} \rightarrow \mathbb{Q}$ , in the sense that  $\text{Tr}_{E/\mathbb{Q}}(\Phi) = \phi$  [Zar83, §2.1]. Hence, the centralizer of  $E$  in  $\text{SO}(T_\mathbb{Q}, \phi)$  coincides with the Weil restriction of scalars  $\text{Res}_{E/\mathbb{Q}}(\text{SO}_E(T_\mathbb{Q}, \Phi))$ , as subgroups of  $\text{GL}(T_\mathbb{Q})$ . On the other hand, by Corollary 2.5,  $\text{MT}(\mathbb{H}_\mathbb{Q}) \cong \text{MT}(T_\mathbb{Q}(1))$  is the centralizer of  $E$  in  $\text{SO}(T_\mathbb{Q}, \phi)$ <sup>2</sup>. Since  $T_\mathbb{Q}$  has odd dimension over  $E$ , every element of  $\text{SO}_E(T_\mathbb{Q}, \Phi)$  has 1 as an eigenvalue. We conclude that  $U_\circ = U$ , as desired.

To see that  $U$  is nonempty, the argument given in [Cha14, Proposition 15(2)] over  $\mathbb{Q}_\ell$  also works over  $\mathbb{Q}$  to produce a  $\mathbb{Q}$ -point of  $U$ . Thus  $U$  is Zariski dense, as desired.  $\square$

For a prime number  $\ell$  and positive integer  $e$ , let  $\alpha$  be a class in  $\text{H}_{\text{et}}^2(\overline{X}, \mu_{\ell^e})$  which is not contained in the image of  $\text{NS}(\overline{X}) \otimes \mathbb{Z}/\ell^e\mathbb{Z}$ . In §4, this will be a Galois-invariant choice of lift of a class in  $\text{Br}(X)[\ell^e]$ .

We fix, for each  $n \geq 1$ , compatible isomorphisms between the  $\text{H}_{\text{et}}^2(\overline{X}, \mu_{\ell^{en}})$  and a system of standard free  $(\mathbb{Z}/\ell^{en}\mathbb{Z})$ -modules  $P_{\ell^e, n}$ . In what follows, we index objects by the pair  $(\ell^e, n)$ ; while the objects depend only on the value  $\ell^{en}$ , we do this to remember that  $\ell^e$  is fixed and  $n$  varies. The image of each Galois representation

$$\rho_{\ell^e, n}: \text{Gal}(\overline{k}/k) \rightarrow \text{Aut}(\text{H}_{\text{et}}^2(\overline{X}, \mu_{\ell^{en}})),$$

then lies in the finite group  $\text{Aut}_{\mathbb{Z}/\ell^{en}\mathbb{Z}}(P_{\ell^e, n})$ . Let  $\Gamma_{\ell^e, n} \subseteq \text{Aut}_{\mathbb{Z}/\ell^{en}\mathbb{Z}}(P_{\ell^e, n})$  denote the image under this identification. We are interested in  $\Gamma_{\ell^e, n}$ -conjugacy classes of elements of  $\Gamma_{\ell^e, n}$  that may be realized as images of Frobenius conjugacy classes  $[\text{Frob}_{\mathfrak{p}}]$  for  $\overline{X}$ , with  $\mathfrak{p}$  a finite place of  $k$ .

Using the universal coefficient theorem and the comparison results for analytic and étale cohomology with torsion coefficients, we have reduction morphisms

$$\text{H}^2(X(\mathbb{C}), \mathbb{Z}(1)) \twoheadrightarrow \text{H}^2(X(\mathbb{C}), (\mathbb{Z}/\ell^{en}\mathbb{Z})(1)) \simeq \text{H}_{\text{et}}^2(\overline{X}, \mu_{\ell^{en}}) \simeq P_{\ell^e, n},$$

whence a map

$$\mathcal{MT}(\text{H}) \subset \text{GL}(\text{H}) \rightarrow \text{Aut}(\text{H}_{\text{et}}^2(\overline{X}, \mu_{\ell^{en}})).$$

Let  $U_{\ell^e, n} \subset \Gamma_{\ell^e, n}$  denote the elements of  $\Gamma_{\ell^e, n}$  whose preimages under this map all lie in the set  $U$  constructed in Proposition 3.1, i.e., the only eigenvalue that is a root of unity is 1, with minimal multiplicity. This is nonempty for  $n \gg 0$  because  $U$  is  $\ell$ -adically open and elements of  $U_{\ell^e, n}$  correspond to  $\ell$ -adic open balls in  $U$ . Finally, let  $A_{\ell^e, n}$  be the set of elements in  $P_{\ell^e, n}$  that are congruent to  $\alpha$  modulo  $\ell^e$ .

**Lemma 3.2.** *There is an  $n > 0$  and a class  $\gamma \in U_{\ell^e, n}$  such that any  $\tilde{\gamma} \in U$  lying over  $\gamma$  has  $(+1)$ -eigenspace whose reduction mod  $\ell^{en}$  contains an element  $\alpha' \in A_{\ell^e, n}$  with  $\gamma(\alpha') = \alpha'$ . In particular, the reduction mod  $\ell^e$  contains  $\alpha$ .*

<sup>2</sup>Please see the Correction below.

*Proof.* By Corollary 2.9, there exists an  $n_0 > 0$  such that  $\text{im}(\rho_{\ell^e, n}) = (\pi_{n, n_0}^{\ell^e})^{-1}(\text{im}(\rho_{\ell^e, n_0}))$  for all  $n \geq n_0$ , and in particular, the image  $\text{im}(\rho_{\ell^e, n}) = \Gamma_{\ell^e, n}$  contains all elements of  $\mathcal{MT}(\mathbb{H})(\mathbb{Z}/\ell^{en}\mathbb{Z})$  that are congruent to the identity mod  $\ell^{en_0}$ .

We first claim that the set

$$S := \{g \in U(\mathbb{Z}/\ell^{en}\mathbb{Z}) : g \bmod \ell^{en_0} = \text{Id}\}$$

is non-empty for some  $n > n_0$ . Indeed, by [Pop14, §1.A.3], the field  $\mathbb{Q}_\ell$  is a large field, and since  $\mathcal{MT}(\mathbb{H})$  has a smooth point, the  $\ell$ -adic points of  $\mathcal{MT}(\mathbb{H})$  are Zariski dense [Pop14, Proposition 2.6]. Since  $U$  is Zariski open, this implies that, despite the fact that  $\text{Id} \notin U$ , the  $\ell$ -adic neighborhood around  $\text{Id}$  contains points in  $U$ . By taking reductions of these points, we find some  $n > n_0$  for which  $S \neq \emptyset$ . Note that  $S \subset \Gamma_{\ell^e, n}$ .

By Proposition 3.1, elements in  $U$  have minimal  $(+1)$ -eigenspace, and for any subspace  $W \subset \mathbb{H}$  of dimension  $\dim_{\mathbb{Q}} E$ , there exists an element of  $U$  with  $(+1)$ -eigenspace equal to  $W$ . We restrict to considering elements in  $U$  whose reductions mod  $\ell^{en}$  are in  $S$ , and note that this congruence condition mod  $\ell^{en_0}$  does not restrict which subspaces of  $\mathbb{H}$  can appear as the  $(+1)$ -eigenspace. Thus, there must exist an element  $\tilde{\gamma} \in U$  whose mod  $\ell^{en}$  reduction  $\gamma$  is in  $S$  and the  $(+1)$ -eigenspace of  $\tilde{\gamma}$  contains a vector which reduces mod  $\ell^e$  to  $\alpha$ . The preimage of  $\gamma$  in  $\mathcal{MT}(\mathbb{H})_{\mathbb{Z}_\ell}$  is an  $\ell$ -adic open ball with center in  $U$ , and hence, after making  $n$  larger and replacing  $\gamma$  if necessary to ensure the preimage of  $\gamma$  is contained in  $U$ , every element in  $U$  lying over  $\gamma$  has  $(+1)$ -eigenspace whose reduction mod  $\ell^e$  contains  $\alpha$ .  $\square$

#### 4. PROOF OF THEOREM 1.1

In this section we use the notation of Theorem 1.1, and suppose  $\alpha \in \text{Br}(X)$  has order  $m$ . Write  $\mathcal{T}$  for a finite set of places of  $k$  containing all the archimedean places, and let  $\mathcal{O}_{k, \mathcal{T}}$  denote the corresponding ring of  $\mathcal{T}$ -integers. Fix a smooth proper model  $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_{k, \mathcal{T}}$  of  $X$  for a suitable  $\mathcal{T}$ . When considering reductions  $X_{\mathfrak{p}} := \mathcal{X} \times_{\mathcal{O}_{k, \mathcal{T}}} \mathbb{F}_{\mathfrak{p}}$  of  $X$  modulo a prime  $\mathfrak{p}$ , we tacitly assume throughout that: (1)  $\mathfrak{p} \notin \mathcal{T}$ , and (2) the element  $\alpha \in \text{Br}(X)[m]$  is cohomologically unramified for  $\mathfrak{p} \notin \mathcal{T}$ , so a lift of  $\alpha$  under the surjection  $\text{H}^2(X, \mu_m) \rightarrow \text{Br}(X)[m]$  can be spread out to an element  $\mathcal{A} \in \text{H}^2(\mathcal{X}, \mu_m)$  (cf. [Poo17, Corollary 6.6.11]). Let  $\alpha_{\mathfrak{p}} \in \text{Br}(X_{\mathfrak{p}})$  be the image of  $\mathcal{A}$  under the specialization  $\text{H}^2(\mathcal{X}, \mu_m) \rightarrow \text{H}^2(X_{\mathfrak{p}}, \mu_m) \rightarrow \text{Br}(X_{\mathfrak{p}})[m]$ . Note that the set

$$\mathcal{S}(X, \alpha) := \{\mathfrak{p} \in \Omega_{k, \mathcal{T}}^0 : \alpha_{\mathfrak{p}} = 0 \in \text{Br}(X_{\mathfrak{p}})\}$$

depends on the choice of model  $\mathcal{X}$ . A different choice of model will produce a set that differs from this one at finitely many places. We will show that for a choice of model  $\mathcal{X}$ ,  $\mathcal{S}(X, \alpha)$  contains a set of positive natural density, in which case the result holds for any choice of smooth proper model.

Consider the image of  $\alpha \in \text{Br}(X)[m]$  under the natural map  $\text{Br}(X) \rightarrow \text{Br}(\overline{X})$ , which we will continue to call  $\alpha$ . Note that in  $\text{Br}(\overline{X})$ ,  $\alpha$  has order  $m'$  for some  $m'$  dividing<sup>3</sup>  $m$ . If  $m' = 1$ , then  $\alpha$  is algebraic. We begin the proof of Theorem 1.1 by handling this case.

<sup>3</sup>It is possible to have  $1 < m' < m$ ; see, e.g., [GS19].

**Lemma 4.1.** *If the class  $\alpha \in \text{Br}(X)[m]$  is algebraic, then there is a set  $\mathcal{S}$  of places of  $k$  of positive natural density such that  $\alpha_{\mathfrak{p}} = 0$  in  $\text{Br}(X_{\mathfrak{p}})$  for all  $\mathfrak{p} \in \mathcal{S}$ .*

*Proof.* Let  $K/k$  be a finite Galois extension that splits  $\text{NS}(\overline{X})$ , i.e., an extension for which  $\text{NS}(X_K) \simeq \text{NS}(\overline{X})$ . The Hochschild–Serre spectral sequence furnishes an isomorphism

$$\text{Br}_1(X_K)/\text{Br}_0(X_K) \xrightarrow{\sim} \text{H}^1(\text{Gal}(\overline{K}/K), \text{NS}(X_K)),$$

where  $\text{Br}_1(X_K) := \ker(\text{Br}(X_K) \rightarrow \text{Br}(\overline{X}))$  and  $\text{Br}_0(X_K) := \text{im}(\text{Br}(K) \rightarrow \text{Br}(X))$ ; see [VA13, §3.4]. Since the action of  $\text{Gal}(\overline{K}/K)$  on  $\text{NS}(X_K)$  is trivial, the Galois cohomology group  $\text{H}^1(\text{Gal}(\overline{K}/K), \text{NS}(X_K))$  vanishes, so  $\alpha$  becomes constant in  $\text{Br}(X_K)$ .

We claim that for  $\mathfrak{p} \subset \mathcal{O}_{k,\mathcal{T}}$  that split completely in  $\mathcal{O}_K$ , the class  $\alpha_{\mathfrak{p}} \in \text{Br}(X_{\mathfrak{p}})$  is trivial. The set of such  $\mathfrak{p}$  has positive natural density, by the Chebotarev density theorem. To see that  $\alpha_{\mathfrak{p}}$  is trivial for such  $\mathfrak{p}$ , let  $\mathfrak{P}$  be any place in  $\mathcal{O}_K$  lying over  $\mathfrak{p}$ . Since  $\mathfrak{p}$  is completely split, we know that  $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_{\mathfrak{P}}$ , hence  $X_{\mathfrak{p}} \cong X_{\mathfrak{P}}$  and  $\text{Br}(X_{\mathfrak{p}}) \cong \text{Br}(X_{\mathfrak{P}})$ . Thus to consider  $\alpha_{\mathfrak{p}}$ , we can first consider  $\alpha$  in  $\text{Br}(X_K)$  and then specialize to  $\text{Br}(X_{\mathfrak{P}})$ . Since  $\alpha$  is algebraic, its image in  $\text{Br}(X_K)$  lies in  $\text{Br}_0(X_K)$ , and so it specializes modulo  $\mathfrak{P}$  into  $\text{Br}_0(X_{\mathfrak{P}}) \subseteq \text{Br}(X_{\mathfrak{P}})$ , which is trivial. Hence  $\alpha$  maps to zero in  $\text{Br}(X_{\mathfrak{p}})$ .  $\square$

By Lemma 4.1, it suffices to prove Theorem 1.1 in the case that  $\alpha$  is transcendental, i.e.  $1 < m' \leq m$ . By smooth base change (cf. Section 2.5), there is an isomorphism  $\text{H}_{\text{et}}^2(\overline{X}, \mu_m) \xrightarrow{\sim} \text{H}_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$ , depending on the choice of model  $\mathcal{X}$ , which determines the following commutative diagram of short exact sequences coming from the Kummer sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{NS}(\overline{X}) \otimes \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \text{H}_{\text{et}}^2(\overline{X}, \mu_m) & \longrightarrow & \text{Br}(\overline{X})[m] \longrightarrow 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow \\ 0 & \longrightarrow & \text{NS}(\overline{X}_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \text{H}_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m) & \longrightarrow & \text{Br}(\overline{X}_{\mathfrak{p}})[m] \longrightarrow 0. \end{array}$$

Note that the image of  $\alpha$  under the map  $\text{Br}(\overline{X})[m] \rightarrow \text{Br}(\overline{X}_{\mathfrak{p}})[m]$  is  $\alpha_{\mathfrak{p}}$ . We will show there is a set of finite places  $\mathfrak{p}$  of positive natural density for which a lift of  $\alpha$  to  $\text{H}_{\text{et}}^2(\overline{X}, \mu_m)$  is, upon reduction, in the image of  $\text{NS}(\overline{X}_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z}$ .

By the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{NS}(X) \otimes \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \text{H}_{\text{et}}^2(X, \mu_m) & \longrightarrow & \text{Br}(X)[m] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\text{NS}(\overline{X}) \otimes \mathbb{Z}/m\mathbb{Z})^{\text{Gal}(\overline{k}/k)} & \longrightarrow & \text{H}_{\text{et}}^2(\overline{X}, \mu_m)^{\text{Gal}(\overline{k}/k)} & \longrightarrow & (\text{Br}(\overline{X})[m])^{\text{Gal}(\overline{k}/k)} \end{array}$$

with exact rows, we can pick a lift of  $\alpha \in \text{Br}(X)$  to a Galois-invariant class in  $\text{H}_{\text{et}}^2(\overline{X}, \mu_m)$ , which we will again call  $\alpha$ . Since the Brauer class is transcendental,  $\alpha \in \text{H}_{\text{et}}^2(\overline{X}, \mu_m)$  is not in the image of  $\text{NS}(\overline{X}) \otimes \mathbb{Z}/m\mathbb{Z}$ , and we can apply the results of Section 3.

Write  $m = \ell_1^{e_1} \cdots \ell_r^{e_r}$  with  $\ell_i$  distinct primes,  $1 \leq i \leq r$ . Analogous to the notation introduced in §§2–3, we will write  $A_{m,n} \subset P_{m,n}$  for  $\alpha' \in P_{m,n}$  which reduce to  $\alpha \bmod m$  and

$\Gamma_{m,n} \subset \text{Aut}_{\mathbb{Z}/m^n\mathbb{Z}}(P_{m,n})$  for the image of  $\rho_{m,n}$ . By identifying  $P_{m,n}$  with  $\bigoplus_{i=1}^r P_{\ell_i^{e_i}, n}$ , we find that  $\alpha = (\alpha_i)_{1 \leq i \leq r}$  for  $\alpha_i \in H_{\text{et}}^2(\overline{X}, \mu_{\ell_i^{e_i}})$ .

For each  $\alpha_i$ ,  $1 \leq i \leq r$ , fix the  $n_i$  and  $\gamma_i \in U_{\ell_i^{e_i}, n_i}$  coming from Lemma 3.2. By taking any of the  $n_i$  larger if necessary, we can assume without loss of generality that  $n_1 = n_2 = \dots = n_r$ , and we will call this common integer  $n$ . Also by Lemma 3.2, there is an  $\alpha'_i \in A_{\ell_i^{e_i}, n}$  such that  $\gamma_i(\alpha'_i) = \alpha_i$ . Set

$$\alpha' := (\alpha'_i)_{1 \leq i \leq r} \in A_{m,n}, \quad \text{and} \quad \gamma := (\gamma_i)_{1 \leq i \leq r},$$

which is in  $\Gamma_{m,n}$  by Corollary 2.11 (enlarge  $n$  if necessary so that  $n > m_0$  in the corollary). By construction, we have  $\gamma(\alpha') = \alpha'$ . We note there is no guarantee that  $\alpha'$  is Galois-invariant. Recall from §2.5 that choosing an embedding  $\bar{k} \hookrightarrow \bar{k}_{\mathfrak{p}}$  is equivalent to choosing an element

$$\text{Frob}_{\mathfrak{p}} \in \text{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}}) \hookrightarrow \text{Gal}(\bar{k}/k)$$

that reduces to  $\text{Frob} \in \text{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}})$ , well-defined up to the inertia group  $I_{\mathfrak{p}}$ .

Since  $\alpha'$  need not be Galois invariant, its mod  $\mathfrak{p}$  image in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n})$  depends on the above choice of element in the conjugacy class  $[\text{Frob}_{\mathfrak{p}}] \subset \text{Gal}(\bar{k}/k)$ . However,  $\alpha'$  reduces to  $\alpha \in P_{m,1} \cong H_{\text{et}}^2(\overline{X}, \mu_m)$ , which is Galois invariant. As a consequence, the image of  $\alpha$  modulo  $\mathfrak{p}$  is well-defined: it is exactly the class in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$  which maps to the base change of  $\alpha_{\mathfrak{p}}$  in  $\text{Br}(\overline{X}_{\mathfrak{p}})[m]$ . For this reason, we will also call it  $\alpha_{\mathfrak{p}} \in H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$ .

**Proposition 4.2.** *Suppose that the  $\Gamma_{m,n}$ -conjugacy class  $\rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$  coincides with the  $\Gamma_{m,n}$ -conjugacy class of  $\gamma$ . Then  $\alpha_{\mathfrak{p}}$  is algebraic.*

*Proof.* We begin with an outline of the proof. First, we show that there is a choice of element  $\sigma \in [\text{Frob}_{\mathfrak{p}}]$ , corresponding to  $\gamma$ , for which  $\alpha'$  maps to an algebraic class under the induced isomorphism  $P_{m,n} \cong H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n})$  of cyclic modules. Then we show that, for any other element in the  $\Gamma_{m,n}$ -conjugacy class  $\rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ , there is a choice of element in  $[\text{Frob}_{\mathfrak{p}}]$  for which a different class in the orbit  $\Gamma_{m,n} \cdot \alpha'$  maps to an algebraic class in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n})$ . The last step is to observe that every class in  $\Gamma_{m,n} \cdot \alpha'$  is equal to  $\alpha \bmod m$ , so independently of the choice of element in  $\rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ , the image of  $\alpha$  in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$  is well-defined and  $\alpha$  maps to an algebraic class in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$ .

Since  $\gamma \in \rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ , choose  $\sigma \in [\text{Frob}_{\mathfrak{p}}]$  so that  $\rho_{m,n}(\sigma) = \gamma$ . This choice determines the horizontal isomorphisms in the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1)) & \xrightarrow{\sim} & \bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1)) \\ \downarrow & & \downarrow \\ H_{\text{et}}^2(\overline{X}, \mu_{m^n}) & \xrightarrow{\sim} & H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n}) \end{array} \quad (4.1)$$

in such a way that the action of  $(\prod_{i=1}^r \rho_{\ell_i})(\sigma)$  on  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1))$  is compatible with the diagonal action of  $\text{Frob}$  on  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1))$  (see §2.5). Note that the vertical maps are surjective because  $H_{\text{et}}^3(\overline{X}, \mathbb{Z}_{\ell_i}(1)) = 0$  for K3 surfaces. The vertical surjections are induced

by the surjection

$$\prod_{i=1}^r \mathbb{Z}_{\ell_i}(1) \twoheadrightarrow \mu_{m^n},$$

so the action of  $(\prod_{i=1}^r \rho_{\ell_i})(\sigma)$  on  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1))$  is compatible via the surjection with the action of  $\rho_{m,n}(\sigma)$  on  $H_{\text{et}}^2(\overline{X}, \mu_{m^n})$ .

Since  $\rho_{\ell_i}(\sigma)$  reduces to  $\rho_{\ell_i^{e_i}, n}(\sigma)$  and  $\rho_{\ell_i^{e_i}, n}(\sigma) = \gamma_i \in U_{\ell_i^{e_i}, n}$  for  $1 \leq i \leq r$ , we have  $\rho_{\ell_i}(\sigma) \in U(\mathbb{Z}_{\ell_i})$ .

By the Integral Tate conjecture, the classes in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1))$  on which Frob acts by roots of unity are exactly the algebraic classes. Hence the classes of  $H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1))$  on which  $\rho_{\ell_i}(\sigma)$  acts by roots of unity are exactly the classes that map to algebraic classes in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1))$ . Since  $\rho_{\ell_i}(\sigma) \in U(\mathbb{Z}_{\ell_i})$ , the only eigenvalue of  $\rho_{\ell_i}(\sigma)$  that is a root of unity is 1.

By the compatibility of the action of  $(\prod_{i=1}^r \rho_{\ell_i})(\sigma)$  on  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1))$  and  $\gamma$  on  $H_{\text{et}}^2(\overline{X}, \mu_{m^n})$  in the above diagram, the image of the eigenspace

$$V := \mathfrak{E} \left( \left( \prod_{i=1}^r \rho_{\ell_i} \right) (\sigma), 1 \right) \subset \bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}, \mathbb{Z}_{\ell_i}(1))$$

in  $H_{\text{et}}^2(\overline{X}, \mu_{m^n})$  is a subspace of the eigenspace  $\mathfrak{E}(\gamma, 1) \subset H_{\text{et}}^2(\overline{X}, \mu_{m^n})$ . On the other hand, since  $\rho_{\ell_i^{e_i}, n}(\sigma) = \gamma_i \in U_{\ell_i^{e_i}, n}$  and  $\gamma$  projects onto  $\gamma_i$  for each  $1 \leq i \leq r$ , the commutative diagram

$$\begin{array}{ccc} \mathcal{MT}(\mathbb{H}) & \longrightarrow & \text{Aut}(H_{\text{et}}^2(\overline{X}, \mu_{m^n})) \\ & \searrow & \downarrow \\ & & \text{Aut}(H_{\text{et}}^2(\overline{X}, \mu_{\ell_i^{e_i n}})) \end{array}$$

implies that all of the preimages of  $\gamma$  in  $\mathcal{MT}(\mathbb{H})$  lie in  $U$ . This forces the rank of  $\mathfrak{E}(\gamma, 1)$  as a  $\mathbb{Z}/m^n\mathbb{Z}$ -module to be minimal, so the image of  $V$  in  $H_{\text{et}}^2(\overline{X}, \mu_{m^n})$  must be all of  $\mathfrak{E}(\gamma, 1)$ . In particular, since  $\gamma(\alpha') = \alpha'$ , we deduce that the image of  $\alpha'$  in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n})$  via (4.1) is the image of a tuple of algebraic classes in  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1))$ . Thus, under the identification determined by  $\gamma \in \rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ ,  $\alpha'$  becomes algebraic modulo  $\mathfrak{p}$ .

Next, consider a different element  $\gamma' = \eta\gamma\eta^{-1} \in \rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$  for some  $\eta \in \Gamma_{m,n}$ , which fixes  $\eta(\alpha') \in P_{m,n}$ . Note that  $\eta(\alpha') \in A_{m,n}$  because  $\alpha$  is Galois-invariant. By the same argument as above, it follows that for this choice, the image of  $\eta(\alpha')$  in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_{m^n})$  is the image of a tuple of algebraic classes in  $\bigoplus_{i=1}^r H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mathbb{Z}_{\ell_i}(1))$ . Thus, under the identification determined by  $\gamma' \in \rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ ,  $\eta(\alpha')$  becomes algebraic modulo  $\mathfrak{p}$ .

Now reduce mod  $m$ , and recall that  $\pi_{n,1}^m$  is the reduction map from level  $m^n$  to level  $m$ . Observe that

$$\pi_{n,1}^m(\Gamma_{m,n} \cdot \alpha') = \{\alpha\},$$

since  $\Gamma_{m,n} \cdot \alpha' \subset A_{m,n}$ . Thus for every element in the conjugacy class  $\rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$ , the image  $\alpha_{\mathfrak{p}}$  of  $\alpha$  in  $H_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$  is an algebraic class. Finally, by the Kummer sequence, this means  $\alpha$  maps to zero in  $\text{Br}(\overline{X}_{\mathfrak{p}})[m]$ . Therefore,  $\alpha_{\mathfrak{p}}$  is algebraic, as desired.  $\square$

**Corollary 4.3.** *There is a set  $\mathcal{S}$  of places of  $k$  of positive natural density such that for each  $\mathfrak{p} \in \mathcal{S}$ ,  $\alpha_{\mathfrak{p}}$  is algebraic.*

*Proof.* Let  $\mathcal{S}$  be the set of  $\mathfrak{p}$  such that  $X$  has good reduction at  $\mathfrak{p}$  and the  $\Gamma_{m,n}$ -conjugacy class  $\rho_{m,n}([\text{Frob}_{\mathfrak{p}}])$  is the  $\Gamma_{m,n}$ -conjugacy class of  $\gamma$ . Since  $\Gamma_{m,n}$  is a finite group, we know by the Chebotarev Density Theorem that  $\mathcal{S}$  has positive natural density. By Proposition 4.2,  $\alpha_{\mathfrak{p}}$  is algebraic for every  $\mathfrak{p} \in \mathcal{S}$ .  $\square$

**Corollary 4.4.** *For every  $\mathfrak{p} \in \mathcal{S}$ ,  $\alpha_{\mathfrak{p}} = 0 \in \text{Br}(X_{\mathfrak{p}})$ .*

*Proof.* First, the group  $\text{Br}(\mathbb{F}_{\mathfrak{p}})$  is trivial since  $\mathbb{F}_{\mathfrak{p}}$  is a finite field; the long exact sequence of low-degree terms for the Hochschild–Serre spectral sequence shows that as a consequence, for  $\Gamma = \text{Gal}(\overline{\mathbb{F}_{\mathfrak{p}}}/\mathbb{F}_{\mathfrak{p}})$ , the natural map

$$\text{NS}(X_{\mathfrak{p}}) \rightarrow \text{NS}(\overline{X}_{\mathfrak{p}})^{\Gamma}$$

is an isomorphism. In the proof of Proposition 4.2, we see that  $\alpha_{\mathfrak{p}}$  comes from a class in  $\text{NS}(\overline{X}_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z}$ , and moreover that  $\alpha_{\mathfrak{p}}$  comes from a class on which  $\text{Frob}_{\mathfrak{p}}$  acts trivially. Thus by continuity of the  $\Gamma$ -action, this implies that  $\alpha_{\mathfrak{p}} \in \text{H}_{\text{et}}^2(\overline{X}_{\mathfrak{p}}, \mu_m)$  lifts to  $(\text{NS}(\overline{X}_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z})^{\Gamma}$  (note that  $\Gamma$  acts trivially on  $\mathbb{Z}/m\mathbb{Z}$  here). As a consequence,  $\alpha_{\mathfrak{p}}$  lifts further to  $\text{NS}(X_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z}$ . By exactness of

$$0 \rightarrow \text{NS}(X_{\mathfrak{p}}) \otimes \mathbb{Z}/m\mathbb{Z} \rightarrow \text{H}_{\text{et}}^2(X_{\mathfrak{p}}, \mu_m) \rightarrow \text{Br}(X_{\mathfrak{p}})[m] \rightarrow 0,$$

we have that  $\alpha_{\mathfrak{p}} \in \text{Br}(X_{\mathfrak{p}})[m]$  must actually be zero.  $\square$

This completes the proof of Theorem 1.1.  $\square$

## 5. APPLICATION: RATIONAL REDUCTIONS OF GENERAL CUBIC FOURFOLDS

We use the notation of §1.2. For a general smooth cubic fourfold  $(Y, K_8)$  in  $\mathcal{C}_{K_8}$ , projection away from a plane  $P$  gives a quadric surface bundle  $\varpi : \tilde{Y} := \text{Bl}_P(Y) \rightarrow \mathbb{P}^2$ , whose relative variety of lines admits a Stein factorization

$$F_1(\varpi) \xrightarrow{\varphi} X \rightarrow \mathbb{P}^2,$$

where  $\varphi$  is an étale  $\mathbb{P}^1$ -bundle and the second arrow is a double cover branched along a plane sextic, making  $X$  into a K3 surface [HVAV11, §5]. If  $\alpha \in \text{Br}(X)[2]$  is the class of the étale  $\mathbb{P}^1$ -bundle  $\varphi$  then  $Y$  is rational whenever  $\alpha = 0$ . [HPT18, §3].

Similarly, a general element  $(Y, K_{18})$  in  $\mathcal{C}_{K_{18}}$  contains a sextic elliptic ruled surface  $T$  [AHTVA19, Theorem 2]; projection away from  $T$  induces a fibration

$$\psi : \tilde{Y} = \text{Bl}_T(Y) \rightarrow \mathbb{P}^2 \tag{5.1}$$

whose general fiber is (a projection onto  $\mathbb{P}^5$  of) a sextic del Pezzo surface [AHTVA19, §§1,2]. By [AHTVA19, Prop. 10], there exists a K3 surface  $X$  of degree two and an element  $\alpha \in \text{Br}(X)[3]$  such that  $Y$  is rational if  $\alpha = 0$ .

*Proof of Theorem 1.5.* Choose a smooth proper model  $\mathcal{Y}$  for  $Y$  over  $\mathcal{O}_{k,\mathcal{T}}$  for  $\mathcal{T}$  a finite set of places. The constructions of  $\varpi$  and  $\phi$  above also work over  $\mathcal{O}_{k,\mathcal{T}}$ , which gives rise to a

relative K3 surface and relative Brauer class associated to  $\mathcal{Y}$ . Using this model in §4, the result follows. Note that in all cases, we have  $\mathrm{NS}(X_{\mathbb{C}}) \simeq \mathbb{Z}$ , so by Remark 1.2 the required hypotheses on the K3 surface  $X$  are satisfied.  $\square$

There exist pairs  $(X, \alpha)$ , defined over  $\mathbb{Q}$ , arising from  $\varpi$  or  $\phi$  above, and satisfying the hypotheses of Theorem 1.1. An example with  $Y \in \mathcal{C}_{K_8}$  is given in [HVAV11]; an example with  $Y \in \mathcal{C}_{K_{18}}$  is given in [BVA20]. In both cases, the Brauer class given is transcendental.

## 6. APPLICATION: REDUCTIONS OF TWISTED DERIVED EQUIVALENCES OF K3 SURFACES

Let  $X$  be a K3 surface of degree  $2d$  with polarization  $h$ . Fix a primitive Mukai vector  $v = (r, c, s) \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ , and let  $M := M_h(v)$  be the moduli space of Gieseker  $h$ -semi-stable sheaves on  $X$  of rank  $r$ , first Chern class  $c$ , and Euler characteristic  $r + s$ . Assume that  $h$  is  $v$ -generic, i.e. every  $h$ -semi-stable sheaf over  $\bar{k}$  is  $h$ -stable. Then  $M$  parametrizes geometrically stable sheaves and is a smooth projective variety. When  $M$  is nonempty (e.g., when  $v$  is effective) and  $v^2 = 0$ ,  $M$  is again a K3 surface [Muk87b, Theorem 1.4]. A reference for these moduli spaces of sheaves is [HL10]. Over nonclosed fields, see [Cha16] or [Fre20].

There is a Brauer class  $\alpha \in \mathrm{Br}(M)$ , of order dividing the gcd over all Mukai vectors  $w$  of  $(r, c, s) \cdot w$  (where  $\cdot$  means the Mukai pairing), which obstructs the existence of a universal sheaf on  $X \times M$ . If  $\alpha = 0$ , then  $M$  is a fine moduli space, and the universal sheaf on  $X \times M$  induces a  $k$ -linear equivalence  $D^b(M) \cong D^b(X)$  [Or97, Theorem 3.11]. More generally, there is an  $\pi_M^* \alpha^{-1}$ -twisted universal sheaf on  $X \times M$ , for  $\pi_M$  the projection onto  $M$ , which induces an equivalence  $D^b(M, \alpha) \cong D^b(X)$ . [Că102, Theorem 1.3] (see also [LO15, Theorem 3.16], [LMS14, Proposition 3.4.2] for non-closed fields).

**Example 6.1.** Let  $X$  be a smooth complete intersection of three quadrics in  $\mathbb{P}^5$ , so that  $X$  is a degree 8 K3 surface, and assume  $\mathrm{NS}(\bar{X}) = \mathbb{Z}h$ . Then  $M := M_h(2, h, 2)$  is a degree 2 K3 surface. Points of  $M$  come from bundles on the quadrics in the net  $\Lambda \cong \mathbb{P}^2$  of quadrics containing  $X$ ; this realizes  $M$  as the double cover of  $\mathbb{P}^2$  branched along the sextic curve corresponding to degenerate quadrics in  $\Lambda$ . The Brauer class  $\alpha \in \mathrm{Br}(M)$  obstructing the existence of a universal sheaf has order 2, and is represented by a Brauer-Severi variety, given by taking the Stein factorization of the universal family of quadrics in  $\Lambda$ . For more details, see [Muk84, Example 0.9], [Muk87a, Example 2.2], [IK13, §3], or [MSTVA17, §3.2].

*Proof of Theorem 1.6.* For  $X$  and  $(M, \alpha)$  as above over a number field  $k$ , there is a finite set of places  $\mathcal{T}$  and a smooth proper model  $\mathcal{X} \times \mathcal{M}$  over  $\mathcal{O}_{k, \mathcal{T}}$  along with a relative twisted universal sheaf on  $\mathcal{X} \times \mathcal{M}$ . For any place  $\mathfrak{p} \notin \mathcal{T}$ , the relative twisted universal sheaf specializes to a twisted universal sheaf on the reduction  $X_{\mathfrak{p}} \times M_{\mathfrak{p}}$ . For the derived equivalence  $D^b(X_{\mathfrak{p}}) \cong D^b(M_{\mathfrak{p}})$  that we seek, we need to know that  $M_{\mathfrak{p}}$  contains only stable sheaves. This is not guaranteed by  $M_{\mathfrak{p}}$  being smooth, but the set of  $\mathfrak{p}$  for which  $M_{\mathfrak{p}}$  contains properly semistable sheaves is finite. Indeed, being geometrically stable is an open condition [HL10, Prop. 2.3.1], and the morphism  $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_{k, \mathcal{T}}$  is projective [Lan04, Theorem 0.2]. Finiteness follows since the generic fiber contains only stable sheaves. Then Theorem 1.1 gives the result.  $\square$

Pairs  $(M, \alpha)$  satisfying the hypotheses of Theorem 1.1 exist. By Remark 1.2, the example in [MSTVA17, §5.4] gives a K3 surface as in Example 6.1 such that the associated twisted K3 surface satisfies the hypotheses of Theorem 1.1. Moreover, the Brauer class is transcendental.

### CORRECTION

We are grateful to Ziquan Yang for pointing out an error in the formulation and proof of Theorem 2.4 and Corollary 2.5. The correct conclusion of the Theorem should be “the Mumford-Tate group is the subgroup

$$\mathbb{Q}^* \cdot \text{Res}_{E/\mathbb{Q}}(\text{SO}_E(\text{T}(X)_{\mathbb{Q}}, \Phi)) \subset \text{GO}(\text{T}(X)_{\mathbb{Q}}, \phi),$$

where  $\mathbb{Q}^*$  acts via scaling”; this follows directly from Zarhin’s original analysis. The mistake is that the torus  $T_E$ , associated with scalar multiplication by  $E$ , is not a subtorus of the orthogonal group associated with  $(\text{T}(X)_{\mathbb{Q}}, \phi)$ . Rather, it lives in the larger group  $\text{GL}(\text{T}(X)_{\mathbb{Q}})$ , as multiplying by  $e \in E$  rescales the quadratic form  $\Phi$  by  $e^2$ . The correct approach is to intersect the centralizer of  $T_E$  in this larger group (which is connected) with  $\text{Res}_{E/\mathbb{Q}}(\text{SO}_E(\text{T}(X)_{\mathbb{Q}}, \Phi))$ ; this intersection need not be connected. The correct formulation of the Corollary should refer to its identity component. This error does not affect the application to our main theorem.

### REFERENCES

- [AHTVA19] N. Addington, B. Hassett, Yu. Tschinkel, and A. Várilly-Alvarado, *Cubic fourfolds fibered in sextic del Pezzo surfaces*, Amer. J. Math. **141** (2019), no. 6, 1479–1500. [↑1, 1.2, 5, 5](#)
- [AIM] *AimPL: Rational subvarieties in positive characteristic*, available at <http://aimpl.org/ratsubvarpos>. [↑1, 1.4](#)
- [And96] Y. André, *On the Shafarevich and Tate conjectures for hyper-Kähler varieties*, Math. Ann. **305** (1996), no. 2, 205–248. [↑2.7](#)
- [AO83] H. Appelgate and H. Onishi, *Similarity problem over  $\text{SL}(n, \mathbf{Z}_p)$* , Proc. Amer. Math. Soc. **87** (1983), no. 2, 233–238. [↑2.5](#)
- [BVA20] J. Berg and A. Várilly-Alvarado, *Odd order obstructions to the Hasse principle on general K3 surfaces*, Math. Comp. **89** (2020), no. 323, 1395–1416. [↑5](#)
- [Bog80] F. A. Bogomolov, *Sur l’algébricité des représentations  $l$ -adiques*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 15, A701–A703 (French, with English summary). [↑2.4](#)
- [CM20] A. Cadoret and B. Moonen, *Integral and adelic aspects of the Mumford-Tate conjecture*, J. Inst. Math. Jussieu **19** (2020), no. 3, 869–890. [↑2.3, 2.10](#)
- [Că102] A. Căldăraru, *Nonfine moduli spaces of sheaves on K3 surfaces*, Int. Math. Res. Not. **20** (2002), 1027–1056. [↑1.3, 6](#)
- [Cha13] F. Charles, *The Tate conjecture for K3 surfaces over finite fields*, Invent. Math. **194** (2013), no. 1, 119–145. [↑2.5](#)
- [Cha14] ———, *On the Picard number of K3 surfaces over number fields*, Algebra Number Theory **8** (2014), no. 1, 1–17. [↑1.1, 1.3, 3](#)
- [Cha16] ———, *Birational boundedness for holomorphic symplectic varieties, Zarhin’s trick for K3 surfaces, and the Tate conjecture*, Ann. of Math. (2) **184** (2016), no. 2, 487–526. [↑6](#)
- [CEJ20] E. Costa, A.-S. Elsenhans, and J. Jahnel, *On the distribution of the Picard ranks of the reductions of a K3 surface*, Res. Number Theory **6** (2020), no. 3, [Paper No. 27, 25 pp.] [↑1.1](#)

- [CT14] E. Costa and Yu. Tschinkel, *Variation of Néron-Severi ranks of reductions of K3 surfaces*, Exp. Math. **23** (2014), no. 4, 475–481. [↑1.1](#)
- [dJK00] A. J. de Jong and N. Katz, *Monodromy and the Tate conjecture: Picard numbers and Mordell-Weil ranks in families. part A*, Israel J. Math. **120** (2000), no. part A, 47–79. [↑1.1](#)
- [Ell04] J. Ellenberg, *K3 surfaces over number fields with geometric Picard number one*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 135–140. [↑1.2](#)
- [EJ21] A.-S. Elsenhans and J. Jahnel, *Frobenius trace distributions for K3 surfaces*, arXiv preprint arXiv:2102.10620 (2021). [↑1.3](#)
- [Fal88] G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299. [↑2.4](#)
- [Fre20] S. Frei, *Moduli spaces of sheaves on K3 surfaces and Galois representations*, Selecta Math. (N.S.) **26** (2020), no. 1, Paper No. 6, 16. [↑6](#)
- [Fre] ———, *Rational reductions of geometrically irrational varieties.*, In preparation. [↑1](#)
- [GS80] F. Grunewald and D. Segal, *Some general algorithms. I. Arithmetic groups*, Ann. of Math. (2) **112** (1980), no. 3, 531–583. [↑2.5](#)
- [GS19] D. Gvirts and A. N. Skorobogatov, *Cohomology and the Brauer groups of diagonal surfaces*, arXiv preprint arXiv:1905.11869 (2019). [↑3](#)
- [Has99] B. Hassett, *Some rational cubic fourfolds*, J. Algebraic Geom. **8** (1999), no. 1, 103–114. [↑1](#)
- [HPT18] B. Hassett, A. Pirutka, and Yu. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, Acta Math. **220** (2018), no. 2, 341–365. [↑1](#), [1.2](#), [5](#)
- [HVAV11] B. Hassett, A. Várilly-Alvarado, and P. Varilly, *Transcendental obstructions to weak approximation on general K3 surfaces*, Adv. Math. **228** (2011), no. 3, 1377–1404. [↑5](#), [5](#)
- [Huy16] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. [↑1.2](#)
- [HL10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. [↑6](#), [6](#)
- [IK13] C. Ingalls and M. Khalid, *Rank 2 sheaves on K3 surfaces: a special construction*, Q. J. Math. **64** (2013), no. 2, 443–470. [↑6.1](#)
- [KMP16] W. Kim and K. Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), Paper No. e28, 34. [↑2.5](#)
- [Kuz10] A. Kuznetsov, *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston, Boston, MA, 2010, pp. 219–243. [↑1](#), [1.2](#)
- [Lan04] A. Langer, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), no. 1, 251–276. [↑6](#)
- [LMS14] M. Lieblich, D. Maulik, and A. Snowden, *Finiteness of K3 surfaces and the Tate conjecture*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 285–308 (English, with English and French summaries). [↑2.5](#), [6](#)
- [LO15] M. Lieblich and M. Olsson, *Fourier-Mukai partners of K3 surfaces in positive characteristic*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 5, 1001–1033 (English, with English and French summaries). [↑6](#)
- [MP15] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668. [↑2.5](#)
- [Mas96] D. Masser, *Specializations of endomorphism rings of abelian varieties*, Bull. Soc. Math. France **124** (1996), no. 3, 457–476 (English, with English and French summaries). MR1415735 [↑1.2](#)
- [MSTVA17] K. McKinnie, J. Sawon, S. Tanimoto, and A. Várilly-Alvarado, *Brauer groups on K3 surfaces and arithmetic applications*, Brauer groups and obstruction problems, Progr. Math., vol. 320, Birkhäuser/Springer, Cham, 2017, pp. 177–218. [↑6.1](#), [6](#)

- [Muk84] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), no. 1, 101–116. [↑1.3](#), [6.1](#)
- [Muk87a] ———, *Moduli of vector bundles on K3 surfaces and symplectic manifolds*, Sūgaku **39** (1987), no. 3, 216–235 (Japanese). Sugaku Expositions **1** (1988), no. 2, 139–174. [↑6.1](#)
- [Muk87b] ———, *On the moduli space of bundles on K3 surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413. [↑6](#)
- [Noo95] R. Noot, *Abelian varieties—Galois representation and properties of ordinary reduction*, Compositio Math. **97** (1995), no. 1-2, 161–171. Special issue in honour of Frans Oort. [↑1.2](#)
- [Orl97] D. O. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381. Algebraic geometry, 7. [↑6](#)
- [Poo17] B. Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. [↑4](#)
- [Pop14] F. Pop, *Little survey on large fields—old & new*, Valuation theory in interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014, pp. 432–463. [↑3](#)
- [Sch11] C. Schnell, *Two lectures about Mumford-Tate groups*, Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), no. 2, 199–216. [↑2.1](#), [2.1](#)
- [Ser13] J.-P. Serre, *Un critère d’indépendance pour une famille de représentations  $\ell$ -adiques*, Comment. Math. Helv. **88** (2013), no. 3, 541–554 (French, with English summary). [↑2.4](#)
- [Ser86] ———, *Letter to K. Ribet of January 1, 1981, in Œuvres. Vol. IV*, Springer, Berlin, 1986 (French). 1985–1998. [↑2.6](#)
- [SGAIV-3] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4); Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. [↑2.5](#)
- [SSTT19] A. N. Shankar, A. Shankar, Y. Tang, and S. Tayou, *Exceptional jumps of Picard ranks of reductions of K3 surfaces over number fields*, arXiv preprint arXiv:1909.07473v1 (2019). [↑1.4](#)
- [Spr09] T. A. Springer, *Linear algebraic groups*, 2nd ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. [↑2.1](#)
- [Tan90] S. G. Tankeev, *Surfaces of K3 type over number fields and the Mumford-Tate conjecture*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 4, 846–861 (Russian); English transl., Math. USSR-Izv. **37** (1991), no. 1, 191–208. [↑2.7](#)
- [Tan95] S. G. Tankeev, *Surfaces of K3 type over number fields and the Mumford-Tate conjecture. II*, Izv. Ross. Akad. Nauk Ser. Mat. **59** (1995), no. 3, 179–206 (Russian, with Russian summary); English transl., Izv. Math. **59** (1995), no. 3, 619–646. [↑2.7](#)
- [Tat68] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 189–214. [↑2.5](#)
- [Ter85] T. Terasoma, *Complete intersections with middle Picard number 1 defined over  $\mathbf{Q}$* , Math. Z. **189** (1985), no. 2, 289–296. [↑1.2](#)
- [Tot16] B. Totaro, *Hypersurfaces that are not stably rational*, J. Amer. Math. Soc. **29** (2016), no. 3, 883–891. [↑1](#)
- [vL07] R. van Luijk, *K3 surfaces with Picard number one and infinitely many rational points*, Algebra Number Theory **1** (2007), no. 1, 1–15, DOI 10.2140/ant.2007.1.1. MR2322921 [↑1.2](#)
- [VA13] A. Várilly-Alvarado, *Arithmetic of del Pezzo surfaces*, Birational geometry, rational curves, and arithmetic, Simons Symp., Springer, Cham, 2013, pp. 293–319. [↑4](#)
- [Zar83] Yu. G. Zarhin, *Hodge groups of K3 surfaces*, J. Reine Angew. Math. **341** (1983), 193–220. [↑1.1](#), [2.2](#), [2.4](#), [2.1](#), [3](#)

DEPARTMENT OF MATHEMATICS MS 136, RICE UNIVERSITY, 6100 S. MAIN ST., HOUSTON, TX  
77005, USA

*Email address:* [sf31@rice.edu](mailto:sf31@rice.edu)

*URL:* <http://math.rice.edu/~sf31>

INSTITUTE FOR COMPUTATIONAL AND EXPERIMENTAL RESEARCH IN MATHEMATICS, 121 SOUTH MAIN  
STREET, 11TH FLOOR, BOX E, PROVIDENCE, RI 02903 USA

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, BOX 1917 151 THAYER STREET, PROVIDENCE,  
RHODE ISLAND 02912 USA

*Email address:* [brendan\\_hassett@brown.edu](mailto:brendan_hassett@brown.edu)

*URL:* <http://http://www.math.brown.edu/bhassett/>

DEPARTMENT OF MATHEMATICS MS 136, RICE UNIVERSITY, 6100 S. MAIN ST., HOUSTON, TX  
77005, USA

*Email address:* [av15@rice.edu](mailto:av15@rice.edu)

*URL:* <http://math.rice.edu/~av15>