FAILURE OF THE HASSE PRINCIPLE ON GENERAL K3 SURFACES

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ABSTRACT. We show that transcendental elements of the Brauer group of an algebraic surface can obstruct the Hasse principle. We construct a general K3 surface X of degree 2 over $\mathbb{Q}$, together with a two-torsion Brauer class α that is unramified at every finite prime, but ramifies at real points of X. Motivated by Hodge theory, the pair (X, α) is constructed from a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified over a hypersurface of bi-degree $(2, 2)$.

1. INTRODUCTION

Let X be a smooth projective geometrically integral variety over a number field k. If X has a k_v-point for every place v of k (equivalently, if its set $X(\mathbf{A}_k)$ of adelic points is nonempty), yet it does not have a k-point, then we say that X does not satisfy the Hasse principle. Manin [Man71] showed that any subset $\mathcal{S}$ of the Brauer group $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ may be used to construct an intermediate set

$$X(k) \subseteq X(\mathbf{A}_k)^{\mathcal{S}} \subseteq X(\mathbf{A}_k)$$

that often explains failures of the Hasse principle, in the sense that $X(\mathbf{A}_k)^{\mathcal{S}}$ may be empty, even if $X(\mathbf{A}_k)$ is not. In this case, we say there is a Brauer-Manin obstruction to the Hasse principle for X. See §4 for the definition of $X(\mathbf{A}_k)^{\mathcal{S}}$.

There is a filtration of the Brauer group $\mathrm{Br}_0(X) \subseteq \mathrm{Br}_1(X) \subseteq \mathrm{Br}(X)$, where

$$\mathrm{Br}_0(X) := \mathrm{im}\left(\mathrm{Br}(k) \to \mathrm{Br}(X)\right),$$
$$\mathrm{Br}_1(X) := \ker\left(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})\right),$$

and $\overline{X} = X \times_k \overline{k}$ for a fixed algebraic closure $\overline{k}$ of k. Elements in $\mathrm{Br}_0(X)$ are said to be constant; class field theory shows that if $\mathcal{S} \subseteq \mathrm{Br}_0(X)$, then $X(\mathbf{A}_k)^{\mathcal{S}} = X(\mathbf{A}_k)$, so these elements cannot obstruct the Hasse principle. Elements in $\mathrm{Br}_1(X)$ are called algebraic; the remaining elements of the Brauer group are transcendental.

There is a large body of literature, spanning the last four decades, on algebraic Brauer classes and algebraic Brauer-Manin obstructions to the Hasse principle and the related notion of weak approximation (i.e., where sets $\mathcal{S} \subseteq \mathrm{Br}_1(X)$ suffice to explain failures of these phenomena); see, for example [Man74, BSD75, CTCS80, CTSSD87, CTKS87, SD93, SD99, KT04, Bri06, BBFL07, Cun07, Cor07, KT08, Log08, VA08, LvL09, EJ10a, EJ10b, Cor10, EJ11a]. The systematic study of these obstructions benefits in no small part from an isomorphism

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \xrightarrow{\sim} H^1\big(k, \mathrm{Pic}(\overline{X})\big), \tag{1}$$

coming from the Hochschild-Serre spectral sequence.

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Obstructions arising from transcendental elements, on the other hand, remain mysterious, because it is difficult to get a concrete handle on transcendental elements of the Brauer group; there is no known analogue of (1) for the group $\operatorname{Br}(X)/\operatorname{Br}_1(X)$.

If X is a curve, or a surface of negative Kodaira dimension, then $\operatorname{Br}\left(\overline{X}\right) = 0$, so the Brauer group is entirely algebraic. On the other hand, in 1996 Harari constructed a 3-fold with a transcendental Brauer-Manin obstruction to the Hasse principle [Har96]. This begs the question: what about algebraic surfaces? Can transcendental Brauer classes obstruct the Hasse principle on an algebraic surface? A natural place to study this question is the class of K3 surfaces; they are arguably some of the simplest surfaces of nonnegative Kodaira dimension in the Castelnuovo-Enriques-Manin classification. The group $\operatorname{Br}(X)/\operatorname{Br}_1(X)$ is finite for a K3 surface [SZ08], but it can be nontrivial.

With arithmetic applications in mind, several authors over the last decade have constructed explicit transcendental elements on K3 surfaces [Wit04, SSD05, HS05, Ier10, Pre10, ISZ11, SZ12]. Wittenberg, Ieronymou and Preu have used these elements to exhibit obstructions to weak approximation (i.e., density of $X(k)$ in $\prod_v X(k_v)$ for the product of the v-adic topologies). In all cases the K3 surfaces considered have elliptic fibrations that play a vital role in the construction of transcendental classes.

Inspired by Hodge-theoretic work of van Geemen and Voisin [vG05, Voi86], we recently constructed a K3 surface with geometric Picard number 1 (and hence no elliptic fibrations), together with a transcendental Brauer class α obstructing weak approximation; see [HVAV11] (joint with Varilly). The pair (X, α) was obtained from a cubic fourfold containing a plane. At the time, we were unable to extend our work to obtain counterexamples to the Hasse principle, in part because we were unable to control the invariants of α at real points of X—ironically, this is precisely the reason we obtain a counterexample to weak approximation! See Remarks 1.3 as well.

Taking advantage of some recent developments (see Remarks 1.3), our goal in this paper is to rectify the above situation and show, once and for all, that transcendental Brauer classes on algebraic surfaces can obstruct the Hasse principle.

Theorem 1.1. *Let X be a K3 surface of degree 2 over a number field k, with function field $\mathbf{k}(X)$, given as a sextic in the weighted projective space $\mathbb{P}(1,1,1,3) = \operatorname{Proj} k[x_0, x_1, x_2, w]$ of the form*

$$w^2 = -\frac{1}{2} \cdot \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}, \tag{2}$$

where $A, \dots, F \in k[x_0, x_1, x_2]$ are homogeneous quadratic polynomials. Then the class $\mathcal{A}$ of the quaternion algebra $(B^2 - 4AD, A)$ in $\operatorname{Br}(\mathbf{k}(X))$ extends to an element of $\operatorname{Br}(X)$.

When $k = \mathbb{Q}$, there exist particular polynomials $A, \dots, F \in \mathbb{Z}[x_0, x_1, x_2]$ such that X has geometric Picard rank 1 and $\mathcal{A}$ gives rise to a transcendental Brauer-Manin obstruction to the Hasse principle on X.

Remark 1.2. In [vG05, §9], Van Geemen showed that *every* Brauer class α of order 2 on a polarized complex K3 surface (X, f) of degree 2 with $\operatorname{Pic}(X) = \mathbb{Z}f$ gives rise to (and must arise from) one of three types of varieties:

- a smooth complete intersection of three quadrics in $\mathbb{P}^5$ (itself a K3 surface), or

- a cubic fourfold containing a plane, or
- a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a hypersurface of bidegree $(2, 2)$.

More precisely, the class α determines a sublattice $T_\alpha \subseteq T_X$ of the transcendental lattice of X which is a polarized Hodge structure, a twist of which is Hodge isometric to a primitive sublattice of the middle cohomology of one the three types of varieties above. See §2 for more details.

The Azumaya algebra $\mathcal{A}$ of Theorem 1.1 represents a class arising from a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a hypersurface of bidegree $(2, 2)$.

Remarks 1.3. We record a few remarks on the computational subtleties behind the second part of Theorem 1.1:

(1) For computational purposes, we go in a direction "opposite" to van Geemen: starting from one of the three types of varieties described in Remark 1.2, defined over a number field k, we recover a K3 surface X over k of degree 2, together with a 2-torsion Azumaya algebra $\mathcal{A}$. Unfortunately, there is no guarantee that X has geometric Picard number $\rho = 1$; in fact, it need not. We use a recent theorem of Elsenhans and Jahnel [EJ11c] to certify that our example has $\rho = 1$.
(2) Curiously, one of the most delicate steps in the proof of Theorem 1.1 is determining the primes of bad reduction of X. We have to factor an integer with 318 decimal digits, whose smallest prime factor turns out to have 66 digits!
(3) We use some of our work on varieties parametrizing maximal isotropic subspaces of families of quadrics admitting at worst isolated singularities to show that $\mathcal{A}$ can ramify only at the real place, 2-adic places and primes of bad reduction for X [HVAV11, §3]. These are thus the only places where the local invariants of $\mathcal{A}$ can be nontrivial.
(4) We rely on recent work of Colliot-Thélène and Skorobogatov [CTS13] to control the local invariants for the algebra $\mathcal{A}$ at odd primes of bad reduction.

Remark 1.4. The Azumaya algebra of Theorem 1.1 looks remarkably similar to the algebra we used in [HVAV11] to exhibit counter-examples to weak approximation. This is not a coincidence: compare Theorem 3.2 with [HVAV11, Theorem 5.1].

Outline of the paper. In §2 we explain the content of Remark 1.2 in detail, following van Geemen [vG05]. The section is not logically necessary for the paper, but we include it for completeness because it explains how to construct, in principle, Azumaya algebras representing every two-torsion Brauer class on a general K3 surface of degree 2.

In §3, we explain how to explicitly construct, from a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a hypersurface of bidegree $(2, 2)$, a pair (X, α) where X is a K3 surface of degree 2 and $\alpha \in \operatorname{Br}(X)[2]$ is an Azumaya algebra. We work mostly over a discrete valuation ring (see Theorem 3.2). This flexibility later affords us control, when working over number fields, of local invariants at places where α ramifies; see Lemma 4.4. In §4, we give a collection of sufficient conditions to control the evaluation maps of α over number fields, specializing ultimately to the case $k = \mathbb{Q}$. Notably, Proposition 4.1 (due to Colliot-Thélène and Skorobogatov) together with Lemma 4.2 show that the evaluation maps of α are constant at non-2-adic finite places of bad reduction of X whenever the singular locus consists of $r < 8$ ordinary double points.

We use this preparatory work to give an example in §5 of a surface witnessing the second part of Theorem 1.1. In §6 we give details of how we found the example of §5, using a computer.

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2. Lattices and Hodge theory

In this section, all varieties are defined over $\mathbb{C}$. Our goal here is to outline van Geemen's geometric constructions representing two-torsion Brauer classes on a K3 surface of degree 2 and Picard rank 1. Strictly speaking, this section is not logically necessary in the proof of Theorem 1.1, and we use only one of the three constructions described. We include it, however, so that readers not acquainted with these ideas get a clear sense of the geometric motivation behind Theorem 1.1.

Let X be a complex K3 surface. Regarding its middle cohomology as a lattice with respect to the intersection form, we can write [LP81, §1]

$$H^2(X,\mathbb{Z}) \simeq U^3 \oplus E_8(-1)^2 =: \Lambda \tag{3}$$

where

$$U = \langle e, f\rangle, \quad \text{with intersections} \quad \begin{array}{c|cc} & e & f \\ \hline e & 0 & 1 \\ f & 1 & 0 \end{array}$$

and E_8 is the positive definite lattice arising from the corresponding root system, i.e., the unique positive definite even unimodular lattice of rank eight. Let e and f denote the generators of the first summand U in (3), and $h \in H^2(X,\mathbb{Z})$ a primitive vector with $h \cdot h = 2d > 0$. The isomorphism (3) can be chosen so that

$$h \mapsto e + df.$$

Writing $v = e - df$, we have

$$h^\perp \simeq \mathbb{Z}v \oplus \Lambda', \qquad \text{where } \Lambda' := U^2 \oplus E_8(-1)^2.$$

Let (X,h) be a polarized K3 surface of degree $2d$, i.e., $h \cdot h = 2d$; assume that $\operatorname{Pic}(X)$ is generated by h. Since $H^3(X,\mathbb{Z}) = 0$, the long exact sequence in cohomology associated to the exponential sequence gives rise to the short exact sequence

$$0 \to H^2(X,\mathbb{Z})/\langle h\rangle \to H^2(X,\mathscr{O}_X) \to \operatorname{Br} X \to 0.$$

Applying the snake lemma to the diagram obtained by multiplication by 2 on this exact sequence, we see that two-torsion elements of the Brauer group of X may be interpreted as elements

$$\alpha \in H^2(X,\mathbb{Z}/2\mathbb{Z})/\langle h\rangle.$$

Under this identification, we can express

$$\alpha = n\bar{f} + \bar{\lambda}_\alpha, \quad n = 0, 1,$$

where $\bar{f}$ is the image of f and $\bar{\lambda}_\alpha$ is the image of some $\lambda_\alpha \in \Lambda'$. Using the non-degenerate cup product on $H^2(X,\mathbb{Z})$, let $\alpha^\perp$ be the kernel of the map $h^\perp \to \mathbb{Z}/2\mathbb{Z}$, $x \mapsto x \cdot (nf + \lambda_\alpha)$. We have

$$\alpha^\perp \subset h^\perp \subset H^2(X,\mathbb{Z}),$$

where the first subgroup has index two when $\alpha \neq 0$.

Assume from now on that $\alpha \neq 0$. If $n = 1$ and $\lambda_\alpha = 0$, then $\alpha^\perp = \mathbb{Z}(2v) \oplus \Lambda'$, a lattice with discriminant group $\mathbb{Z}/8d\mathbb{Z}$, generated by $v/2d$. If $\lambda_\alpha \neq 0$, choose $\mu \in \Lambda'$ satisfying

$$\mu \cdot \lambda_\alpha \equiv 1 \bmod 2. \tag{4}$$

In this case

$$\alpha^\perp = \mathbb{Z}(v + \mu) + \{\lambda' \in \Lambda' : \lambda' \cdot \lambda_\alpha \equiv 0 \bmod 2\},$$

a lattice with discriminant group generated by $(-v + 2d\lambda_\alpha)/4d$, which is therefore $\mathbb{Z}/8d\mathbb{Z}$.

If $n = 0$ then $\lambda_\alpha \neq 0$ and we can also choose $\mu \in \Lambda'$ satisfying (4). We have

$$\alpha^\perp = \mathbb{Z}v \oplus \{\lambda' \in \Lambda' : \lambda' \cdot \lambda_\alpha \equiv 0 \bmod 2\},$$

a lattice with discriminant group $(\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^2$, where the last two summands are generated by $\lambda_\alpha/2$ and μ.

General results on quadratic forms (see, for example, [Nik79]) make it possible to classify even indefinite quadratic forms with prescribed rank and discriminant group H, provided the rank of the form is significantly larger than the number of generators of H. In particular, van Geemen [vG05, Proposition 9.2] classifies isomorphism classes of lattices $\alpha^\perp$ arising from this construction:

- if $n = 0$ there is a unique such lattice, up to isomorphism;
- if $n = 1$ and d is even then there is a unique such lattice up to isomorphism;
- if $n = 1$ and d is odd then there are two such lattices up to isomorphism, depending on the parity of $\lambda_\alpha \cdot \lambda_\alpha/2$.

He goes further when $d = 1$, offering geometric constructions of varieties having primitive Hodge structure isomorphic to $\alpha^\perp$. We elaborate on his description:

Case $n = 0$: Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ denote a smooth hypersurface of bidegree $(2,2)$ and $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ the double cover branched along W. Let h_1 and h_2 denote the divisors on Y obtained by pulling back the hyperplane classes from the factors. We have intersections:

	h_1^2	h_1h_2	h_2^2
h_1^2	0	0	2
h_1h_2	0	2	0
h_2^2	2	0	0

The non-zero Hodge numbers of Y are:

$$h^{00} = h^{44} = 1, h^{11} = h^{33} = 2, h^{13} = h^{31} = 1, h^{22} = 22.$$

Consider the weight-two Hodge structure

$$\langle h_1^2, h_1h_2, h_2^2 \rangle^\perp \subset H^4(Y)(1)$$

having underlying lattice M, with respect to the intersection form. The lattice M is even indefinite, and it has the same rank and discriminant group as $\alpha^\perp$. Thus

$$M \simeq \alpha^\perp.$$

Case $n = 1, \lambda_\alpha \cdot \lambda_\alpha \equiv 0 \pmod 4$: In this case, there exists a *primitive* embedding

$$\alpha^\perp \hookrightarrow U^3 \oplus E_8(-1)^2,$$

unique up to automorphisms of the source and target. We can interpret the image as the primitive cohomology of a polarized K3 surface (S, f) with $f \cdot f = 8$.

Case $n = 1, \lambda_\alpha \cdot \lambda_\alpha \equiv 2 \pmod 4$: Let Y be a cubic fourfold containing a plane P, with hyperplane class h. We have the intersections:

	h^2	P
h^2	3	1
P	1	3

The non-zero Hodge numbers of Y are

$$h^{00} = h^{44} = 1, h^{11} = h^{33} = 1, h^{13} = h^{31} = 1, h^{22} = 21.$$

The weight-two Hodge structure

$$\langle h^2, P \rangle^\perp \subset H^4(Y)(1)$$

has underlying lattice isomorphic to $\alpha^\perp$.

The last two geometric constructions yield explicit unramified Azumaya algebras over the degree two K3 surface. The connection between cubic fourfolds containing planes and quaternion algebras over the K3 surface can be found in [HVAV11]; the other construction goes back to Mukai [Muk84]: A degree eight K3 surface S is generally a complete intersection of three quadrics in $\mathbb{P}^5$, and the discriminant curve of the corresponding net is a smooth plane sextic. Let X be a degree-two K3 surface obtained as the double cover of $\mathbb{P}^2$ branched along this sextic. The variety $\mathcal{F}$ parametrizing maximal isotropic subspaces of the quadrics cutting out S admits a morphism (cf. [HVAV11, §3]) $\mathcal{F} \to X$, which is smooth with geometric fibers isomorphic to $\mathbb{P}^3$.

In this paper, we focus on the first case, and use the resulting Azumaya algebra for arithmetic purposes.

3. Unramified conic bundles

Let k be an algebraically closed field of characteristic $\neq 2$, and let W be an irreducible type $(2, 2)$ divisor on $\mathbb{P}^2 \times \mathbb{P}^2$, that is, hypersurface of bidegree $(2, 2)$. The two projections $\pi_1 \colon W \to \mathbb{P}^2$ and $\pi_2 \colon W \to \mathbb{P}^2$ define conic bundle structures on W. Let $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ be the double cover branched along W. Composing this map with the projections onto the factors we obtain two quadric surface bundles $q_i \colon Y \to \mathbb{P}^2$. Note that the π_i and q_i need not be flat morphisms.

Let x_0, x_1, x_2 and y_0, y_1, y_2 denote homogeneous coordinates on the $\mathbb{P}^2$'s. The equation for W may be expressed as a quadratic form in the y_j's with coefficients quadratic in the x_j's (or vice versa). The determinant of the associated symmetric 3×3 matrix of quadratic forms gives the locus over which π_1 (or π_2) ramifies. This determinant might be identically zero. Otherwise, we obtain plane sextic curves C_1 and C_2 over which π_1 and π_2 (as well as q_1 and q_2) are ramified. Let $\phi_i \colon X_i \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ branched over C_i; if C_i is a smooth curve then X_i is a K3 surface of degree 2.

If W has at worst isolated singularities then the geometric generic fiber of π_i (and hence of q_i) is smooth. Indeed, if the geometric generic fiber of π_i were singular then there would exist a generically étale $V \to \mathbb{P}^2$ such that $V \times_{\mathbb{P}^2} W$ is singular over the generic point of V. Here we are using the fact that the characteristic $\neq 2$. Thus it follows that C_1 and C_2 are curves.

Lemma 3.1. *Retain the notation above and assume that C_1 and C_2 are curves. If Y (equivalently, W) is not smooth then neither C_1 nor C_2 is smooth. On the other hand, if W is smooth then each π_i is flat and C_i is singular if π_i has a geometric fiber of rank 1, $i = 1, 2$.*

Proof. An easy application of the Jacobian criterion shows that Y is smooth if and only if W is smooth. We use the latter scheme to prove the remaining claims of the lemma.

Let $w \in W$ be a singular point, $c_i \in C_i$ its images under projection, (u_0, u_1) local coordinates of the first $\mathbb{P}^2$ centered at c_1, and (v_0, v_1) local coordinates of the second $\mathbb{P}^2$ centered at c_2. The defining equation of W takes the form

$$a(u_0, u_1)v_0^2 + b(u_0, u_1)v_0v_1 + 2d(u_0, u_1)v_1^2 + c(u_0, u_1)v_0 + e(u_0, u_1)v_1 = 0,$$

where the coefficients are quadratic in u_0 and u_1, and $c(0, 0) = e(0, 0) = 0$. Note that W would be singular if the coefficients $a(u_0, u_1), \ldots, e(u_0, u_1)$ were all zero. The defining equation for C_1 is therefore

$$\det \begin{pmatrix} 2a & b & c \\ b & 2d & e \\ c & e & 0 \end{pmatrix} = 0.$$

Expanding this out, we get

$$bce - ae^2 - c^2d = 0,$$

where each term vanishes to order ≥ 2 at $c_1 = (0, 0)$.

The last statement of the lemma is a consequence of [Bea77, Prop. 1.2]. □

Theorem 3.2. *Let $\mathcal{O}$ denote a discrete valuation ring with residue field $\mathbb{F}$ of characteristic $\neq 2$. Let $\mathcal{W}$ be a type $(2, 2)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ flat over $\operatorname{Spec} \mathcal{O}$, and $\mathcal{Y} \to \mathbb{P}^2 \times \mathbb{P}^2$ a double cover simply branched along $\mathcal{W}$. For $i = 1, 2$, let $q_i\colon \mathcal{Y} \to \mathbb{P}^2$ denote the quadric surface bundle obtained by projecting onto the i-th factor, and let $\mathcal{C}_i \subset \mathbb{P}^2$ be its discriminant divisor. Assume that for some $j \in \{1, 2\}$, $\mathcal{C}_j$ is flat over $\mathcal{O}$, and that $(\mathcal{C}_j)_{\mathbb{F}}$ is smooth.*

Let $r_j\colon \mathcal{F}_j \to \mathbb{P}^2$ be the relative variety of lines of q_j. Then the Stein factorization

$$r_i\colon \mathcal{F}_j \to \mathcal{X}_j \xrightarrow{\phi_j} \mathbb{P}^2$$

consists of a smooth $\mathbb{P}^1$-bundle followed by a degree-two cover of $\mathbb{P}^2$, which is a K3 surface.

Proof. By Lemma 3.1, smoothness of $(\mathcal{C}_j)_{\mathbb{F}}$ implies smoothness of $\mathcal{W}_{\mathbb{F}}$, and hence of $\mathcal{Y}_{\mathbb{F}}$. The same lemma shows that the fibers of $(q_j)_{\mathbb{F}}$ have at worst isolated singularities. On the other hand, the morphism $q_j\colon \mathcal{Y} \to \mathbb{P}^2$ is flat, and thus $\mathcal{Y}$ is a regular scheme. Geometric fibers of q_j over the generic point of $\operatorname{Spec} \mathcal{O}$ with non-isolated singularities specialize to geometric fibers over the closed point with non-isolated singularities. Hence the fibers of q_j have isolated singularities. The theorem now follows directly from [HVAV11, Proposition 3.3]: The Stein factorization of the variety of maximal isotropic subspaces of a family of even-dimensional quadric hypersurfaces with (at worst) isolated singularities is isomorphic to the discriminant double cover of the base.

Since the morphism $\mathcal{C}_j \to \mathbb{P}^2$ is flat, smoothness of $(\mathcal{C}_j)_{\mathbb{F}}$ implies that $\mathcal{C}_j$ is regular. Hence $\mathcal{X}_j$ is a K3 surface over $\operatorname{Spec} \mathcal{O}$. □

The smooth $\mathbb{P}^1$-bundle $r_j \colon \mathcal{F}_j \to \mathcal{X}_j$ may be interpreted as a two-torsion element of $\operatorname{Br}(\mathcal{X}_j)$. Without loss of generality, assume in the hypotheses of Theorem 3.2 that $(\mathcal{C}_1)_{\mathbb{F}}$ is smooth; we omit the subscript $j = 1$ from here on in. We give an explicit quaternion algebra over $\mathbf{k}(\mathcal{X})$ representing the Brauer class of $\mathcal{F} \to \mathcal{X}$. Express

$$\mathbb{P}^2 \times \mathbb{P}^2 = \operatorname{Proj} \mathcal{O}[x_0, x_1, x_2] \times_{\mathcal{O}} \operatorname{Proj} \mathcal{O}[y_0, y_1, y_2]$$

so the equation for $\mathcal{W}$ takes the form

$$\begin{aligned} 0 &= A(x_0,x_1,x_2)y_0^2 + B(x_0,x_1,x_2)y_0y_1 + C(x_0,x_1,x_2)y_0y_2 \\ &\quad + D(x_0,x_1,x_2)y_1^2 + E(x_0,x_1,x_2)y_1y_2 + F(x_0,x_1,x_2)y_2^2, \end{aligned} \tag{5}$$

for some homogeneous quadratic polynomials $A, \dots, F \in \mathcal{O}[x_0, x_1, x_2]$. The coefficients are unique modulo multiplication by a common unit in $\mathcal{O}$.

Consider the bigraded ring $\mathcal{O}[x_0, x_1, x_2, y_0, y_1, y_2, v]$ where

$$\deg(x_i) = (1,0), \quad \deg(y_i) = (0,1), \quad \deg(v) = (1,1),$$

and let

$$R := \bigoplus_{n\in\mathbb{Z}} \mathcal{O}[x_0, x_1, x_2, y_0, y_1, y_2, v]_{(n,n)}$$

denote the graded subring generated by elements of bidegree (n, n) for some n. Then an equation for $\mathcal{Y} \subset \operatorname{Proj} R$ is

$$\begin{aligned} v^2 &= A(x_0,x_1,x_2)y_0^2 + B(x_0,x_1,x_2)y_0y_1 + C(x_0,x_1,x_2)y_0y_2 \\ &\quad + D(x_0,x_1,x_2)y_1^2 + E(x_0,x_1,x_2)y_1y_2 + F(x_0,x_1,x_2)y_2^2. \end{aligned} \tag{6}$$

The quadric surface bundle $q \colon \mathcal{Y} \to \mathbb{P}^2$ is ramified over the curve

$$\mathcal{C}: \quad \det \begin{pmatrix} 2A & B & C & 0 \\ B & 2D & E & 0 \\ C & E & 2F & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = 0. \tag{7}$$

Thus, after rescaling, the K3 surface $\mathcal{X}$ is described by the hypersurface

$$w^2 = -\frac{1}{2} \cdot \det(M) \tag{8}$$

in $\mathbb{P}(1,1,1,3)$, where $M \in \operatorname{Mat}_3(\mathcal{O}[x_0, x_1, x_2])$ is the leading 3×3 principal minor of the matrix in (7). The factor $-\frac{1}{2}$ is dictated by the interpretation of the structure sheaf of $\mathcal{X}$ as the discriminant algebra of our quadratic form in four variables (cf. [HVAV11, §3.1]).

The discussion in [HVAV11, §3.3] shows that the generic fiber of the map $\mathcal{F} \to \mathcal{X}$ is the Severi-Brauer conic in $\operatorname{Proj} \mathbf{k}(\mathcal{X})[Y_0, Y_1, Y_2]$ given by

$$AY_0^2 + BY_0Y_1 + CY_0Y_2 + DY_1^2 + EY_1Y_2 + FY_2^2 = 0. \tag{9}$$

Essentially, given a smooth quadric surface whose discriminant double cover is split, each component of the variety of lines on the surface is isomorphic to a smooth hyperplane section of the surface. Let

$$M_A := 4DF - E^2, \quad M_D := 4AF - C^2, \quad \text{and} \quad M_F := 4AD - B^2.$$

Completing squares in (9), and renormalizing, we obtain

$$Y_0^2 = -\frac{M_F}{4A^2}Y_1^2 - \frac{\det(M)}{2A \cdot M_F}Y_2^2.$$

Hence, by [GS06, Corollary 5.4.8], the conic (9) corresponds to the Hilbert symbol

$$\left(-\frac{M_F}{4A^2}, -\frac{\det(M)}{2A \cdot M_F}\right). \tag{10}$$

Write $\mathcal{A}$ for the class of this symbol in $\mathrm{Br}(\mathbf{k}(\mathcal{X}))$; $\mathcal{A}$ is unaffected by multiplication by squares in either entry of a representative symbol. Since $-\frac{1}{2}\det(M)$ is a square in $\mathbf{k}(\mathcal{X})^\times$, we see that

$$(-M_F, A \cdot M_F) = (-M_F, A)$$

is another representative of $\mathcal{A}$ (the equality uses the multiplicativity of the Hilbert symbol and the relation $(-M_F, M_F) = 1$ [Ser73, III, Proposition 2]). Here we have the usual abuse of notation: the entries are not rational functions, though they are homogeneous polynomials of even degree.

Depending on how we complete squares and renormalize (9), we may obtain several representatives of $\mathcal{A}$:

$$\begin{aligned} &(-M_F, A), \quad (-M_D, A), \quad (-M_F, D), \\ &(-M_A, D), \quad (-M_D, F), \quad (-M_A, F). \end{aligned} \tag{11}$$

Proposition 3.3. *Let X be a K3 surface of degree 2 over a number field k, given as a sextic in the weighted projective space $\mathbb{P}(1,1,1,3) = \operatorname{Proj} k[x_0, x_1, x_2, w]$ of the form*

$$w^2 = -\frac{1}{2} \cdot \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix},$$

where $A, \ldots, F \in k[x_0, x_1, x_2]$ are homogeneous quadratic polynomials. Then the class $\mathcal{A}$ of the quaternion algebra $(B^2 - 4AD, A)$ in $\mathrm{Br}(\mathbf{k}(X))$ extends to an element of $\mathrm{Br}(X)$.

Proof. Let $\mathcal{O}$ be the valuation ring at some finite place of k where X has good reduction. The proposition follows directly from Theorem 3.2 and the subsequent discussion, keeping track of what is happening over the generic point of $\operatorname{Spec} \mathcal{O}$. Indeed, define $\mathcal{W}$ and $\mathcal{Y}$, respectively, by (5) and (6). The resulting curve $(\mathcal{C}_1)_k$ is the branch curve of the double cover $X \to \mathbb{P}^2$, which is smooth because X is a K3 surface, by hypothesis. $\square$

Remark 3.4. The assortment of quaternion algebras (11) representing the class $\mathcal{A}$ of Proposition 3.3 is useful for the computation of the invariant map on the image of the evaluation map $\mathrm{ev}_{\mathcal{A}} \colon X(\mathbf{A}_k) \to \bigoplus_v \mathrm{Br}(k_v), (P_v) \mapsto \big(\mathcal{A}(P_v)\big)$. The industrious reader can check that at every local point of X, either the first, fourth or fifth representative of $\mathcal{A}$ in our list is well-defined; we shall not use this observation directly.

4. Local Invariants

Let X be a smooth projective geometrically integral variety over a number field k. For $\mathcal{S} \subseteq \operatorname{Br}(X)$, let

$$X(\mathbf{A}_k)^{\mathcal{S}} := \left\{ (P_v) \in X(\mathbf{A}_k) : \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0 \text{ for all } \mathcal{A} \in \mathcal{S} \right\}.$$

The inclusion $X(k) \subseteq X(\mathbf{A}_k)^{\mathcal{S}}$ follows from class field theory. See [Sko01, §5.2] for details. The local invariants $\operatorname{inv}_v \mathcal{A}(P_v)$ can be nonzero only at a finite number of places: the archimedean places of k, the places of bad reduction of X, and places where the class $\mathcal{A}$ is ramified.

We begin this section by explaining how recent work of Colliot-Thélène and Skorobogatov [CTS13] shows that local invariants are *constant* at certain finite places v of bad reduction for X where the singular locus satisfies a technical hypothesis. Specializing to the case where X is a K3 surface over a number field k as in Proposition 3.3, this technical hypothesis is satisfied provided the singular locus at v consists of $r < 8$ ordinary double points (Lemma 4.2).

We then show that the class $\mathcal{A}$ of Proposition 3.3 can ramify only over infinite places, 2-adic places, and places of bad reduction for X. Finally, in the special case $k = \mathbb{Q}$, we give sufficient conditions for local invariants of $\mathcal{A}$ to be trivial at 2-adic points and nontrivial at real points.

4.1. **Places of bad reduction with mild singularities.** In this section we use the following notation: k is a finite extension of $\mathbb{Q}_p$ with a fixed algebraic closure $\overline{k}$, $\mathcal{O}$ denotes the ring of integers of k, and $\mathbb{F}$ denotes its residue field. We let X be a smooth, proper, geometrically integral variety over k and write $\pi\colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}$ for a flat proper morphism with $X = \mathcal{X} \times_{\mathcal{O}} k$.

The following proposition is a straightforward refinement of [CTS13, Proposition 2.4], using ideas in the remark on the case of bad reduction in [CTS13, §2]. We include the details here for the reader's convenience.

Proposition 4.1. *Let $\ell \neq p$ be a prime. Assume that $\mathcal{X}$ is regular with geometrically integral fibers over $\operatorname{Spec} \mathcal{O}$, and that the smooth locus $\mathcal{X}_{\mathbb{F}}^{\mathrm{sm}}$ of the closed fiber is geometrically irreducible and has no connected unramified cyclic geometric coverings of degree ℓ. If $X(k) \neq \emptyset$, then, for $\mathcal{A} \in \operatorname{Br}(X)\{\ell\}$, the image of the evaluation map $\operatorname{ev}_{\mathcal{A}}\colon X(k) \to \operatorname{Br}(k)$ consists of one element.*

Proof. Let $\mathcal{Z}$ be the largest open subscheme of $\mathcal{X}$ that is smooth over $\operatorname{Spec} \mathcal{O}$; note that $\mathcal{Z} \times_{\mathcal{O}} k = X$. Write $\mathcal{Z}_{\mathbb{F}}$ for its closed fiber, and note that $\mathcal{Z}_{\mathbb{F}} = \mathcal{X}_{\mathbb{F}}^{\mathrm{sm}}$. Let $\mathcal{Z}_{\mathbb{F}}^{(1)}$ denote the set of closed integral subvarieties of $\mathcal{Z}_{\mathbb{F}}$ of codimension 1. In [Kat86, Prop. 1.7], Kato shows there is a complex

$$\operatorname{Br}(X)[\ell^n] \xrightarrow{\text{res}} H^1\big(\mathbf{k}(\mathcal{Z}_{\mathbb{F}}), \mathbb{Z}/\ell^n\mathbb{Z}\big) \to \bigoplus_{Y \in \mathcal{Z}_{\mathbb{F}}^{(1)}} H^0\big(\mathbf{k}(Y), \mathbb{Z}/\ell^n\mathbb{Z}(-1)\big).$$

(In Kato's notation, take $q = -1$, $i = -2$, $n \mapsto \ell^n$, and $X \mapsto \mathcal{Z}$.) We claim that for $\mathcal{A} \in \operatorname{Br}(X)[\ell^n]$, the residue $\operatorname{res}(\mathcal{A}) \in H^1\big(\mathbf{k}(\mathcal{Z}_{\mathbb{F}}), \mathbb{Z}/\ell^n\mathbb{Z}\big)$ lies in the subgroup $H^1_{\text{ét}}\big(\mathcal{Z}_{\mathbb{F}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$. Indeed, the group $H^1\big(\mathbf{k}(\mathcal{Z}_{\mathbb{F}}), \mathbb{Z}/\ell^n\mathbb{Z}\big)$ classifies connected cyclic covers of $\mathcal{Z}_{\mathbb{F}}$. By Kato's complex, the cover $\mathcal{W} \to \mathcal{Z}_{\mathbb{F}}$ corresponding to $\operatorname{res}(\mathcal{A})$ is unramified in codimension one, and hence, by the Zariski-Nagata purity theorem [SGA03, Exposé X, Théorème 3.1], $\mathcal{W} \to \mathcal{Z}_{\mathbb{F}}$ is unramified.

The long exact sequence of low degree terms associated to the spectral sequence

$$E_2^{p,q} := H^p\big(\mathbb{F}, H^q_{\text{ét}}\big(\mathcal{Z}_{\overline{\mathbb{F}}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)\big) \implies H^{p+q}_{\text{ét}}(\mathcal{Z}_{\mathbb{F}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

starts as follows:

$$0 \to H^1\big(\mathbb{F}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \to H^1_{\text{ét}}\big(\mathcal{Z}_{\mathbb{F}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \to H^1_{\text{ét}}\big(\mathcal{Z}_{\overline{\mathbb{F}}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) \tag{12}$$

Since, by hypothesis, $\mathcal{Z}_{\mathbb{F}}$ has no connected unramified cyclic geometric coverings of degree ℓ, we have $H^1_{\text{ét}}\big(\mathcal{Z}_{\overline{\mathbb{F}}}, \mathbb{Z}/\ell^n\mathbb{Z}\big) = 0$.

We claim there is an element $\alpha \in \operatorname{Br}(k)\{\ell\}$ such that $\mathcal{A} - \alpha$ has trivial residues along any codimension one subvariety of $\mathcal{Z}$. Such a residue depends only on the local ring at the generic point of the subvariety, so it suffices to consider residues along codimension one subvarieties of generic fiber X of $\mathcal{Z} \to \operatorname{Spec} \mathcal{O}$, as well as the residue along the generic point of the special fiber $\mathcal{Z}_{\mathbb{F}}$. Since $\mathcal{A}$ belongs to $\operatorname{Br}(X)$, it can have a nonzero residue only at the generic point of the special fiber $\mathcal{Z}_{\mathbb{F}}$. As above, this residue lies in $H^1_{\text{ét}}\big(\mathcal{Z}_{\mathbb{F}}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$. On the other hand, local class field theory shows that the invariant map $\operatorname{Br}(k)[\ell^n] \to H^1\big(\mathbb{F}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$ is an isomorphism. We pick $\alpha \in \operatorname{Br}(k)\{\ell\}$ so that its invariant in $H^1\big(\mathbb{F}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$ coincides with the preimage of $\operatorname{res}(\mathcal{A})$ in $H^1\big(\mathbb{F}, \mathbb{Z}/\ell^n\mathbb{Z}\big)$ for the map in (12).

By Gabber's purity theorem [Fuj02], it follows that $\mathcal{A} - \alpha \in \operatorname{Br}(\mathcal{Z})\{\ell\} \subseteq \operatorname{Br}(X)\{\ell\}$. A valuation argument shows that $X(k) = \mathcal{X}(\mathcal{O}) = \mathcal{Z}(\mathcal{O})$; see [Sko96, proof of Lemma 1.1(b)]. Since $\operatorname{Br}(\mathcal{O}) = 0$, we conclude that the images of the evaluation maps $\operatorname{ev}_{\mathcal{A}}$ and ev_α in $\operatorname{Br}(k)$ coincide; the latter consists of one element. $\square$

Lemma 4.2. *Suppose that $p \neq 2$. Let X be a K3 surface defined over k, and let $\pi \colon \mathcal{X} \to \operatorname{Spec} \mathcal{O}$ be a flat proper morphism from a regular scheme with $X = \mathcal{X} \times_{\mathcal{O}} k$. Assume that the singular locus of the closed fiber $\mathcal{X}_0 := \mathcal{X}_{\overline{\mathbb{F}}}$ has $r < 8$ points, each of which is an ordinary double point. Then the smooth locus $U \subset \mathcal{X}_0$ has no connected unramified cyclic covers of prime degree $\ell \neq p$.*

Proof. Consider an algebraically closed field F of characteristic different from ℓ. Let Y be a separated integral scheme over F with $\Gamma(Y, \mathscr{O}_Y^*) = F^*$; this is the case if Y is proper, or a dense open subset of a proper scheme with complement of codimension ≥ 2. Then degree ℓ cyclic étale covers of Y are classified by $H^1_{\text{ét}}(Y, \mu_\ell)$ [Mil80, ch.III]. The Kummer exact sequence [Mil80, p.125] implies that $H^1_{\text{ét}}(Y, \mu_\ell) = \operatorname{Pic}(Y)[\ell]$, the ℓ-torsion subgroup.

Combining the canonical homomorphism from the Picard group to the Weil class group and the restriction homomorphism on class groups yields

$$\operatorname{Pic}(\mathcal{X}_0) \subset \operatorname{Cl}(\mathcal{X}_0) \simeq \operatorname{Cl}(U) \simeq \operatorname{Pic}(U).$$

The quotient $\operatorname{Pic}(U)/\operatorname{Pic}(\mathcal{X}_0)$ is two-torsion. Indeed, ordinary double points are étale locally isomorphic to quadric cones, whose local class group equals $\mathbb{Z}/2\mathbb{Z}$ (generated by the ruling). Thus for each closed point $x \in \mathcal{X}_0$, the quotient $\operatorname{Pic}(\operatorname{Spec} \mathscr{O}_{\mathcal{X}_0,x} \setminus \{x\})/\operatorname{Pic}(\operatorname{Spec} \mathscr{O}_{\mathcal{X}_0,x})$ is annihilated by two [Lip69, §14]. If $\ell \neq 2$ then this computation shows that $\operatorname{Pic}(U)[\ell] = \operatorname{Pic}(\mathcal{X}_0)[\ell]$, whence degree ℓ cyclic étale covers of U extend to $\mathcal{X}_0$.

We claim that $\operatorname{Pic}(\mathcal{X}_0)[\ell] = 0$ for each prime $\ell \neq p$. To prove this, replace k by the ramified quadratic extension k' with ring of integers $\mathcal{O}'$, so that $\mathcal{X}' = \mathcal{X} \times_{\mathcal{O}} \mathcal{O}'$ is singular over the double points of $\mathcal{X}_0$. Concretely, given $\mathfrak{p}$ a uniformizer of $\mathcal{O}$, $\mathfrak{p}' = \sqrt{\mathfrak{p}}$ the corresponding uniformizer of $\mathcal{O}'$, and $x \in \mathcal{X}_0$ an ordinary double point, then the étale local equation of $\mathcal{X}$

$$\mathfrak{p} = uv + w^2$$

pulls back to

$$(\mathfrak{p}')^2 = uv + w^2.$$

Let $\tilde{\mathcal{X}} \to \mathcal{X}'$ denote the blow-up of the resulting singularities, with central fiber the union of the proper transform of $\mathcal{X}_0$ and the exceptional divisors

$$\tilde{\mathcal{X}}_0 = S \cup E_1 \cup \cdots \cup E_r, \quad E_j \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

(At the cost of passing to an algebraic space, we could blow down $E_1, \ldots, E_r$ along one of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$.) Note that S is the K3 surface obtained by resolving the ordinary double points of $\mathcal{X}_0$. The pull-back homomorphism

$$\mathrm{Pic}(\mathcal{X}_0) \to \mathrm{Pic}(S)$$

is injective since we may regard U as an open subset of both $\mathcal{X}_0$ and S. However, $\mathrm{Pic}(S)$ has no ℓ-torsion: the specialization homomorphism [Ful98, §20.3]

$$\mathrm{Pic}(X_{\bar{k}}) \to \mathrm{Pic}(S)$$

is injective and any torsion of its cokernel is annihilated by p [MP09, Prop. 3.6].

We now focus on the case $\ell = 2$. Let $F_1, \ldots, F_r$ denote the exceptional divisors of $S \to \mathcal{X}_0$, which satisfy

$$F_1^2 = \cdots = F_r^2 = -2, \quad F_iF_j = 0, i \neq j,$$

because $\mathcal{X}_0$ has ordinary double points.

There may exist étale double covers of U that fail to extend to étale covers of $\mathcal{X}_0$. Given an étale double cover $V \to U$, let $\varpi : T \to S$ denote the normalization of S in the function field of V. Since T is normal, ϖ is a flat morphism [Eis95, Ex. 18.17], étale away from $F_1 \cup \cdots \cup F_r$. Moreover, by purity of the branch locus, ϖ is branched over some subset

$$\{F_{j_1}, \ldots, F_{j_s}\} \subset \{F_1, \ldots, F_r\}.$$

Since the characteristic is odd, ϖ is simply branched over these curves. Consequently, $\sum_{i=1}^{s} F_{j_i} = 2D$ for some $D \in \mathrm{Pic}(S)$, hence $4D^2 = -2s$. Since D^2 is even, we conclude that $s \equiv 0 \pmod 4$, i.e., $s = 0$ or 4. The case $s = 0$ is impossible, since this would mean that S admits an étale cyclic cover with degree prime to the characteristic. The case $s = 4$ is also impossible: By Riemann-Roch, we have

$$\chi(\mathscr{O}_S(-D)) = 1$$

but

$$h^2(\mathscr{O}_S(-D)) = h^0(\mathscr{O}_S(D)) = 0,$$

as any effective divisor supported in the F_j (like $2D$) is rigid. On the other hand, since a divisor and its negative cannot both be effective, we find

$$h^0(\mathscr{O}_S(-2D)) = 0 \text{ which implies } h^0(\mathscr{O}_S(-D)) = 0.$$

Therefore $h^1(\mathscr{O}_S(-D)) = -1$, which is a contradiction. $\square$

Remark 4.3. When $r = 8$, it is possible that a smooth resolution of $\mathcal{X}_0$ is a K3 surface with a Nikulin involution, in which case the smooth locus $U \subset \mathcal{X}_0$ has a connected unramified cyclic double cover [vGS07].

4.2. **Places where $\mathcal{A}$ can ramify.**

Lemma 4.4. *Let X be a K3 surface over a number field k as in Proposition 3.3. Let v be a finite place of good reduction for X, and assume that v is not 2-adic. Then $\mathcal{A}$ does not ramify at v. Consequently,* $\operatorname{inv}_v \mathcal{A}(P) = 0$ *for all $P \in X(k_v)$.*

Proof. We may assume without loss of generality that the coefficients of $A, \dots, F$ are integral. Let $\mathcal{O}_v$ be the ring of integers of k_v, and $\mathbb{F}_v$ its residue field. Since X is smooth and proper over k and has good reduction at v, there is a smooth proper morphism $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_v$ with $X_{k_v} = \mathcal{X} \times_{\mathcal{O}_v} k_v$. We will show that the class $\mathcal{A} \otimes k_v$ can be spread out to a class in $\operatorname{Br}(\mathcal{X})$. Since, by the valuative criterion of properness, we have $\mathcal{X}(\mathcal{O}_v) = X(k_v)$, it will follow that $\mathcal{A}(P) \in \operatorname{Br}(\mathcal{O}_v) = 0$ for every point $P \in X(k_v)$, establishing all the claims of the proposition.

Define $\mathcal{W}$ and $\mathcal{Y}$ over $\mathcal{O}_v$, respectively, by (5) and (6). The quadric surface bundle $(q_1)_{\mathbb{F}} \colon \mathcal{Y}_{\mathbb{F}} \to \mathbb{P}^2_{\mathbb{F}}$ ramifies over the discriminant curve of $\mathcal{X}_{\mathbb{F}_v} \to \mathbb{P}^2_{\mathbb{F}_v}$, which is smooth, because X has good reduction at v. By Theorem 3.2, there exists a smooth $\mathbb{P}^1$ bundle $\mathcal{F} \to \mathcal{X}$, whose corresponding two-torsion class in $\operatorname{Br}(\mathcal{X})$ is represented by the quaternion algebra $(B^2 - 4AB, A)$, by the discussion following Theorem 3.2. Thus $\mathcal{A} \in \operatorname{Br}(\mathcal{X})$, as claimed. □

4.3. **Real and 2-adic invariants.** In this section we use the notation of Proposition 3.3, specializing to the case $k = \mathbb{Q}$. The following lemma gives a sufficient condition to guarantee that the local invariants of $\mathcal{A}$ at real points of X are always non-trivial.

Lemma 4.5. *Suppose that the quadratic forms A, B, C, D, E and F satisfy*

(1) *A, D and F are negative definite,*
(2) *B, C and E are positive definite.*

Then, for any real point of X, we have

$$M_A > 0, \quad M_D > 0 \quad \textit{and} \quad M_F > 0.$$

Proof. First, observe that we can write $\frac{1}{2}\det(M)$ as

$$A \cdot M_A - (C^2 D + B^2 F - BCE). \tag{13}$$

Let P be a real point of X, so that $\frac{1}{2}\det(M) \le 0$ holds at P. Our hypotheses on $A, \dots, F$ imply that

$$(C^2 D + B^2 F - BCE)(P) < 0.$$

Suppose first that $M_A \le 0$. Then at P we have

$$\frac{1}{2}\det(M) = \underbrace{A}_{<0} \cdot \underbrace{M_A}_{\le 0} - (\underbrace{C^2 D + B^2 F - BCE}_{<0}) > 0,$$

a contradiction. Hence $M_A > 0$ at P. A similar argument shows the remaining two cases. □

Corollary 4.6. *Suppose the hypotheses of Lemma 4.5 hold. Then the local invariant of $\mathcal{A}$ at every real point of X is nontrivial.*

Proof. It suffices to show that, for any real point P of X, there is a quaternion algebra representing $\mathcal{A}$ whose entries are both negative at P. Using the six representatives (11) of $\mathcal{A}$, together with Lemma 4.5, the result follows. □

Next, we write down a sufficient condition to guarantee that the local invariant map on $\mathcal{A}$ is constant and trivial on 2-adic points. Write $v_2\colon \mathbb{Q}_2 \to \mathbb{Z}\cup\{\infty\}$ for the standard 2-adic valuation. Recall that $a \in \mathbb{Q}_2^\times$ is a square if and only if $v_2(a)$ is even and if $a/2^{v_2(a)} \equiv 1 \bmod 8$.

Let $P = [x_0 : x_1 : x_2 : w]$ denote a 2-adic point of X. We may assume without loss of generality that x_0, x_1 or x_2 are elements of $\mathbb{Z}_2$, at least one of which is a 2-adic unit. Suppose first that x_0 is a 2-adic unit, so that $v_2(x_0) = 0$. We use the representative $(-M_F, A)$ of $\mathcal{A}$ to evaluate invariants at P. Write

$$A = A_1x_0^2 + A_2x_0x_1 + A_3x_0x_2 + A_4x_1^2 + A_5x_1x_2 + A_6x_2^2,$$

and suppose that the coefficients of A satisfy

$$A_1 \equiv 1 \bmod 8, \quad \text{and} \quad v_2(A_i) \geq 3 \text{ for } i = 2,\dots,6.$$

Then, at P, we have $A \equiv 1 \bmod 8$ (since $v_2(x_0) = 0$) so A is a 2-adic square. It follows that $\operatorname{inv}_2 \mathcal{A}(P) = 0$, provided that $M_F(P) \neq 0$. To ensure this, we impose restrictions on the coefficients of the quadratic form

$$B = B_1x_0^2 + B_2x_0x_1 + B_3x_0x_2 + B_4x_1^4 + B_5x_1x_2 + B_6x_2^2.$$

Suppose that

$$v_2(B_1) = 0, \quad \text{and} \quad v_2(B_i) \geq 1 \text{ for } i = 2,\dots,6.$$

Then, since $v_2(x_0) = 0$, it follows that

$$v_2(M_F(P)) = v_2(B(P)) = 0$$

and hence $M_F \neq 0$ at P.

To ensure that 2-adic invariants of $\mathcal{A}$ are trivial at points where $v_2(x_1) = 0$, we use the representative $(-M_A, D)$ of $\mathcal{A}$ and constrain the coefficients of D and E, respectively, in a manner analogous to how we constrained the coefficients of A and B. We proceed similarly for 2-adic points with $v_2(x_2) = 0$, this time using the representative $(-M_D, F)$ and we constrain the coefficients of C and F. We summarize our discussion in the following lemma.

Lemma 4.7. *Write*

$$\begin{aligned}
A &= A_1x_0^2 + A_2x_0x_1 + A_3x_0x_2 + A_4x_1^2 + A_5x_1x_2 + A_6x_2^2,\\
B &= B_1x_0^2 + B_2x_0x_1 + B_3x_0x_2 + B_4x_1^2 + B_5x_1x_2 + B_6x_2^2,\\
C &= C_1x_0^2 + C_2x_0x_1 + C_3x_0x_2 + C_4x_1^2 + C_5x_1x_2 + C_6x_2^2,\\
D &= D_1x_0^2 + D_2x_0x_1 + D_3x_0x_2 + D_4x_1^2 + D_5x_1x_2 + D_6x_2^2,\\
E &= E_1x_0^2 + E_2x_0x_1 + E_3x_0x_2 + E_4x_1^2 + E_5x_1x_2 + E_6x_2^2,\\
F &= F_1x_0^2 + F_2x_0x_1 + F_3x_0x_2 + F_4x_1^2 + F_5x_1x_2 + F_6x_2^2.
\end{aligned}$$

Suppose that the coefficients of these quadratic forms satisfy:

(1) $A_1 \equiv 1 \bmod 8$*, and* $v_2(A_i) \geq 3$ *for* $i \neq 1$.
(2) $v_2(B_1) = 0$*, and* $v_2(B_i) \geq 1$ *for* $i \neq 1$.
(3) $v_2(C_6) = 0$*, and* $v_2(C_i) \geq 1$ *for* $i \neq 6$.
(4) $D_4 \equiv 1 \bmod 8$*, and* $v_2(D_i) \geq 3$ *for* $i \neq 4$.

(5) $v_2(E_4) = 0$, *and* $v_2(E_i) \geq 1$ *for* $i \neq 4$.
(6) $F_6 \equiv 1 \bmod 8$, *and* $v_2(F_i) \geq 3$ *for* $i \neq 6$.

Then, for every 2-adic point P of X, we have $\operatorname{inv}_2 \mathcal{A}(P) = 0$. □

5. An example

Consider the quadrics

$$\begin{aligned}
A &:= -7x_0^2 - 16x_0x_1 + 16x_0x_2 - 24x_1^2 + 8x_1x_2 - 16x_2^2 \\
B &:= 3x_0^2 + 2x_0x_2 + 2x_1^2 - 4x_1x_2 + 4x_2^2 \\
C &:= 10x_0^2 + 4x_0x_1 + 4x_0x_2 + 4x_1^2 - 2x_1x_2 + x_2^2 \\
D &:= -16x_0^2 + 8x_0x_1 - 23x_1^2 + 8x_1x_2 - 40x_2^2 \\
E &:= 4x_0^2 - 4x_0x_2 + 11x_1^2 - 4x_1x_2 + 6x_2^2 \\
F &:= -40x_0^2 + 32x_0x_1 - 40x_1^2 - 8x_1x_2 - 23x_2^2.
\end{aligned} \tag{14}$$

Let $W \subset \operatorname{Proj} \mathbb{Q}[x_0, x_1, x_2] \times \operatorname{Proj} \mathbb{Q}[y_0, y_1, y_2]$ be the type $(2,2)$ divisor given by the vanishing of the bihomogeneous polynomial

$$\begin{aligned}
&A(x_0, x_1, x_2)y_0^2 + B(x_0, x_1, x_2)y_0y_1 + C(x_0, x_1, x_2)y_0y_2 \\
&\qquad + D(x_0, x_1, x_2)y_1^2 + E(x_0, x_1, x_2)y_1y_2 + F(x_0, x_1, x_2)y_2^2.
\end{aligned} \tag{15}$$

As in §3, the projections $\pi_i \colon W \to \mathbb{P}^2$ give conic bundle structures on W ramified over plane sextics C_i, $i = 1, 2$. An equation for C_1 is then given by $f := -\frac{1}{2}\det(M) = 0$, with M as in (8). An equation for C_2 can be found analogously. The Jacobian criterion shows that both C_1 and C_2 are smooth; thus, for $i = 1, 2$, the double cover $X_i \to \mathbb{P}^2$ ramified along C_i is a K3 surface of degree 2.

5.1. **Primes of bad reduction.** The primes of bad reduction of X_1 and C_1 coincide. The latter divide the generator m of the ideal obtained by saturating

$$\left\langle f, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right\rangle \subset \mathbb{Z}[x_0, x_1, x_2]$$

by the irrelevant ideal and eliminating x_0, x_1 and x_2. We obtain

$$\begin{aligned}
m = {} &1115508232640214856843363784231663793779083264535962688555888430968 \\
&8933364438401787008291918987282105867611490800785997644322303281186 \\
&8922614222749465991103128446037422257623280138072129654879995620391 \\
&0907629715637695773281604080143775185215794393627484442538367517916 \\
&8651952191024387026109016400178074232186309443422761817391984342483 \\
&34511814400.
\end{aligned}$$

Standard factorization methods quickly reveal a few small prime power factors of m:

$$m = 2^8 \cdot 5^2 \cdot 7 \cdot 89 \cdot 173 \cdot 257^2 \cdot 263 \cdot 650779^2 \cdot m'.$$

The remaining factor m' has 318 decimal digits. Factoring m' with present day mathematical and computational technology is a difficult problem. However, the presence of the second

K3 surface X_2 supplies a backdoor solution: by Lemma 3.1, a prime of bad reduction for W is a prime of bad reduction for *both* X_1 and X_2.

Another Groebner basis calculation shows that the primes of bad reduction of X_2 divide

$$
\begin{aligned}
n := {} & 1846844538670477411689751271343875632264637432426913448131563435566 0\\
& 592161986539274104685992121309053984914995555340459305944952630349 81\\
& 501008813533526650956496316776134120792930449734464067645096940531 12\\
& 104716314390705483403586684931173345823145746749262233154399099550 21\\
& 69734958675148542099295443193821166161408 00
\end{aligned}
$$

Again, standard factorization methods give a few small prime power factors of n:

$$n = 2^{11} \cdot 5^2 \cdot 7 \cdot 89 \cdot 173 \cdot 263 \cdot 461^2 \cdot 6547^2 \cdot n',$$

where n' has 290 decimal digits. Our observation says that we may reasonably expect that m' and n' have a large greatest common divisor (which is easily calculated using the Euclidean algorithm). This is indeed the case:

$$
\begin{aligned}
\gcd(m', n') := {} & 809147864157687938441948148614369785987783654943839689121548451\\
& 788111145202992792430023470932052297439515068068797124401938255\\
& 799311490342451172887433057574480263654457987109316488649107.
\end{aligned}
$$

Here a small miracle happens: $\gcd(m', n')$ *is a prime number!* This claim is rigorously verified using elliptic curve primality proving algorithms [AM93], implemented in both SAGE and `magma`. We are now in a position to complete the factorization of m, and hence compute the primes of bad reduction for X_1, which are:

$$
\begin{aligned}
& 2, 5, 7, 89, 173, 257, 263, 650779,\\
& 52121973867809622086857396991358254666084809926031949922459992273 9,\\
& \gcd(m', n').
\end{aligned}
$$

We note that the penultimate prime in the list above occurs with multiplicity 2 in the factorization of m. We will write q for this prime number in Table 1.

Remark 5.1. Our numerical experiments yield several "viable" pairs $(X_1, \mathcal{A})$ that could be counter-examples to the Hasse principle explained by a transcendental Brauer-Manin obstruction arising from $\mathcal{A}$, in the following sense: X_1 has geometric Picard rank 1, and we can control the real and 2-adic invariants of $\mathcal{A}$ (using Corollary 4.6 and Lemma 4.7). Out of a dozen or so viable candidates that our initial search yielded, the example we present is the only one we found for which $\gcd(m', n')$ is a prime number. One can obtain further examples by computing 2-adic invariants by "brute force" instead of using Lemma 4.7.

5.2. **Local points.** By the Weil Conjectures, if $p > 22$ is a prime such that X_1 has smooth reduction $(X_1)_p$ at p, then $(X_1)_p$ has a smooth $\mathbb{F}_p$-point, which can be lifted by Hensel's lemma to a smooth $\mathbb{Q}_p$-point. Thus, to show X_1 is locally soluble, it suffices to verify that X_1 has local points at $\mathbb{R}$ (clear), and at $\mathbb{Q}_p$ for primes $p \leq 19$ and primes $p > 19$ where X_1 has bad reduction. This is indeed the case: we substitute integers with small absolute value for x_0, x_1, and x_2, and check if $-\frac{1}{2}\det(M)$ is a square in $\mathbb{Q}_p$. The results are recorded in Table 1.

p	x_0	x_1	x_2	$-\frac{1}{2}\det(M)$
2	0	0	-1	57872
3	-1	-1	1	1622952
5	-1	-1	-1	736256
7	-1	-1	0	256575
11	-1	-1	-1	736256
13	-1	-1	-1	736256
17	-1	-1	1	1622952
19	-1	-1	-1	736256
89	-1	0	-1	80019
173	-1	-1	0	256575
257	-1	-1	-1	736256
263	-1	-1	0	256575
650779	-1	-1	1	1622952
q	-1	-1	-1	736256
$\gcd(m', n')$	-1	-1	-1	736256

TABLE 1. Verifying X_1 has $\mathbb{Q}_p$-points at small p and primes of bad reduction. The numerical entries in the rightmost column are all squares in the appropriate p-adic field.

5.3. **Picard Rank 1.** In this section we show X_1 has (geometric) Picard rank 1. This will allow us to conclude that the obstruction to the Hasse principle arising from $\mathcal{A}$ is genuinely transcendental. Until recently, the method to prove a K3 surface has odd Picard rank, devised by van Luijk and refined by Kloosterman, and Elsenhans and Jahnel [vL07, Klo07, EJ11b], required point counting over extensions of the residue field at two primes of good reduction. A recent result of Elsenhans and Jahnel allows us to prove odd Picard rank using information at two primes, but counting points over extensions of a single residue field.

Theorem 5.2 ([EJ11c]). *Let $f\colon X \to \operatorname{Spec}\mathbb{Z}$ be a proper, flat morphism of schemes. Suppose there is a rational prime $p \neq 2$ such that the fiber X_p of f at p satisfies $H^1(X_p, \mathscr{O}_{X_p}) = 0$. Then the specialization homomorphism $\operatorname{Pic}(X_{\overline{\mathbb{Q}}}) \to \operatorname{Pic}(X_{\overline{\mathbb{F}}_p})$ has torsion-free cokernel.*

We deduce the following generalization of [EJ11c, Example 1.6].

Proposition 5.3. *Let X be a K3 surface of degree 2 over $\mathbb{Q}$, given as a double cover $\pi\colon X \to \mathbb{P}^2$ ramified over a smooth plane sextic curve C. Let p and p' denote two odd primes of good reduction for X. Assume that there exists a line ℓ that is tritangent to the curve C_p, and suppose further that $\operatorname{Pic}(\overline{X}_p)$ has rank 2 and is generated by the curves in $\pi_p^{-1}(\ell)$. If there are no tritangent lines to the curve $C_{p'}$, then $\operatorname{Pic}(\overline{X})$ has rank 1.*

Proof. Since $\operatorname{Pic}(\overline{X})$ injects into $\operatorname{Pic}(\overline{X}_p)$, if $\operatorname{Pic}(\overline{X})$ has rank 2, then we claim the tritangent line ℓ must lift to a tritangent line L in characteristic 0. To see this, note that by Theorem 5.2, the components of the pullback of ℓ to $\overline{X}_p$ lift to divisors C and C' on $\overline{X}$ such that $C^2 = C'^2 = -2$ and $C \cdot h = C' \cdot h = 1$, where h is the pullback of the class of a line from $\mathbb{P}^2$. By Riemann-Roch, either C or $-C$ is effective. Since $C \cdot h > 0$, we conclude that C is effective;

likewise for C'. The projection L of C or C' to $\mathbb{P}^2$ is a tritangent line in characteristic zero, as claimed.

Degree considerations show that L cannot break upon reduction modulo p'. This contradicts the assumption that the curve $C_{p'}$ has no tritangent lines. □

Remarks 5.4. Proposition 5.3 is computationally useful because:

(1) Checking the existence of a tritangent line modulo p' is an easy Groebner basis calculation; see [EJ08, Algorithm 8].
(2) Given a K3 surface of degree 2 over $\mathbb{Q}$, we can quickly *search* for small primes p of good reduction over which the branch curve $C_{p'}$ of the double cover $X_{p'} \to \mathbb{P}^2_{\mathbb{F}_{p'}}$ has a tritangent line.

Our particular surface X_1 reduces modulo 3 to the (smooth) K3 surface

$$\begin{aligned} w^2 = 2x_1^2(x_0^2 + 2x_0x_1 + 2x_1^2)^2 + (2x_0 + x_2)(x_0^5 + x_0^4x_1 + x_0^3x_1x_2 + x_0^2x_1^3 + x_0^2x_1^2x_2 \\ + +2x_0^2x_2^3 + x_0x_1^4 + 2x_0x_1^3x_2 + x_0x_1^2x_2^2 + x_1^5 + 2x_1^4x_2 + 2x_1^3x_2^2 + 2x_2^5). \end{aligned}$$

From the expression on the right hand side, it is clear that $2x_0 + x_2 = 0$ is a tritangent line to the branch curve of the double cover. The components of the pullback of this line generate a rank 2 sublattice of $\operatorname{Pic}\big((\overline{X_1})_3\big)$. Let $N_n := \#X_1(\mathbb{F}_{3^n})$; counting points we find

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
7	79	703	6607	60427	532711	4792690	43068511	387466417	3486842479

This is enough information to determine the characteristic polynomial f of Frobenius on $H^2((X_1)_{\overline{\mathbb{F}}_3}, \overline{\mathbb{Q}}_\ell)$; see, for example [vL07] (the sign of the functional equation for f is negative—a positive sign gives rise to roots of f of absolute value $\neq 3$). Setting $f_3(t) = 3^{-22}f(3t)$, we obtain a factorization into irreducible factors as follows:

$$\begin{aligned} f_3(t) = \frac{1}{3}(t-1)(t+1)(3t^{20} + 3t^{19} + 5t^{18} + 5t^{17} + 6t^{16} + 2t^{15} + 2t^{14} - 3t^{13} - 4t^{12} - 8t^{11} \\ - 6t^{10} - 8t^9 - 4t^8 - 3t^7 + 2t^6 + 2t^5 + 6t^4 + 5t^3 + 5t^2 + 3t + 3). \end{aligned}$$

The number of roots of $f_3(t)$ that are roots of unity give an upper bound for $\operatorname{Pic}((X_1)_{\overline{\mathbb{F}}_3})$ (see, e.g., [vL07, Corollary 2.3]). The roots of the degree 20 factor of $f_3(t)$ are not integral, so they are not roots of unity. We conclude that $\operatorname{rk}\operatorname{Pic}((X_1)_{\overline{\mathbb{F}}_3}) = 2$.

A computation shows that X_1 has no line tritangent to the branch curve when we reduce modulo $p' = 11$ (see Remark 5.4(i)). Note that the surface is not smooth at $p' = 5, 7$. Applying Proposition 5.3, we obtain the following result.

Proposition 5.5. *The surface X_1 has geometric Picard rank* 1. □

5.4. **Local invariants.** In this section we compute the local invariants of the algebra $\mathcal{A}$ for our particular surface X_1.

Proposition 5.6. *Let $p \leq \infty$ be a place of $\mathbb{Q}$. For any $P \in X_1(\mathbb{Q}_p)$, we have*

$$\operatorname{inv}_p\big(\mathcal{A}(P)\big) = \begin{cases} 0, & \text{if } \mathbb{Q}_p \neq \mathbb{R}, \\ 1/2, & \text{if } \mathbb{Q}_p = \mathbb{R}. \end{cases}$$

Proof. Whenever $p \neq 2$ is a finite prime of good reduction for X_1, we have $\operatorname{inv}_p\big(\mathcal{A}(P)\big) = 0$ for all P, by Lemma 4.4.

At every odd prime of bad reduction of X_1, the singular locus consists of $r < 8$ ordinary double points: for most of these primes p the claim follows because the valuation at p of the discriminant of X_1 is one, by our work in §5.1, so the singular locus consists of a single ordinary double point. For the remaining primes, a straightforward computer calculation does the job.

Together with Proposition 4.1 and Lemma 4.2, this implies that $\operatorname{inv}_p\big(\mathcal{A}(P)\big)$ is independent of P; it thus suffices to evaluate these invariants at a single point P. We use the local points listed in Table 1 to verify that all the local invariants vanish.

Finally, the quadrics (14) are readily seen to satisfy the hypotheses of Lemmas 4.5 and 4.7, which establishes the claim for real and 2-adic points of X_1, using Corollary 4.6. ☐

5.5. **Proof of Theorem 1.1.** The first part of the Theorem is just Proposition 3.3. We specialize now to the case $k = \mathbb{Q}$.

Let $A, \dots, F$ be as in (14), so that X is the surface X_1 considered throughout this section. The cohomology group $H^1\big(\mathbb{Q}, \operatorname{Pic}(\overline{X})\big)$ is trivial, because $\operatorname{Pic}(\overline{X}) \cong \mathbb{Z}$, with trivial Galois action, by Proposition 5.5. By (1), we have $\operatorname{Br}_1(X) = \operatorname{Br}_0(X)$. Hence, the class $\mathcal{A} \in \operatorname{Br}(X)$ is transcendental, if it is not constant.

We established in §5.2 that $X(\mathbb{A}) \neq \emptyset$. On the other hand, $X(\mathbb{A})^{\mathcal{A}} = \emptyset$, by Proposition 5.6. This shows that $\mathcal{A}$ is nonconstant, and that $X(\mathbb{A})^{\mathrm{Br}} = \emptyset$. ☐

6. Computations

In the interest of transparency, we briefly outline the computations that led to the example witnessing the second part of Theorem 1.1. The basic idea is to construct "random" K3 surfaces of the form (2), and perform a series of tests that guarantee the statement of Theorem 1.1 holds. Any surface left over after Step 7 below is a witness to this theorem.

Step 1: Seed polynomials. Generate random homogeneous quadratic polynomials

$$A, B, C, D, E, \text{ and } F \in \mathbb{Z}[x_0, x_1, x_2],$$

with coefficients in a suitable range, subject to the constraints imposed by the hypotheses of Lemma 4.7. We also require that the signs of x_0^2, x_1^2 and x_2^2 are positive for B, C and E, and negative for A, D and F, to improve the chances that the hypotheses of Lemma 4.5 are satisfied. If these hypotheses are not satisfied, then start over.

Step 2: Smoothness. Compute $f := -\frac{1}{2}\det(M)$, where M is the matrix in (8). This is an equation for the curve C_1. Use the Jacobian criterion to check smoothness of C_1 over $\mathbb{Q}$ and $\mathbb{F}_3$ (the latter will be needed to certify that the K3 surface X_1 has Picard rank 1). If either condition is not satisfied, then start over.

Step 3: Tritangent lines. Here we have the hypotheses of Proposition 5.3 in mind. Over $\mathbb{F}_3$, use [EJ08, Algorithm 8] to test for the existence of a tritangent line to C_1. Let

$$S := \{p : 5 \leq p \leq 100 \text{ a prime of good reduction for } C_1\}.$$

Find $p \in S$, such that C_1 over $\mathbb{F}_p$ has no tritangent line. If either test fails, then start over.

Step 4: Local points. For primes $p \leq 22$ and $p = \infty$, test for $\mathbb{Q}_p$-points of $X_1/\mathbb{Q} : w^2 = f$ by plugging in integers with small absolute value (typically 1 or 2) for x_0, x_1 and x_2, and determining whether f is a p-adic square. If this test fails, then it is plausible that X_1 has no local points (false negatives are certainly possible); start over.

Step 5: Point Counting. Use [EJ08, Algorithm 15] to determine $X_1(\mathbb{F}_{3^n})$ for $n = 1, \ldots, 10$. This algorithm counts *Galois orbits* of points, saving a factor of n when counting $\mathbb{F}_{3^n}$-points. Use [EJ08, Algorithms 21 and 23] to determine an upper bound $\rho_{\rm up}$ for the geometric Picard number of the surface X_1 over $\mathbb{F}_3$. If $\rho_{\rm up} > 2$, then start over. Otherwise, Proposition 5.3 guarantees that X_1 has geometric Picard number 1, by our work in Step 3.

Step 6: Primes of bad reduction. Proceeding as in the beginning of §5.1, compute the integer m whose prime factors give the places of bad reduction of C_1 (and hence those of X_1). Compute an equation for C_2, as well as the analogous integer n giving its primes of bad reduction. Typically, m and n will be very large. Proceed as in §5.1 to factorize them. If the factorization is not feasible (e.g., the integer $\gcd(m', n')$ as in §5.1 not is prime), then start over.

Step 7: Computations at places of bad reduction. At odd places of bad reduction, check for local points, as in Step 4. Determine the (geometric) singular locus. If at any prime in question the locus does not consist of $r < 8$ ordinary double points, then start over. Use to the local points found to compute the (constant) value the invariants of $\mathcal{A}$ takes at these places. If there is no Brauer-Manin obstruction, then start over.

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