POTENTIAL DENSITY OF RATIONAL POINTS FOR K3 SURFACES OVER FUNCTION FIELDS

BRENDAN HASSETT AND YURI TSCHINKEL

Abstract. We give examples of non-isotrivial K3 surfaces over complex function fields with Zariski-dense rational points and Néron-Severi rank one.

1. Introduction

Let F be a number field or a function field of a curve over the complex numbers. For each variety X defined over F, we can ask whether the rational points $X(F)$ are dense in X. We say that rational points of X are *potentially dense* if there exists a finite extension E/F with $X(E)$ dense in X. Potential density of rational points is expected to be a geometric property, depending (for smooth projective X) on the positivity of the canonical class K_X. For example, if X is a curve or a surface with K_X negative, rational points are potentially dense; indeed, such varieties are known to be rational over some finite extension of F. On the other hand, potential density fails for curves and for certain classes of higher-dimensional varieties with K_X positive, by a theorem of Faltings. We refer the reader to [5] for a general discussion of the connections between classification theory and potential density questions.

The intermediate case where $K_X = 0$ remains mysterious. Already the case of K3 surfaces is open, in general. Potential density has been proved for abelian varieties, Enriques surfaces, and special classes of K3 surfaces, e.g., those with an elliptic fibration or an infinite group of automorphisms [10] [4] [3] [12]. However, all the existing examples of K3 surfaces with dense rational points have geometric Néron-Severi rank at least two.

Our main result is:

Theorem 1. *Let B be a complex curve and $F = \mathbb{C}(B)$ its function field. There exist non-isotrivial K3 surfaces over F, with geometric Néron-Severi rank one, and a Zariski dense set of F-rational points.*

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This partially answers a question of Campana [5, Sect. 11].

Over a number field, no examples of K3 surfaces with geometric Néron-Severi rank one and a dense set of rational points are known. Our method uses the fact that the ground field is uncountable (see Remark 11); it does not yield examples over $\bar{\mathbb{Q}}(B)$.

We summarize the contents of this paper. Sections 2 through 6 contain the proof of Theorem 1. The reader may want to consult the heuristic discussion in Section 8 before tackling the main argument. Our general strategy is to deform reducible curves–unions of sections with smooth rational curves in the fibers–to new sections of larger degree (see Proposition 8). Sections 2 and 3 review general results on deformations of curves and moduli spaces of K3 surfaces. Our examples are pencils of K3 surfaces of low degree; Section 4 describes these pencils and Proposition 7 indicates the situations where our construction applies. It is an open problem to find examples of large degree with the properties described in Theorem 1. Indeed, recent results of Gritsenko, Hulek, and Sankaran [9] imply that the moduli space of K3 surfaces of large degree is of general type; in particular, there are no pencils through the generic point. Section 7 partially explains why pencils play a special rôle in our analysis. Section 9 extends our results to higher-dimensional Calabi-Yau hypersurfaces.

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2. Deformation theory

In this section we recall basic facts from deformation theory used in subsequent proofs. Throughout, we work over an algebraically closed field k.

Let X be a smooth projective variety over k, $C \subset X$ a closed subscheme. The tangent space to the Hilbert scheme $\mathcal{H}ilb(X)$ at $[C]$ is

$$\mathcal{T}_{\mathcal{H}ilb(X)}|_{[C]} = \mathrm{Hom}_X(\mathcal{I}_C, \mathcal{O}_C) = \mathrm{Hom}_C(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C).$$

We have an exact sequence

$$\mathcal{I}_C/\mathcal{I}_C^2 \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_C \to \Omega_C^1 \to 0;$$

when C is reduced, the left arrow is injective at generic points of C [17] pp. 188, [11] II.8 Ex.1.

Assume now that C is a reduced local-complete-intersection subscheme of X, so that $\mathcal{I}_C/\mathcal{I}_C^2$ is a locally free $\mathcal{O}_C$-module. Since C is reduced, an $\mathcal{O}_C$-linear homomorphism of locally-free modules is injective provided it is injective at each generic point, and we obtain

$$0 \to \mathcal{I}_C/\mathcal{I}_C^2 \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_C \to \Omega_C^1 \to 0.$$

Dualizing our exact sequence yields

$$\begin{aligned} 0 \to \mathcal{H}om_C(\Omega_C^1, \mathcal{O}_C) \to \mathcal{H}om_C(\Omega_X^1, \mathcal{O}_C) \to \mathcal{H}om_C(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C) \qquad (\dagger) \\ \to \mathcal{E}xt_C^1(\Omega_C^1, \mathcal{O}_C) \to 0 \end{aligned}$$

on the level of coherent sheaves and

$$\begin{aligned} 0 \to \mathrm{Hom}_C(\Omega_C^1, \mathcal{O}_C) \to \mathrm{Hom}_C(\Omega_X^1, \mathcal{O}_C) \to \mathrm{Hom}_C(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C) \\ \to \mathrm{Ext}_C^1(\Omega_C^1, \mathcal{O}_C) \end{aligned}$$

on cohomology. We write

$$\mathcal{T}_C = \mathcal{H}om_C(\Omega_C^1, \mathcal{O}_C) \quad \mathcal{T}_X|_C = \mathcal{H}om_C(\Omega_X^1, \mathcal{O}_C) \quad \mathcal{N}_{C/X} = \mathcal{H}om_C(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C);$$

the last two of these are locally free. We can rewrite (†) as

$$0 \to \mathcal{T}_C \to \mathcal{T}_X|_C \to \mathcal{N}_{C/X} \to \mathcal{E}xt_C^1(\Omega_C^1, \mathcal{O}_C) \to 0. \tag{2.1}$$

Moreover, we have

$$\mathcal{T}_{\mathcal{H}ilb(X)}|_{[C]} = \Gamma(C, \mathcal{N}_{C/X}).$$

Assume that C is a nodal curve and fix a node $p \in C$. We can choose local coordinates $x_1, \ldots, x_n$ for X centered about p so that

$$C = \{x_1 x_2 = x_3 = \ldots = x_n = 0\}.$$

We give local presentations for the terms in exact sequence (2.1). The tangent bundle of X restricted to C is freely generated by

$$\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n.$$

The subsheaf $\mathcal{T}_C$ is generated by $x_1\partial/\partial x_1 - x_2\partial/\partial x_2$ which induces the relation

$$x_1\partial/\partial x_1 = x_2\partial/\partial x_2$$

in the quotient sheaf $\mathcal{T}_X|_C/T_C$. The sheaf $\mathcal{E}xt_C^1(\Omega_C^1, \mathcal{O}_C)$ has fiber at p equal to the versal deformation space of the isolated singularity $(C, p) \subset$

X, which in this case is one-dimensional. The normal bundle $\mathcal{N}_{C/X}$ is an extension of this Ext-group by the quotient $\mathcal{T}_X|_C/T_C$, with free generators

$$x_2^{-1}\partial/\partial x_1 = x_1^{-1}\partial/\partial x_2, \partial/\partial x_3, \dots, \partial/\partial x_n. \tag{2.2}$$

Let $\nu : C^\nu \to C$ denote the normalization of C at the point p, with conductor $\nu^{-1}(p) = \{p', p''\}$. Consider the exact sequence

$$0 \to \mathcal{T}_{C^\nu} \to \mathcal{T}_X|_{C^\nu} \to \mathcal{N}_{C^\nu/X} \to 0.$$

There is an induced extension (see [8]§2, [1]§25-27):

$$0 \to \mathcal{N}_{C^\nu/X} \to \nu^*\mathcal{N}_{C/X} \to Q \to 0, \tag{2.3}$$

where Q is torsion, supported at $\{p', p''\}$ with length one at each point. We interpret $\nu^*\mathcal{N}_{C/X}$ in (2.3) as the sheaf of sections of $\mathcal{N}_{C^\nu/X}$ with simple poles at p' (resp. p'') in the direction $\mathcal{T}_{C^\nu}|_{p''}$ (resp. $\mathcal{T}_{C^\nu}|_{p'}$.) The elements listed in (2.2) may be regarded as free generators for both $\nu^*\mathcal{N}_{C^\nu/X}|_{p'}$ and $\nu^*\mathcal{N}_{C^\nu/X}|_{p''}$; the resulting identification expresses the compatibility condition satisfied by sections of $\nu^*\mathcal{N}_{C/X}$ which descend to $\mathcal{N}_{C/X}$.

3. K3 surfaces

In this section we work over $\mathbb{C}$.

Let S_0 be a smooth proper complex variety and $D \subset S_0$ a smooth divisor. There is a natural exact sequence

$$0 \to \mathcal{T}_{S_0}\langle -D\rangle \to \mathcal{T}_{S_0} \to \mathcal{N}_{D/S_0} \to 0,$$

where the first term is derivations with logarithmic zeroes along D. This induces a long exact sequence

$$\to H^1(S_0, \mathcal{T}_{S_0}\langle -D\rangle) \to H^1(S_0, \mathcal{T}_{S_0}) \xrightarrow{\phi} H^1(D, \mathcal{N}_{D/S_0}) \to$$

where the second term parametrizes the first order deformations of S_0 and the first term parametrizes first order deformations of the pair (S_0, D) (see [14]).

Let $\pi : \mathcal{S} \to \Delta$ be a deformation of S_0 over $\Delta = \mathrm{Spec}(\mathbb{C}[[z]])$, with $S_0 = \pi^{-1}(0)$ so that $\mathcal{N}_{S_0/\mathcal{S}} \simeq \mathcal{O}_{S_0}$. Consider the extension

$$0 \to \mathcal{N}_{D/S_0} \to \mathcal{N}_{D/\mathcal{S}} \to \mathcal{N}_{S_0/\mathcal{S}}|_D \to 0 \tag{3.1}$$

and the corresponding extension class

$$\eta \in \mathrm{Ext}^1_D(\mathcal{N}_{S_0/\mathcal{S}}|_D, \mathcal{N}_{D/S_0}) \simeq \mathrm{Ext}^1_D(\mathcal{O}_D, \mathcal{N}_{D/S_0}) = H^1(D, \mathcal{N}_{D/S_0}).$$

Let

$$\kappa \in H^1(S_0, \mathcal{T}_{S_0})$$

be the Kodaira-Spencer class of d/dz; using the isomorphisms

$$H^1(S_0, \mathcal{T}_{S_0}) \simeq \mathrm{Ext}^1_{S_0}(\mathcal{O}_{S_0}, \mathcal{T}_{S_0}) \quad \mathcal{N}_{S_0/\mathcal{S}} \simeq \mathcal{O}_{S_0}$$

we see that κ classifies the extension

$$0 \to \mathcal{T}_{S_0} \to \mathcal{T}_{\mathcal{S}}|_{S_0} \to \mathcal{N}_{S_0/\mathcal{S}} \to 0.$$

Restricting this to D and taking quotients by $\mathcal{T}_D$, we obtain extension (3.1). Thus we obtain:

Lemma 2. *The classes $\phi(\kappa)$ and η are proportional by a non-zero constant.*

Proposition 3. *Let $\pi : \mathcal{S} \to \Delta$ be a deformation of a K3 surface S_0, containing a smooth rational curve ℓ. Assume that the Kodaira-Spencer class κ is not in the tangent space to the locus of K3 surfaces containing some deformation of ℓ. Then*

$$\mathcal{N}_{\ell/\mathcal{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. The moduli space of K3 surfaces is smooth, with tangent space $H^1(S_0, \mathcal{T}_{S_0})$ at S_0; the subspace parametrizing surfaces containing a deformation of ℓ is also smooth with tangent space equal to the image of

$$H^1(\mathcal{T}_{S_0}\langle -\ell \rangle) \to H^1(\mathcal{T}_{S_0}).$$

Moreover, κ is not in this subspace precisely when $\phi(\kappa) \neq 0$.

We have the extension

$$0 \to \mathcal{N}_{\ell/S_0} \to \mathcal{N}_{\ell/\mathcal{S}} \to \mathcal{N}_{S_0/\mathcal{S}}|_\ell \to 0$$

with $\mathcal{N}_{\ell/S_0} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{N}_{S_0/\mathcal{S}}|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}$. The extension class η is in the group

$$\mathrm{Ext}^1_\ell(\mathcal{N}_{S_0/\mathcal{S}}|_\ell, \mathcal{N}_{\ell/S_0}) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$$

hence

$$\mathcal{N}_{\ell/\mathcal{S}} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } \eta \neq 0 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \text{if } \eta = 0 \end{cases}.$$

Lemma 2 and the hypothesis guarantee that $\eta \neq 0$. □

For each $g \geq 2$, let $\mathcal{K}_{2g-2}$ denote the moduli space of K3 surfaces S with a primitive polarization f of degree $2g-2$. For each $d > 0$, let $\mathcal{D}_{g,d}$ denote the moduli space of triples (S, f, ℓ), where $(S, f) \in \mathcal{K}_{2g-2}$ and

$$\Lambda_{g,d} := \langle f, \ell \rangle = \mathbb{Z}f + \mathbb{Z}\ell \subset \mathrm{NS}(S)$$

is a saturated sublattice with intersection form

$$\begin{array}{c|cc} & f & \ell \\ \hline f & 2g-2 & d \\ \ell & d & -2 \end{array}.$$

Let $\iota : \mathcal{D}_{g,d} \to \mathcal{K}_{2g-2}$ be the induced projection, which is finite onto its image.

Each $\mathcal{D}_{g,d}$ is an open subspace of the moduli spaces of lattice-polarized K3 surfaces of Nikulin [19] (see [7] for detailed discussion). Now $\Lambda_{g,d}$ admits a primitive imbedding into the cohomology lattice of a K3 surface, which is unique up to conjugation by automorphisms of the cohomology lattice [20] 1.14.4. Surjectivity of the period map and the global Torelli theorem [2] Exposés X,IX guarantee $\mathcal{D}_{g,d}$ is nonempty and irreducible.

Proposition 4. *Consider $(S, f, \ell) \in \mathcal{D}_{g,d}$ such that $\mathrm{NS}(S) = \langle f, \ell \rangle$; this is the case for a very general such triple. Then there is a smooth rational curve in S with divisor class ℓ.*

Proof. Our hypothesis implies that $\mathrm{NS}(S)$ has discriminant

$$\mathrm{Disc}\,\langle f, \ell \rangle = -(f \cdot f)(\ell \cdot \ell) + (f \cdot \ell)^2 = 4(g-1) + d^2 > 4.$$

In case (2), we have

$$\mathrm{Disc}\,\langle E, \Gamma \rangle = -(E \cdot E)(\Gamma \cdot \Gamma) + (E \cdot \Gamma)^2 = 1;$$

in the latter case of (3), the discriminant is 4.

Recall the description of the monoid of effective classes of a polarized K3 surface (S, f) [16] 1.6: It is generated by

(1) $D \in \mathrm{NS}(S)$ with $D^2 \geq 0$ and $D \cdot f > 0$;
(2) $\Gamma \in \mathrm{NS}(S)$ with $\Gamma^2 = -2$ and $\Gamma \cdot f > 0$.

Moreover, indecomposable effective classes Γ with $\Gamma^2 = -2$ represent smooth rational curves. Thus ℓ is effective, and represents a smooth rational curve if it is indecomposable.

Assume the contrary, so there exists an indecomposable effective class m with $\ell - m$ effective and nonzero. As f is ample, we have $f \cdot \ell > f \cdot m$. Since m is indecomposable, $m \cdot m \geq -2$. The discriminants satisfy

$$\mathrm{Disc}\,\langle f, m \rangle = -(f \cdot f)(m \cdot m) + (f \cdot m)^2 < -(f \cdot f)(\ell \cdot \ell) + (f \cdot \ell)^2 = \mathrm{Disc}\,\langle f, \ell \rangle\,.$$

However, since $\langle f, m \rangle \subset \langle f, \ell \rangle$ we have

$$\mathrm{Disc}\,\langle f, m \rangle \geq \mathrm{Disc}\,\langle f, \ell \rangle\,,$$

a contradiction. □

4. Pencils of K3 surfaces

Definition 5. A *pencil of K3 surfaces* consists of

(1) a projective irreducible threefold $Y \subset \mathbb{P}^N$;
(2) a codimension-two linear subspace $\Lambda \subset \mathbb{P}^N$ meeting Y transversally along a smooth curve Z, so that a generic hyperplane $\Lambda \subset H \subset \mathbb{P}^N$ intersects Y in a K3 surface.

Projection from Λ

$$p_\Lambda : \mathrm{Bl}_\Lambda \mathbb{P}^N \to \mathbb{P}^1$$

induces a flat morphism

$$\pi : \mathcal{S} := \mathrm{Bl}_Z Y \to \mathbb{P}^1$$

with generic fiber a K3 surface.

Let E denote the exceptional divisor of the blow-up $\beta : \mathrm{Bl}_Z Y \to Y$.

(1) Since Z is a complete intersection in Y we have

$$E \simeq \mathbb{P}(\mathcal{N}_{Z/Y}) \simeq \mathbb{P}(\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)) \simeq \mathbb{P}^1 \times Z;$$

projection onto the first factor coincides with $\pi|_E$. Thus for each $p \in \mathbb{P}^1$, the fiber $\mathcal{S}_p = \pi^{-1}(p)$ intersects E along $\{p\} \times Z$.
(2) For each $z \in Z$, the fiber $E_z := \beta^{-1}(z)$ meets each member of the pencil in one point and thus yields a section $\mathsf{s}_z : \mathbb{P}^1 \to \mathcal{S}$ of π.
(3) The normal bundle of this section is computed by the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{N}_{E_z/E} & \to & \mathcal{N}_{E_z/\mathcal{S}} & \to & \mathcal{N}_{E/\mathcal{S}}|_{E_z} & \to & 0 \\ 0 & \to & \mathcal{O}_{\mathbb{P}^1} & \to & \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \to & \mathcal{O}_{\mathbb{P}^1}(-1) & \to & 0. \end{array}$$

(4) For each smooth fiber $\mathcal{S}_p$ and each curve $\ell \subset \mathcal{S}_p$, the intersection $Z \cap \ell \neq \emptyset$. Given $z \in Z \cap \ell$, E_z meets ℓ transversally at z, i.e., the tangent spaces of ℓ and E_z at z are transverse.

Example 6. Let $Y \subset \mathbb{P}^N$ be a threefold with isolated singularities and $\omega_Y = \mathcal{O}_Y(-1)$; if Y is smooth then it is a Fano threefold. We can choose Λ so that it meets Y along a smooth curve, which yields a pencil of K3 surfaces as in Definition 5.

Proposition 7. *Fix $2 \leq g \leq 10$. There exists a projective variety $X \subset \mathbb{P}^M$ so that the generic K3 surface of degree $2g-2$ can be realized as an intersection of X with a linear subspace of codimension $\dim(X)-2$, and thus as a member of a pencil of K3 surfaces.*

Proof. The generic K3 surface of degree 2 is a hypersurface of degree 6 in weighted projective space $\mathbb{P}(1,1,1,2)$, which has isolated singularities. Imbed

$$X := \mathbb{P}(1,1,1,2) \hookrightarrow \mathbb{P}^M$$

using weighted-homogeneous forms of degree six. Smooth hyperplane sections are degree 2 K3 surfaces.

For $g = 3,4,5,6,7,8,9,10$, the generic K3 surface of degree $2g-2$ is a complete intersection in a generalized flag variety [18]. We extract the variety X in each case:

- $g = 3$ quartic hypersurface in $\mathbb{P}^3$: take X to be the 4-fold Veronese reimbedding of $\mathbb{P}^3$;
- $g = 4$ complete intersection of a quadric and a cubic in $\mathbb{P}^4$: take X to be the 3-fold Veronese reimbedding of a smooth quadric hypersurface $Q \subset \mathbb{P}^4$;
- $g = 5$ a generic complete intersection of three quadrics: take X to be the 2-fold Veronese reimbedding of $\mathbb{P}^5$;
- $g = 6$ a complete intersection of $\mathrm{Gr}(2,5)$ with a quadric and a codimension-three linear space: take X to be the 2-fold Veronese reimbedding of the generic codimension-three linear section of $\mathrm{Gr}(2,5)$;
- $g = 7$ a codimension-eight linear section of the isotropic Grassmannian $\mathrm{IGr}(5,10) \subset \mathbb{P}^{15}$;
- $g = 8$ a codimension-six linear section of the Grassmannian $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$;
- $g = 9$ a codimension-four linear section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6) \subset \mathbb{P}^{13}$;
- $g = 10$ a codimension-three linear section of the flag variety of dimension five, associated with the adjoint representation of G_2, imbedded in $\mathbb{P}^{13}$.

We obtain pencils of K3 surfaces of degree $2g-2$ as follows: Taking generic subspaces

$$\Lambda_{N-2} \subset \Lambda_N \subset \mathbb{P}^M$$

of codimensions $\dim(X)-1$ and $\dim(X)-3$ respectively, there is a pencil

$$\Lambda_{N-2} \subset \Lambda_{N-1}(p) \subset \Lambda_N, \quad p \in \mathbb{P}^1$$

of $(N-1)$-dimensional linear spaces. Taking $Y = \Lambda_N \cap X$ and $Z = \Lambda_{N-2} \cap X$, we obtain a pencil as in Definition 5. □

5. Deforming sections in pencils

Proposition 8. *Let $\pi : \mathcal{S} \to \mathbb{P}^1$ be a pencil of K3 surfaces. Suppose there exists a point $p \in \mathbb{P}^1$ so that $\mathcal{S}_p$ is smooth and contains a smooth rational curve ℓ with normal bundle $\mathcal{N}_{\ell/\mathcal{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then there exists a one-parameter family of sections of π, the closure of which contains ℓ.*

Proof. Let Z be the base locus of the pencil. Choose a point $z \in \ell \cap Z \subset \mathcal{S}_p$ and consider the nodal curve

$$C = \ell \cup_z E_z$$

with normalization $\nu : C^\nu \to C$. Consider the exact sequence

$$0 \to \mathcal{N}_{C^\nu/\mathcal{S}} \to \nu^* \mathcal{N}_{C/\mathcal{S}} \to Q \to 0$$

with Q a torsion sheaf, of length one at each of the points $\{z', z''\}$ over z.

It suffices to show that $\mathcal{N}_{C/\mathcal{S}}$ has no higher cohomology and admits a global section not mapping to zero in Q (see Section 2): Then the Hilbert scheme is smooth at C and contains deformations of C smoothing the node z (cf. [13], proof of Proposition 24 and [8] §2). Moreover, these deformations meet the generic fiber of π at one point, and thus are sections of π not contained in Z. Finally, since ℓ is a component of a degeneration of these sections, it lies in the closure of the surface traced out by these sections.

We evaluate $\mathcal{N}_{C/\mathcal{S}}$ on each component of C: On E_z we have

$$0 \to \mathcal{N}_{E_z/\mathcal{S}} \to \mathcal{N}_{C/\mathcal{S}}|_{E_z} \to Q(E_z) \to 0 \tag{5.1}$$

with $Q(E_z)$ of length one at z; furthermore, $\mathcal{N}_{E_z/\mathcal{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ with the non-negative summand corresponding to the normal direction to E_z in E. Since the tangent vector to ℓ is not contained in E, the $\mathcal{O}_{\mathbb{P}^1}$ component is saturated in the extension (5.1), and the $\mathcal{O}_{\mathbb{P}^1}(-1)$ component is *not* saturated ([13] Proposition 23, cf. Sublemma 27). In particular, we conclude

$$\mathcal{N}_{C/\mathcal{S}}|_{E_z} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}.$$

As for ℓ, we have $\mathcal{N}_{\ell/\mathcal{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ and an extension

$$0 \to \mathcal{N}_{\ell/\mathcal{S}} \to \mathcal{N}_{C/\mathcal{S}}|_\ell \to Q(\ell) \to 0$$

with $Q(\ell)$ of length one. In this case, the only possibility is

$$\mathcal{N}_{C/\mathcal{S}}|_\ell \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

We observe that

(1) $\mathcal{N}_{C/\mathcal{S}}|_{E_z}$ is globally generated and has no higher cohomology;
(2) $\mathcal{N}_{C/\mathcal{S}}|_{\ell}$ has no higher cohomology and each nonzero global section is nonzero in $Q(\ell)$.

Since C is obtained by gluing E_z and ℓ at a single point, $\mathcal{N}_{C/\mathcal{S}}$ also has no higher cohomology [15, II.7.5]. Fixing a nonzero section t of $\mathcal{N}_{C/\mathcal{S}}|_{\ell}$, we can find a section of $\mathcal{N}_{C/\mathcal{S}}|_{E_z}$ agreeing with t at z. □

6. Construction of the examples

Theorem 9. *Let $\pi : \mathcal{S} \to \mathbb{P}^1$ be a pencil of K3 surfaces of degree $2g-2$ with base locus Z. Assume that there exists an infinite sequence of positive integers*

$$d_1 < d_2 < \dots$$

such that $\mathbb{P}^1$ intersects $\mathcal{D}_{g,d_j}$ transversally at some point p_j and the class $\ell_j \in \Lambda_{g,d_j}$ is represented by a smooth rational curve $\ell_j \subset \mathcal{S}_{p_j}$. Then $\mathcal{S}$ has a Zariski dense set of sections of π.

Proof. We have

$$\mathcal{N}_{\ell_j/\mathcal{S}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

by Proposition 3. Proposition 8 gives a one-parameter family of sections

$$\begin{array}{ccc} \Sigma_j & \stackrel{\sigma_j}{\to} & \mathcal{S} \\ {\scriptstyle \psi_j}\downarrow & & \\ R_j & & \end{array},$$

with $\sigma_j : \psi_j^{-1}(r) \to \mathcal{S}$ a section for $r \in R_j$ generic, and $\sigma_j(\psi_j^{-1}(r_0)) \supset \ell_j$ for the distinguished point $r_0 \in R_j$ around which the deformation was produced. It follows that the irreducible surface $\widetilde{\Sigma}_j := \sigma_j(\Sigma_j)$ contains ℓ_j.

We claim that $\cup_j \widetilde{\Sigma}_j$ is dense in $\mathcal{S}$. Let Ξ denote the closure of all these surfaces. If $\Xi \subsetneq \mathcal{S}$ then Ξ is a union of irreducible surfaces, each dominating $\mathbb{P}^1$. Thus the degree $\Xi \cap \mathcal{S}_p \subset \mathcal{S}_p$ is bounded; in particular, Ξ cannot contain ℓ_j when $d_j \gg 0$, a contradiction. □

Corollary 10. *Let $\mathcal{S} \to \mathbb{P}^1$ be a very general pencil of degree $2g-2$, for $2 \le g \le 10$. Then the hypothesis of Theorem 9 are satisfied, and sections are Zariski dense.*

Proof. By Proposition 7, the moduli space $\mathcal{K}_{2g-2}$ is dominated by an open subset of the Grassmannian $\mathrm{Gr}(N, M+1)$ parametrizing the $(N-$

1)-dimensional linear subspaces in $\mathbb{P}^M$. For $d \gg 0$, $\mathcal{D}_{g,d}$ determines a nonempty divisor $\mathcal{D}'_{g,d} \subset \mathrm{Gr}(N, M+1)$, which is necessarily ample. Each pencil corresponds to a $\mathbb{P}^1 \subset \mathrm{Gr}(N, M+1)$, so the pencils are parametrized by a suitable Hilbert scheme. For fixed d, consider the $\mathbb{P}^1 \subset \mathrm{Gr}(N, M+1)$ meeting $\mathcal{D}'_{g,d}$ transversely at a point corresponding to a K3 surface containing a smooth rational curve ℓ of degree d. Proposition 4 implies that this is a nonempty open subset of the Hilbert scheme. Take the countable intersection of these open subsets in the space of all pencils; applying the Baire category theorem, we are left with a nonempty dense subset of the space of all pencils. □

Remark 11. Our argument does not preclude the (unlikely) possibility that every pencil defined over $\overline{\mathbb{Q}}$ is tangent to $\mathcal{D}_{g,d}$ for each $d \gg 0$.

We complete the argument for Theorem 1:

Proof. Corollary 10 gives pencils of K3 surfaces with dense rational points. Proposition 7 implies that the generic K3 surface of degree $2g-2$, $2 \leq g \leq 10$ arises in such a pencil; in particular, K3 surfaces with Néron-Severi rank one occur. The Néron-Severi group of the geometric generic fiber injects in the Néron-Severi group of each smooth member of the pencil; indeed, the map on homology is an isomorphism. Thus the Néron-Severi group of the geometric generic fiber is also of rank one.

For simplicity, we will only discuss non-isotriviality in the degree four case: Any pair of quartic surfaces

$$S_i = \{f_i(w, x, y, z) = 0\} \subset \mathbb{P}^3, \quad i = 1, 2$$

are contained in a pencil, e.g.,

$$\mathcal{S} = \{s_1 f_1 + s_2 f_2 = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1.$$

This is non-isotrivial whenever $S_1 \not\cong S_2$, and completes the proof when $B = \mathbb{P}^1$.

For an arbitrary smooth complex curve B, we can express $\mathbb{C}(B)$ as a finite extension of $\mathbb{C}(\mathbb{P}^1)$. Any K3 surface with dense $\mathbb{C}(\mathbb{P}^1)$-rational points *a fortiori* has dense $\mathbb{C}(B)$-rational points. □

7. Limitations

In this section, we discuss constraints to deforming sections in K3 fibrations. We hope to explain why pencils are a natural source of examples of K3 surfaces with potentially-dense rational points.

Let $\pi : \mathcal{S} \to B$ be a fibration satisfying the following

(1) $\mathcal{S}$ is smooth and the generic fiber is a K3 surface;
(2) the singular fibers have at worst rational double points;
(3) π is non-isotrivial and projective.

Let $\mu : B \to \overline{\mathcal{K}}_{2g-2}$ be the classifying map into the Baily-Borel compactification of the corresponding moduli space of polarized K3 surfaces. Recall that the pull-back of the natural polarization on the Baily-Borel compactification is $\mathcal{L} := \pi_*\omega_\pi$. Since the fibers of π have trivial dualizing sheaf, ω_π is the pull-back of a line bundle from B, which is necessarily $\mathcal{L}$ (by the projection formula).

Suppose we have a section $\sigma : B \to \mathcal{S}$. Consider the exact sequence

$$0 \to \mathcal{T}_{\sigma(B)} \to \mathcal{T}_{\mathcal{S}}|_{\sigma(B)} \to \mathcal{N}_{\sigma(B)/\mathcal{S}} \to 0.$$

It follows that

$$c_1(\mathcal{N}_{\sigma(B)/\mathcal{S}}) = c_1(\mathcal{T}_{\mathcal{S}}) - c_1(\mathcal{T}_{\sigma(B)}) = -c_1(\omega_\pi),$$

whence

$$\deg(\mathcal{N}_{\sigma(B)/\mathcal{S}}) = -\deg(\mu(B)) < 0.$$

(For pencils of quartic surfaces the degree is -1, the smallest possible value.) Hence $\mathcal{N}_{\sigma(B)/\mathcal{S}}$ cannot be globally generated at the generic point: If it were then $\det(\mathcal{N}_{\sigma(B)/\mathcal{S}})$ would have a nonvanishing section, contradicting our degree computation. In particular, we cannot deform a section in a two-parameter family sweeping out a dense subset of $\mathcal{S}$.

8. Projective geometry of pencils of quartic surfaces

Corollary 10 says that a very general pencil of quartic surfaces has a Zariski-dense set of sections. Here we interpret these in terms of classical projective geometry.

Recall the set-up of Theorem 9:

$$\begin{array}{ccc} \mathcal{S} & \stackrel{\pi}{\to} & \mathbb{P}^1 \\ {\scriptstyle\beta}\downarrow & & \\ \mathbb{P}^3 & & \end{array}$$

where β is the blowup along the base locus Z of the pencil. We analyze the proper transforms in $\mathbb{P}^3$ of the sections of π produced by Proposition 8.

Let ℓ_1 be a line in some fiber of π; $\beta(\ell_1)$ is a four-secant line to Z. If $z \in Z \cap \beta(\ell_1)$ and $E_z := \beta^{-1}(z)$ then $\beta(\ell_1) \cup_z E_z$ deforms to a 3-secant line to Z. These 3-secants move in a 1-parameter family. Similarly, if ℓ_d is a smooth rational curve of degree d in some fiber of π then $\beta(\ell_d)$

is a rational curve of degree d meeting Z in $4d$ points. The nodal curve $\beta(\ell_d) \cup_z E_z$ deforms to a 1-parameter family of rational curves in $\mathbb{P}^3$ of degree d meeting Z in $4d-1$ points.

Essentially, the proof of Proposition 8 shows that the space of rational curves of degree d in $\mathbb{P}^3$ meeting Z in at least $4d-1$ points (counted with multiplicities) has at least one irreducible component of the expected dimension and the generic such curve meets Z in $4d-1$ points.

9. Higher-dimensional Calabi-Yau hypersurfaces

We have seen in Corollary 10 that a very general pencil of quartic surfaces has a Zariski-dense set of sections. The following generalization is due to Coskun and Starr:

Theorem 12. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a very general pencil of hypersurfaces of degree $n+1$ in $\mathbb{P}^n$, with $n \geq 4$. Then sections of π are Zariski dense.*

The first step is the following extension of Proposition 8 to higher-dimensional pencils (which has an analogous proof):

Proposition 13. *Let $\pi : \mathcal{X} \to \mathbb{P}^1$ be a pencil of hypersurfaces of degree $n+1$ in $\mathbb{P}^n$. Suppose there exists a point $p \in \mathbb{P}^1$ so that $\mathcal{X}_p$ is smooth and contains a smooth rational curve ℓ with normal bundle*

$$\mathcal{N}_{\ell/\mathcal{X}} \simeq \mathcal{O}_{\mathbb{P}^1}^{n-3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Then there exists an $(n-2)$-parameter family of sections of π; its closure contains the deformations of ℓ in $\mathcal{X}_p$.

Just as in Theorem 9, Proposition 13 reduces Theorem 12 to the following result, which is a natural extension of Theorem 1.27 of [6] for quintic threefolds:

Theorem 14. *Let X be a very general hypersurface of degree $n+1$ in $\mathbb{P}^n$, with $n \geq 4$. Then there exists an infinite sequence of integers*

$$d_1 < d_2 < \dots$$

such that X contains a rational curve ℓ of degree d_j with

$$\mathcal{N}_{\ell/X} \simeq \mathcal{O}_{\mathbb{P}^1}^{n-3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Furthermore, deformations of these curves are Zariski dense in X.

It suffices to produce one hypersurface X with the desired properties; even a singular X will work, as long as it is smooth along each ℓ.

The proof proceeds by induction, following [6], which establishes the base case $n = 4$. Consider a hypersurface $X \subset \mathbb{P}^n$ of degree $n+1$ containing the codimension-two linear subspace $\{x_0 = x_n = 0\} \subset \mathbb{P}^n$. The defining equation for X can be written uniquely

$$x_0 f_n(x_0, \dots, x_{n-1}) + x_n g_n(x_0, \dots, x_n) = 0$$

where f_n and g_n have degree n. Let H denote the hyperplane $\{x_0 = 0\}$ and

$$W = \{f_n(x_0, \dots, x_{n-1}) = 0\} \subset \{x_n = 0\} \simeq \mathbb{P}^{n-1},$$

a hypersurface of degree n. The singular locus $\mathrm{Sing}(X)$ is defined by the equations

$$f_n + x_0 \frac{\partial f_n}{\partial x_0} + x_n \frac{\partial g_n}{\partial x_0} = x_0 \frac{\partial f_n}{\partial x_i} + x_n \frac{\partial g_n}{\partial x_i} = x_n \frac{\partial g_n}{\partial x_n} + g_n = 0,$$

where $i = 1, \dots, n-1$. This contains the locus

$$\Sigma := \{x_0 = x_n = f_n = g_n = 0\};$$

if this is a smooth complete intersection then $\mathrm{Sing}(X) = \Sigma$.

We make the following generality assumptions on f_n:

- W satisfies the inductive hypothesis, so there is an infinite sequence of integers $\{d_j\}$ so that W contains a rational curve ℓ_j of degree d_j with normal bundle

$$\mathcal{N}_{\ell_j/W} \simeq \mathcal{O}_{\mathbb{P}^1}^{n-4} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

- H intersects W and each ℓ_j transversely.

The Bertini theorem allows us to achieve the last condition for a very general choice of f_n. We also make assumptions on g_n:

- Σ is a smooth complete intersection, so that $\mathrm{Sing}(X) = \Sigma$.
- $\Sigma \cap \ell_j = \emptyset$ for each j.

In particular, the ℓ_j avoid the singularities of X.

We can compute the normal bundle $\mathcal{N}_{\ell_j/X}$ using the extension

$$0 \to \mathcal{N}_{\ell_j/W} \to \mathcal{N}_{\ell_j/X} \to \mathcal{N}_{W/X}|_{\ell_j} \to 0.$$

We have

$$\mathcal{N}_{W \cup H/X} = \mathcal{O}_{W \cap H}(+1)$$

and since H is Cartier away from Σ

$$\mathcal{N}_{W/X}|_{\ell_j} = \mathcal{N}_{W \cup H/X}|_{\ell_j}(-H) = \mathcal{O}_{\ell_j}.$$

The extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}^{n-4} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2 \to \mathcal{N}_{\ell_j/X} \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

is necessarily trivial and

$$\mathcal{N}_{\ell_j/X} \simeq \mathcal{O}_{\mathbb{P}^1}^{n-3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2,$$

as desired.

The deformations of each ℓ_j sweep out a divisor in W; the union of these divisors is dense in W. It remains to check that the deformations of the ℓ_j are dense in X. Each ℓ_j has deformations in X which are not contained in W. Indeed, we have shown that ℓ_j admits infinitesimal deformations that are nonzero in $\mathcal{N}_{W/X}|_{\ell_j}$. In particular, deformations of ℓ_j sweep out an irreducible divisor $D_j \subset X$ not equal to W. If the union $\cup_{j=1}^{\infty} D_j$ were a proper subvariety $\Xi \subsetneq X$, then $\Xi \cap W$ would not be dense in W, contradicting the inductive hypothesis.

Added in proof: Claire Voisin has suggested a variation on Theorem 9 yielding examples of K3 surfaces over $F = \bar{\mathbb{Q}}(t)$ that are non-isotrivial with geometric Náron-Severi rank one, and admit a Zariski-dense set of F-rational points. Fix a single K3 surface S_0 admitting an infinite number of smooth rational curves of degrees $d_1 < d_2 < \dots$ Choose a pencil containing S_0 and meeting an infinite number of the corresponding $\mathcal{D}_{g,d_j}$ transversely.

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Department of Mathematics, Rice University, MS 136, Houston, TX 77251-1892
E-mail address: hassett@rice.edu

Mathematisches Institut, Bunsenstr. 3-5, 37073 Göttingen, Germany
E-mail address: yuri@uni-math.gwdg.de, tschinkel@cims.nyu.edu
Current address: Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012