

# RATIONALITY OF FORMS OF $\overline{\mathcal{M}}_{0,n}$

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ABSTRACT. We study equivariant geometry and rationality of moduli spaces of points on the projective line, for twists associated with permutations of the points.

## 1. INTRODUCTION

In this note, we strengthen a theorem of Florence–Reichstein [FR18] concerning rationality of moduli spaces. They consider *forms* of  $\overline{\mathcal{M}}_{0,n}$ , i.e., varieties over nonclosed fields  $F$  which are isomorphic to the moduli space of  $n$  points on  $\mathbb{P}^1$  over an algebraic closure of  $F$ . These forms are obtained by twisting via Galois actions permuting the points over  $F$ . The main results of [FR18] are:

- if  $n \geq 5$  is odd, and  $F$  is infinite of characteristic  $\neq 2$ , then every form over  $F$  is rational;
- if  $n \geq 6$  is even, and  $F$  has nontrivial 2-torsion in its Brauer group and contains fourth roots of unity, then there exists a form  $X$  of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  such that  $X$  is not retract rational over  $F$ .

These were inspired by a classical theorem of Enriques, Manin, and Swinnerton-Dyer concerning rationality of twists of  $\overline{\mathcal{M}}_{0,5}$ , a del Pezzo surface of degree 5, over any field  $F$ . The proof for  $n \geq 5$  uses (a twisted form of) the Gelfand-MacPherson correspondence, and techniques developed in connection with Noether’s problem for twisted forms of the groups in question.

By [FR18], every form over an infinite field  $F$  is unirational over  $F$ . It is known that every form of  $\overline{\mathcal{M}}_{0,6}$  over  $\mathbb{R}$  is rational [Avi20, Proposition 2.9]; see Corollary 21 for generalizations.

Here, we strengthen their conclusions in two directions: we prove rationality in several situations not addressed in [FR18]. On the other

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hand, we show failure of rationality via Galois cohomology in instances not covered by [FR18], e.g., where the Brauer group of  $F$  is trivial.

An important ingredient throughout is a theorem of [BM13]:

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}) = \mathfrak{S}_n, \quad n \geq 5,$$

acting via permutations of the  $n$  points on  $\mathbb{P}^1$ . In particular, Galois twists of  $\overline{\mathcal{M}}_{0,n}$  factor through subgroups of  $\mathfrak{S}_n$ , and there is a close link between rationality of twists and linearizability of  $G$ -actions on  $\overline{\mathcal{M}}_{0,n}$ ; see [DR15] for a general discussion of such connections. In both situations, there is an action of a finite group on the geometric Picard group

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,n}),$$

via a subgroup of  $\mathfrak{S}_n$ .

We present several stable rationality and linearizability results, including Propositions 3 and 5 (based on the Kapranov construction) and Theorem 24 (using torsors and quotients). Section 3 focuses on geometric constructions. One rationality construction uses Schubert calculus and the geometry of Grassmannians; Theorem 14 extends results of [FR18] to small fields (Corollary 16) and some point configurations in higher-dimensional projective spaces (Corollary 17). Another relies on fibration structures; see Theorem 20. We close with a comprehensive discussion of the  $n = 6$  case (Theorem 34).

For nonrationality/nonlinearizability, we focus on situations where the twisted moduli spaces are toric via the Losev-Manin construction [LM00]. We utilize cohomological **(H1)** and **(SP)**-obstructions (see Section 5): In the arithmetic context, the group is replaced by the absolute Galois group of the ground field  $F$  and the Picard module by the geometric Picard module. We focus on *even*  $n$ :

**Theorem 1** (Corollary 29 and Theorem 30). *For every even  $n \geq 6$  there exists a subgroup  $G = C_2^2 \subset \mathfrak{S}_n$  such that*

$$H^1(G, \mathrm{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2.$$

*In particular,*

- *for all subgroups of  $\mathfrak{S}_n$  containing  $G$ , the corresponding action is not stably linearizable,*
- *for all fields  $F$  admitting a Galois extension  $L/F$  with Galois group  $\mathrm{Gal}(L/F) \simeq G$  there exists a form  $X$  of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  such that  $X$  is not retract rational over  $F$ .*

Indeed, nonvanishing group cohomology is an obstruction to (stable) linearizability, see, e.g., [BP13, Corollary 2.5.2.]. In the context of nonclosed fields, one can find a twist  $X$  of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  so that the corresponding Galois action on the geometric Picard group of  $X$  factors through the prescribed action of  $G$ . This yields nontrivial Galois cohomology, which in turn obstructs retract rationality of  $X$  over  $F$ . In particular, our result applies to fields  $F$  with *trivial* Brauer group, e.g.,  $F = \mathbb{C}(t)$ .

**Remark 2.** Florence and Reichstein have pointed out that the proof of [FR18, Theorem 1.2(b)] – giving forms of  $\overline{\mathcal{M}}_{0,n}$  that are not retract rational – implicitly assumes that the base field contains fourth roots of unity. These are needed to harmonize sign choices in the quaternion algebras constructed in [FR18, Section 7]. Indeed, the field  $\mathbb{R}$  has Brauer group  $\mathbb{Z}/2\mathbb{Z}$  but real forms of  $\overline{\mathcal{M}}_{0,n}$  are rational (see Corollary 21).

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## 2. $\mathfrak{S}_n$ -EQUIVARIANT GEOMETRY

We recall some terminology: Let  $G$  be a finite group acting regularly on a projective variety  $X$ . Assume the action is generically free. The action is *linearizable* if  $X$  is equivariantly birational to the projectivization  $\mathbb{P}(V)$  of a linear representation  $V$  of  $G$  on a vector space. It is *stably linearizable* if  $X \times \mathbb{P}^r$  – where  $G$  acts trivially on the second factor – is linearizable. By the No-Name Lemma, this is equivalent to saying that  $X \times V$  is linearizable for some linear representation  $V$  of  $G$ , or that the total space of a  $G$ -equivariant vector bundle  $E \rightarrow X$  is linearizable.

Stable linearizability and stable rationality of forms are tightly linked [DR15, Theorem 1.1(d)]: A  $G$ -action on  $X$  is stably linearizable over  $F$  if and only if for every infinite field  $K/F$  and every form of  $X$  over  $K$  obtained via twisting by the  $G$ -action, the resulting variety is stably rational.

**Kapranov blowup.** We make use of the Kapranov blowup realization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}, \quad n \geq 4,$$

where  $\beta_n$  is an iterated blowup of  $n - 1$  general points on  $\mathbb{P}^{n-3}$ , lines through pairs of points, etc., see, e.g., [HT02, Section 3.1]. Precisely, we regard

$$\mathbb{P}^{n-3} = \mathbb{P}(k[\mathfrak{S}_{n-1}]/(1, \dots, 1)),$$

so that the  $\mathfrak{S}_{n-1}$ -action is linear. Boundary divisors  $D_I$  are labeled by partitions

$$[1, \dots, n] = I \sqcup I^c, \quad |I|, |I^c| \geq 2.$$

Recall that the Picard group  $\text{Pic}(\overline{\mathcal{M}}_{0,n})$  has rank  $2^{n-1} - \binom{n}{2} - 1$ , and an explicit basis is given by

$$\{H, E_{i_1}, E_{i_1, i_2}, \dots, E_{i_1, \dots, i_{n-4}}\},$$

where  $H$  is the (pullback of the) hyperplane class on  $\mathbb{P}^{n-3}$ , and the other elements are (classes of) exceptional divisors from blowups of points, lines, etc. The boundary divisors  $D_I$  expressed in this basis are

$$D_{i_1, \dots, i_k, n} = E_{i_1, \dots, i_k}, \quad \{i_1, \dots, i_k\} \subset \{1, \dots, n-1\}, \quad k \leq n-4,$$

and

$$[D_{i_1, \dots, i_{n-3}, n}] = H - E_{i_1} - E_{i_2} - \dots - E_{i_1, \dots, i_{n-4}} - E_{i_2, \dots, i_{n-3}}.$$

The  $\mathfrak{S}_n$ -action on  $\text{Pic}(\overline{\mathcal{M}}_{0,n})$  is best understood in terms of the natural  $\mathfrak{S}_n$ -action on the boundary divisors via permutations of indices of  $D_I$ . In particular, there is a distinguished  $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$  acting via permutation of indices on  $E_i$ , for  $i \in \{1, \dots, n-1\}$ .

The Kapranov construction has applications to linearizability:

**Proposition 3.** *Suppose that  $G \subseteq \mathfrak{S}_{n-1}$  acts on  $\overline{\mathcal{M}}_{0,n}$  leaving the  $n$ th point invariant. Then the action of  $G$  is linearizable.*

*For  $n = 2m + 1$  and  $G \subseteq \mathfrak{S}_{2m+1}$ , the  $G$ -action on  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.*

*More generally, for  $G \subseteq \mathfrak{S}_n$  leaving an odd cycle invariant, the  $G$ -action on  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.*

*Proof.* The first assertion reflects the fact that the Kapranov morphism  $\beta_n$  is  $\mathfrak{S}_{n-1}$  invariant and the  $\mathfrak{S}_{n-1}$ -action on  $\mathbb{P}^{n-3}$  is linear. The second assertion is a special case of the third. For the third statement, consider the universal curve

$$\overline{\mathcal{C}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,n}.$$

**Lemma 4.** *Let  $G \subset \mathfrak{S}_n$  act on  $\overline{\mathcal{M}}_{0,n}$  by permutation of the marked points. Then there is a canonical lift of the action to the universal curve*

$$\phi : \overline{\mathcal{C}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,n}.$$

We prove the lemma. Interpreting  $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$ , we have

$$\mathrm{Aut}(\overline{\mathcal{C}}_{0,n}) = \mathfrak{S}_{n+1} \supset \mathfrak{S}_n \hookrightarrow \mathrm{Aut}(\overline{\mathcal{M}}_{0,n}),$$

with the last inclusion an equality when  $n \geq 5$ . The induced action on  $\mathrm{Aut}(\overline{\mathcal{C}}_{0,n})$  is equivariant under forgetting the  $(n+1)$ st point.

Returning to the Proposition, we assume that  $G$  leaves an odd cycle invariant. Then the forgetting morphism  $\phi$  – an étale  $\mathbb{P}^1$ -bundle over  $\mathcal{M}_{0,n}$  – admits a multisection of odd degree. It must therefore be the projectivization of a rank-two  $G$ -equivariant vector bundle over  $\mathcal{M}_{0,n}$ . However, we have already seen that the  $G$ -action on  $\overline{\mathcal{C}}_{0,n} = \overline{\mathcal{M}}_{0,n+1}$  is linearizable. We conclude then that  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.  $\square$

A similar argument yields dividends for the Galois-theoretic question:

**Proposition 5.** *Let  $L/F$  be a Galois extension with Galois group  $\Gamma$ . Fix a representation*

$$\rho : \Gamma \rightarrow \mathfrak{S}_n$$

*and let  ${}^\rho\overline{\mathcal{M}}_{0,n}$  denote the corresponding twist of  $\overline{\mathcal{M}}_{0,n}$  defined over  $F$ .*

- *If  $\rho$  factors through an  $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$  then  ${}^\rho\overline{\mathcal{M}}_{0,n}$  is rational over  $F$ .*
- *If  $n$  is odd then  $\mathbb{P}^1 \times {}^\rho\overline{\mathcal{M}}_{0,n}$  is rational. The same holds if  $\rho$  leaves an odd cycle invariant.*

This gives a weaker version of [FR18, Theorem 1.2]; however, our statement is valid over a finite field as well. See Remark 22 below for a related result.

*Proof.* The Kapranov morphism  $\beta : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$  is equivariant for  $\mathfrak{S}_{n-1}$ , which acts linearly on the target. Thus it descends to

$${}^\rho\overline{\mathcal{M}}_{0,n} \xrightarrow{\sim} \mathbb{P}^{n-3}$$

over  $F$ , proving rationality. For the second assertion, the Kapranov construction yields

$${}^\rho\overline{\mathcal{C}}_{0,2m+1} \xrightarrow{\sim} \mathbb{P}^{2m-1};$$

moreover

$${}^\rho\overline{\mathcal{C}}_{0,2m+1} \rightarrow {}^\rho\overline{\mathcal{M}}_{0,2m+1}$$

is a  $\mathbb{P}^1$ -bundle over a Zariski open subspace of the base. (The generic fiber is a smooth genus zero curve with a cycle of odd degree.) In particular,  $\mathbb{P}^1 \times {}^\rho \overline{\mathcal{M}}_{0,2m+1}$  is rational over  $F$ .  $\square$

**Example 6.** Let  $\mathfrak{S}_n$  act on  $\overline{\mathcal{M}}_{0,n}$ , for  $n \geq 5$ . This action is not linearizable since  $\mathfrak{S}_n$  does not act linearly and generically freely on  $\mathbb{P}^{n-3}$ . Indeed, the smallest faithful representation of  $\mathfrak{S}_n$  has dimension  $n-1$ . When  $n = p$  is a prime, then even the action of the Frobenius subgroup  $\mathfrak{F}_p = \text{Aff}_1(\mathbb{F}_p) \subset \mathfrak{S}_p$  is not linearizable, for the same reason.

**The Losev-Manin construction.** This construction [LM00], [Has03, Section 6.4] is a distinguished factorization

$$\beta_n : \overline{\mathcal{M}}_{0,n} \rightarrow \overline{L}_n \rightarrow \mathbb{P}^{n-3},$$

where we blow up linear subspaces spanned by just  $(n-2)$  points in linear general position. (Note that our indexing of  $\overline{L}_n$  differs from [LM00].) The first arrow contracts the boundary divisors

$$D_{i_1, \dots, i_k, (n-1), n}, \{i_1, \dots, i_k\} \subset \{1, \dots, n-2\}, \quad k \leq n-5,$$

by allowing points indexed by

$$\{1, \dots, n-2\} \setminus \{i_1, \dots, i_k\}$$

to coincide.

We record some properties:

- $\overline{L}_n$  is toric [LM00, Section 2.6];
- the Losev-Manin construction is equivariant under  $\mathfrak{S}_{n-2} \times \mathfrak{S}_2 \subset \mathfrak{S}_n$ , realized as permutations of  $\{1, \dots, n-2\}$  and  $\{n-1, n\}$  [LM00, Theorem 2.5(b)].

The constructions of Losev-Manin give an explicit realization of the torus  $\mathbb{T}$  and its character module  $\mathfrak{X}^*(\mathbb{T})$ . Let  $P$  denote the permutation module for  $\mathfrak{S}_{n-2}$  associated with the first  $n-2$  letters and  $L$  the non-trivial rank-one module for  $\mathfrak{S}_2$  corresponding to  $n-1$  and  $n$ . We regard these as modules for  $\mathfrak{S}_{n-2} \times \mathfrak{S}_2$ . Consider the short exact sequence

$$0 \rightarrow P_0 \rightarrow P \rightarrow \mathbb{Z} \rightarrow 0$$

associated with summing over the  $n-2$  letters. Then we have

$$(2.1) \quad \mathfrak{X}^*(\mathbb{T}) = L \otimes P_0.$$

Indeed, we may describe the open torus orbit in  $\overline{L}_n$  in geometric terms: We identify the points  $n-1$  and  $n$  as 0 and  $\infty$  and the first  $n-2$  points as elements of

$$\text{Hom}(P, \mathbb{P}^1 \setminus \{0, \infty\}) = \text{Hom}(P, \mathbb{T}_L),$$

where  $\mathbb{T}_L$  is the rank-one torus associated with  $L$ . To get moduli, we quotient out by the diagonal action of  $\mathbb{T}_L$ .

We record one last observation: Consider the Kapranov blowups associated with points  $n-1$  and  $n$ :

$$\beta_n[n-1], \beta_n[n] : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}.$$

These two maps are related by an elementary Cremona transformation

$$\text{Cr} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$$

associated with the points indexed by  $\{1, \dots, n-2\}$ . This is equivariant for the  $\mathbb{T}$ -actions and we obtain a birational contraction

$$\overline{L}_n \rightarrow \text{Graph}(\text{Cr}).$$

We summarize this as follows:

**Proposition 7.** *Consider a twist of  $\overline{\mathcal{M}}_{0,n}$  associated with a subgroup of  $\mathfrak{S}_n$  leaving a pair of points invariant. This variety is necessarily toric, realized as a twist of the Losev-Manin space.*

This applies in both equivariant and Galois-theoretic situations.

**The Gelfand-MacPherson correspondence.** Our main source is Kapranov [Kap93].

Let  $\text{Mat}(2, n)$  denote the  $2 \times n$  matrices. The group  $\text{GL}_2$  acts via multiplication from the left

$$A \cdot M \mapsto AM$$

and the torus  $\mathbb{T} = \mathbb{G}_m^n$  acts via multiplication from the right

$$M \cdot \mathbb{T} \mapsto M\mathbb{T}, \quad \mathbb{T} = \text{diag}(t_1, \dots, t_n).$$

Considering the action by the product  $\text{GL}_2 \times \mathbb{G}_m^n$ , with the elements

$$(t^{-1} \text{I}_2, \text{diag}(t, t, \dots, t))$$

in the kernel, we obtain a faithful action of the quotient group

$$(\text{GL}_2 \times \mathbb{G}_m^n) / \mathbb{G}_m.$$

We have an exact sequence

$$1 \rightarrow \mu_2 \rightarrow \text{SL}_2 \times \mathbb{G}_m^n \rightarrow (\text{GL}_2 \times \mathbb{G}_m^n) / \mathbb{G}_m \rightarrow 1,$$

where

$$\mu_2 = (-\text{I}_2, \text{diag}(-1, -1, \dots, -1)).$$

The invariant theory quotient is

$$\text{SL}_2 \backslash \text{Mat}(2, n) = \text{CGr}(2, n),$$

the cone over the Grassmannian  $\mathrm{Gr}(2, n)$  in its Plücker imbedding. The residual action of  $\mathbb{G}_m^n$  on this cone has generic stabilizer  $\mu_2$ ; the action on the Grassmannian has generic stabilizer  $\mathbb{G}_m = \mathrm{diag}(t, t, \dots, t)$ . On the other hand, the geometric invariant theory quotient

$$\mathrm{Mat}(2, n) // \mathbb{G}_m, \quad \mathbb{G}_m = \mathrm{diag}(t, t, \dots, t)$$

yields  $(\mathbb{P}^1)^n$  with factors induced by the columns of the matrix. The residual  $\mathrm{SL}_2$  acts on this product with the distinguished linearization introduced above, which is  $\mathfrak{S}_n$ -symmetric. Again, this action fails to be faithful, as  $\mu_2 \subset \mathrm{SL}_2$  acts trivially.

The Gelfand-MacPherson construction yields isomorphisms

$$(2.2) \quad (\mathrm{CGr}(2, n) \setminus \{0\}) / \mathbb{G}_m^n \xrightarrow{\sim} \mathrm{SL}_2 \backslash (\mathbb{P}^1)^n,$$

where both sides are interpreted as GIT quotients [Kap93, 2.4.7]. Note that we have numerous choices for how to linearize the actions on the left- and right-hand sides, reflecting linearizations of the torus action and ample line bundles on the product; Kapranov's result makes clear how to identify these choices. Let  $X_n$  denote the quotient arising from the  $\mathfrak{S}_n$ -symmetric linearization.

Recall that the stable and strictly semistable loci on  $(\mathbb{P}^1)^n$  are easily identified

$$(2.3) \quad (p_1, \dots, p_n) \text{ stable if there is no point with multiplicity } \geq \frac{n}{2}.$$

It is semistable if all points have multiplicity  $\leq \frac{n}{2}$ . For odd  $n$ , stable and semistable coincide; for even  $n = 2m$ , collections of points where  $m$  indices coincide are strictly semistable, with closed orbits consisting of collections where

$$p_{i_1} = \dots = p_{i_m}, \quad p_{i_{m+1}} = \dots = p_{i_{2m}}, \quad \{i_1, \dots, i_{2m}\} = \{1, \dots, 2m\}.$$

In particular,  $X_{2m}, m \geq 3$  has  $\frac{1}{2} \binom{2m}{m}$  distinguished singular points over which the orbits are identified.

The stable loci on the Grassmannian  $\mathrm{Gr}(2, n)$  for the action of  $\mathbb{G}_m^n \cap \mathrm{SL}_n$  may be described as well: Choose a basis diagonalizing the torus action and let  $(A_{ij}), 1 \leq i < j \leq n$  denote the associated Plücker coordinates. The point  $(A_{ij})$  is stable if there are

- (1) no index  $i$  with  $A_{ij} = 0$  for every  $j$ ; and
- (2) no subset  $I \subset \{1, \dots, n\}$  with  $|I| \geq \frac{n}{2}$  and  $A_{ij} = 0$  for all  $i, j \in I$ .

These descriptions yield an  $\mathfrak{S}_n$ -equivariant stratified blowup [Kap93, 0.4.3, 4.1.8]

$$\beta : \overline{\mathcal{M}}_{0,n} \rightarrow X_n.$$

This blows down all the boundary divisors  $D_I$  except those where  $|I|$  or  $|I^c| = 2$ . The divisors  $D_I$  with  $2|I| = n$  are collapsed to the distinguished singular points  $\Sigma \subset X_{2m}$  where  $m = |I|$  and  $n = 2m$ .

The Gelfand-MacPherson construction is a powerful tool for computing class groups. The induced homomorphism

$$(2.4) \quad \beta_* : \text{Pic}(\overline{\mathcal{M}}_{0,n}) = \text{Cl}(\overline{\mathcal{M}}_{0,n}) \rightarrow \text{Cl}(X_n)$$

is surjective because  $\beta$  is a fibration away from the distinguished singular points. Thus we get an exact sequence

$$(2.5) \quad 0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0,$$

where

$$N = \ker(\beta_*), \quad M = \text{Pic}(\overline{\mathcal{M}}_{0,n}).$$

In particular,  $N$  is generated by the  $D_I$  where  $|I|, |I^c| \neq 2$ . We can easily compute  $Q$  is well. Write

$$\mathfrak{X}^*(\mathbb{G}_m^n) = \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n,$$

so the quotient acting faithfully on the  $\text{CGr}(2, n)$  has characters

$$\left\{ \sum a_i g_i : a_i \in \mathbb{Z}, \sum a_i \equiv 0 \pmod{2} \right\}.$$

These give rise to line bundles on  $X_n \setminus \Sigma$  and divisor classes on the full space. Thus we deduce that

$$Q \subset \mathbb{Z}[\mathfrak{S}_n / \mathfrak{S}_{n-1}]$$

as an index-two subgroup. Note that the element  $g_{i_1} + g_{i_2}, i_1 \neq i_2$  corresponds to the boundary divisor  $D_{i_1 i_2}$ ; indeed, this locus is cut out by the  $2 \times 2$  determinant on  $\mathbb{P}_{i_1}^1 \times \mathbb{P}_{i_2}^1$ . Since  $Q$  is an index-two subgroup of a permutation module, we have

$$(2.6) \quad H^1(G, Q) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad H^1(G, M) = 0 \text{ or } \mathbb{Z}/2\mathbb{Z}.$$

When  $n$  is odd, i.e.,  $n = 2m + 1$ , then  $X_{2m+1}$  is nonsingular,

$$\text{Pic}(X_{2m+1}) = \text{Cl}(X_{2m+1}),$$

and  $\beta$  is the iteration of a sequence of blowups along smooth disjoint centers. Precisely, we blow up the strata where  $m$  points coincide, then where  $m - 1$  points coincide, etc. (see [Has03, §8]); this is naturally

equivariant under the  $\mathfrak{S}_{2m+1}$ -action. By the blowup formula [Ful98, Prop. 6.7], we have

$$\mathrm{Pic}(\overline{\mathcal{M}}_{0,2m+1}) = \mathrm{Pic}(X_{2m+1}) \oplus \{\text{free group on the exceptional divisors}\}.$$

We summarize this in algebraic terms:

**Proposition 8.** *For odd  $n = 2m + 1$ , the exact sequence (2.5) splits  $\mathfrak{S}_{2m+1}$ -equivariantly:*

$$M \simeq N \oplus Q.$$

On the other hand, for  $n$  even, e.g.,  $n = 6$ , there are examples of  $G \subset \mathfrak{S}_n$  such that the sequence does not split equivariantly, since in those cases  $H^1(G, Q) \neq 0$  while  $H^1(G, M) = 0$  (see Example 27).

We return to the isomorphism (2.2) over nonclosed fields. Up to this point, we have been working with schemes but this is compatible with the  $\mu_2$ -gerbe structure over the dense open subset where this is the full stabilizer. When  $n = 2m$  the stabilizers may be larger, e.g., where the sequence in  $(\mathbb{P}^1)^{2m}$  consists of  $m$  copies of a pair of points conjugate over a quadratic extension. In the cone over the Grassmannian,  $2\binom{m}{2} = m^2 - m$  coordinates vanish and the  $m^2$  remaining coordinates are equal to the determinant of the conjugate pair.

We can apply the same analysis to nonsplit actions. This includes working over nonclosed fields, where the  $n$  points are a Galois orbit, or in the equivariant context, where the  $n$  points are invariant under the action of a finite group. In the former situation, over a ground field  $F$  of characteristic zero, let  $E/F$  be an étale algebra of degree  $n$  classified by a representation of the Galois group  $\Gamma_F \rightarrow \mathfrak{S}_n$ . We replace the group  $(\mathrm{GL}_2 \times \mathbb{G}_m^n)/\mathbb{G}_m$  with  $(\mathrm{GL}_2 \times R_{E/F}\mathbb{G}_m)/\mathbb{G}_m$  and  $(\mathbb{P}^1)^n$  with  $R_{E/F}\mathbb{P}^1$  (see [FR18, §4]). Note however that twisting  $\mathrm{Mat}(2, n) = \mathbb{A}^{2n}$  yields a variety isomorphic to  $\mathbb{A}^{2n}$ , albeit with an action of a nonsplit torus.

The  $\mu_2$ -gerbe has an explicit geometric interpretation along  $\mathcal{M}_{0,n}$ : It is encoded by the universal family

$$\phi : \mathcal{C}_{0,n} \rightarrow \mathcal{M}_{0,n},$$

a conic fibration, in general.

### 3. RATIONALITY CONSTRUCTIONS

In this section, we work over an arbitrary field  $F$ , and we let  $\Gamma$  be the absolute Galois group of  $F$ .

**Schubert calculus background.** Our reference is [Kly85].

Consider the Grassmannian  $\text{Gr} = \text{Gr}(p, p+q)$  of  $p$ -dimensional subspaces of a vector space of dimension  $p+q$ . The maximal torus  $\mathbf{T} = \mathbb{G}_m^{p+q}$  acts diagonally on the vector space. Let  $X$  be a generic orbit in  $\text{Gr}$ .

We set combinatorial notation: Consider shuffles of  $\{1, \dots, p+q\}$

$$I = \{i_1 < \dots < i_p\}, \quad J = \{j_1 < \dots < j_q\}.$$

For each such shuffle, record the pairs  $(k, \ell)$ ,  $k = 1, \dots, p$ ,  $\ell = 1, \dots, q$ , such that  $i_k > j_\ell$ . Write

$$\lambda_{p+1-k} = \#\{\ell : j_\ell < i_k\}$$

and note that

$$q \geq \lambda_1 \geq \dots \geq \lambda_p.$$

Write  $\lambda = (\lambda_1, \dots, \lambda_p)$  and use the same notation for the associated Young diagram, which fits into a  $p \times q$  rectangle. The *height*  $\text{ht}(\lambda)$  is the number of indices  $i$  with  $\lambda_i > 0$ . Set  $|\lambda| = \lambda_1 + \dots + \lambda_p$  and let  $\sigma_\lambda$  denote the associated Schubert cycle on  $\text{Gr}$ , a class in  $H^{2|\lambda|}(\text{Gr}, \mathbb{Z})$ .

We recall dimension formulae for representations. Let  $V$  be an  $n$ -dimensional vector space and  $\lambda = (\lambda_1, \dots, \lambda_n)$  a partition of  $|\lambda|$  as above; in particular,  $n \geq \text{ht}(\lambda)$ . The Schur functor  $\mathbb{S}_\lambda(V)$  is a representation of  $\text{SL}(V)$  with dimension [FH91, Theorem 6.3, Exercise 6.4]:

$$\begin{aligned} d_n(\lambda) &:= \dim \mathbb{S}_\lambda(V) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \\ &= \prod_{(a,b)} \frac{n - a + b}{h_{ab}}, \end{aligned}$$

where  $a = 1, \dots, n$  labels the rows of  $\lambda$  (from top to bottom),  $b$  labels the columns (from left to right), and  $h_{ab}$  labels the “hook length”. This is defined as the number of boxes immediately below and to the right of a given box, including the box. For  $n < \text{ht}(\lambda)$  we set  $d_n(\lambda) = 0$ .

For example, when  $\lambda = (\lambda_1, \lambda_2, 0, \dots)$  and  $n \geq 2$ ,

$$\begin{aligned} d_n(\lambda_1, \lambda_2) &= \frac{(n-1+1) \cdots (n-1+\lambda_1)}{1 \cdots (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2 + 2) \cdots (\lambda_1 + 1)} \frac{(n-2+1) \cdots (n-2+\lambda_2)}{1 \cdots \lambda_2} \\ &= \binom{n-1+\lambda_1}{\lambda_1} \binom{n-2+\lambda_2}{\lambda_2} \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1}. \end{aligned}$$

For instance,

$$d_n(2, 1) = \frac{(n+1)n(n-1)}{3}, \quad n \geq 1.$$

Another combinatorial quantity is

$$m_k(\lambda) := \sum_{i=0}^k (-1)^i \binom{|\lambda|+1}{i} d_{k-i}(\lambda).$$

If  $\lambda$  has height  $k$  then  $m_k(\lambda) = d_k(\lambda)$ , as the terms in the sum with  $i > 0$  are zero.

We record a fact that we will use repeatedly in examples:

**Proposition 9.** *Fix an integer  $d \geq 0$ . If  $f(x)$  is a polynomial of degree  $\leq d$  then the  $(d+1)$ th iterated difference*

$$\sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} f(x-i) = 0.$$

When  $\lambda = (\lambda_1, \lambda_2, 0, \dots)$  we have:

$$\begin{aligned} m_k(\lambda_1, \lambda_2) &= \\ \sum_{i=0}^k (-1)^i \binom{\lambda_1 + \lambda_2 + 1}{i} \binom{k-i-1+\lambda_1}{\lambda_1} \binom{k-i-2+\lambda_2}{\lambda_2} \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1}. \end{aligned}$$

For instance, when  $\lambda_1 = 2$  and  $\lambda_2 = 1$  we have

$$\begin{aligned} m_k(2, 1) &= \sum_{i=0}^k (-1)^i \binom{4}{i} \frac{(k-i+1)(k-i)(k-i-1)}{3} \\ &= 2 \left( \binom{k+1}{3} - 4 \binom{k}{3} + 6 \binom{k-1}{3} - 4 \binom{k-2}{3} + \binom{k-3}{3} \right) \\ &= \begin{cases} 2 & \text{if } k = 2, \\ 0 & \text{if } k \geq 3. \end{cases} \end{aligned}$$

For general  $\lambda_1$  and  $\lambda_2$ ,

$$m_2(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2 + 1$$

and

$$m_3(\lambda_1, \lambda_2) = \frac{\lambda_1(\lambda_2 - 1)(\lambda_1 - \lambda_2 + 1)}{2}.$$

**Theorem 10.** [Kly85, Theorem 5] *If  $X$  is the generic torus orbit in  $\text{Gr} = \text{Gr}(p, p+q)$  and  $\lambda$  is a partition with  $|\lambda| = p+q-1$  then*

$$[X] \cdot \sigma_\lambda = m_p(\lambda).$$

For example, take  $p = 2$ . For  $q = 2$

$$[X] \cdot \sigma_{21} = 2$$

and when  $q = 3$  we have

$$[X] \cdot \sigma_{22} = 1, \quad [X] \cdot \sigma_{31} = 3.$$

For general  $q$ , we have  $\lambda_1 \geq \lambda_2 = q+1-\lambda_1 \geq 0$ , i.e.,

$$\frac{q+1}{2} \leq \lambda_1 \leq q+1.$$

Here we have

$$[X] \cdot \sigma_{\lambda_1 q+1-\lambda_1} = 2\lambda_1 - q;$$

in particular, when  $q = 2m-1$  and  $\lambda_1 = m$  we find

$$[X] \cdot \sigma_{mm} = 1.$$

**Remark 11.** The signs in the formula for  $m_k(\lambda)$  obscure the positivity of the result. An alternate formula [BF17, Theorem 5.1] makes this clearer:

$$[X] = \sum_{\lambda \subset (q-1)^{p-1}} \sigma_\lambda \sigma_{\tilde{\lambda}},$$

where  $\tilde{\lambda}$  is the complement to  $\lambda$  in the rectangle  $(q-1)^{p-1}$ :

$$\lambda = (\lambda_1, \dots, \lambda_{p-1}), \quad \tilde{\lambda} = (q-1-\lambda_{p-1}, \dots, q-1-\lambda_1).$$

We refer the reader to [Lia24] for the combinatorics directly relating these formulas.

This extends to general  $p \in \mathbb{N}$ :

**Proposition 12.** *Let  $V$  be a vector space with  $\dim(V) = mp+1$  so that*

$$q = (m-1)p+1 \quad \text{and} \quad (p-1)(q-1) = (m-1)(p-1)p.$$

*Consider the coefficient of*

$$\underbrace{\sigma_{(m-1)(p-1) \dots (m-1)(p-1)}}_{p \text{ times}}$$

*in the expansion of  $[X]$  in  $H^{2(p-1)(q-1)}(\text{Gr}(p, p+q))$ . This equals 1, i.e.,*

$$[X] \cdot \underbrace{\sigma_m \dots m}_{p \text{ times}} = 1.$$

Indeed, this follows from Klyachko's formula (Theorem 10) and

$$m_p(\underbrace{m, \dots, m}_{p \text{ times}}) = d_p(\underbrace{m, \dots, m}_{p \text{ times}}) = 1.$$

**Example 13.** When  $\dim(V) = 3m + 1$  the generic orbit  $X$  for the action of  $T$  on  $\mathrm{Gr}(3, V)$  has codimension  $3(3m - 2) - 3m = 6(m - 1)$  and

$$[X] \cdot \sigma_{m m m} = m_3(m, m, m) = d_3(m, m, m) = 1.$$

This is not the case when  $\dim(V) = 3m + 2, m > 1$ , e.g., for  $m = 2$

$$[X] = 10\sigma_{5,3} + 8\sigma_{5,2,1} + 15\sigma_{4,4} + 15\sigma_{4,3,1} + 6\sigma_{4,2,2} + 3\sigma_{3,3,2}.$$

### Grassmann geometry and rationality.

**Theorem 14.** *Let  $\mathbb{T}$  be a maximal torus – possibly nonsplit – for  $\mathrm{SL}_{pm+1}$  over a field  $F$ . Take  $\mathrm{Gr}(p, V)$  for  $\dim_F(V) = pm + 1$  with the resulting  $\mathbb{T}$ -action. Choose a subspace  $W \subset V$  with*

$$\dim_F(W) = (p - 1)m + 1$$

*and transverse to  $\mathbb{T}$  in the sense that  $\mathrm{Gr}(p, W) \subset \mathrm{Gr}(p, V)$  meets some stable  $\mathbb{T}$ -orbit properly. Then  $\mathrm{Gr}(p, W)$  is a rational section of the quotient*

$$\mathrm{Gr}(p, V) \dashrightarrow \mathrm{Gr}(p, V)/\mathbb{T}.$$

*Since  $\mathrm{Gr}(p, W)$  is rational the same holds true of the quotient.*

Florence [Flo13, §3] has obtained similar results when  $V$  carries a suitable  $F$ -algebra structure. An analog of Theorem 14 holds in the equivariant case, where  $\mathbb{T}$  is stable under the action of a finite group: If  $\mathrm{Gr}(p, W)$  is linearizable or stably linearizable then  $\mathrm{Gr}(p, V)/\mathbb{T}$  is as well.

*Proof.* The stability assumption guarantees that the quotient map is defined over a non-empty open subset of  $\mathrm{Gr}(p, W)$ . Properness of the intersection – which has degree one by Proposition 12 – implies  $\mathrm{Gr}(p, W)$  is mapped birationally to the quotient.  $\square$

**Proposition 15.** *Retain the notation of Theorem 14.*

*If  $F$  is infinite then  $\mathrm{Gr}(p, V)$  admits a codimension- $m$  subspace  $W \subset V$  satisfying the transversality condition.*

*If  $F$  is finite and  $p = 2$  then  $\mathrm{Gr}(2, V)$  admits a stable  $F$ -rational point.*

*If  $F$  is arbitrary and  $p = 2$  then for each stable point there exists a subspace  $W$  satisfying the transversality assumption.*

Combining with Theorem 14 gives a generalization of the results of [FR18]:

**Corollary 16.** *Let  $F$  be a finite field and  $\rho$  a representation of its Galois group in  $\mathfrak{S}_{2m+1}$ . Then  ${}^{\rho}\overline{\mathcal{M}}_{0,2m+1}$  is rational over  $F$ .*

We also obtain analogs in higher dimensions:

**Corollary 17.** *Let  $m \geq 1$  and  $p \geq 2$  be integers. Consider the moduli space of  $pm + 1$  points in  $\mathbb{P}^{p-1}$  up to projective equivalence. Let  $X$  be a variety obtained by twisting via permutations of the points, over an infinite field  $F$ . Then  $X$  is rational.*

*Proof of Proposition 15.* Assume  $F$  is infinite; here we use [Kap93, §1.2]. While Kapranov assumes the ground field has characteristic zero, the toric constructions and interpretation of  $\overline{\mathcal{M}}_{0,n}$  as a Chow quotient for the  $\mathrm{PGL}_2$ -action are valid in positive characteristic [GG14].

The Grassmannian is rational over  $F$  so its  $F$ -rational points are Zariski dense. We note that the torus action determines a collection of  $\overline{F}$ -subspaces

$$V_I \subset V, \quad \emptyset \neq I = \{i_1, \dots, i_r\} \subset \{0, \dots, mp\},$$

spanned by eigenvectors of the torus. Consider the

$$W \in \mathrm{Gr}(mp + 1 - m, mp + 1)$$

meeting some of these improperly, i.e.,

$$\dim(W \cap V_I) > \dim(W) + \dim(V_I) - \dim(V).$$

This is a Zariski closed proper subset of the Grassmannian, defined over  $F$ ; its complement has  $F$ -rational points. Given such a subspace  $W \subset V$ , choose

$$w \in \Lambda \subset W, \quad \dim(\Lambda) = p,$$

defined over  $F$ , with  $w$  not contained in any of the  $V_I \subsetneq V$  and  $\Lambda$  meeting all the  $V_I$  properly. Thus  $\Lambda$  is stable for the torus action and the torus orbit of  $\Lambda$  meets  $\mathrm{Gr}(p, W)$  transversally there.

Now assume that  $F$  is finite and  $p = 2$ . We use the stability criterion (2.3) for points on  $\mathbb{P}^1$  and Kapranov's analysis of the Gelfand-MacPherson correspondence. Here the Galois action  $\rho$  on the  $2m + 1$  points is encoded by a single element  $\sigma \in \mathfrak{S}_{2m+1}$ . Express  $\sigma$  as a product of  $r$  disjoint cycles of lengths  $\ell_i$  with

$$\ell_1 + \dots + \ell_r = 2m + 1, \quad \ell_1 \geq \ell_2 \geq \dots \geq \ell_r.$$

Only  $\ell_1$  can possibly be greater than  $m$ ; if  $\ell_1 \leq m$  then we have  $r \geq 3$ . When  $\ell_1 > m$ , choose a configuration of  $\ell_1$  points defined over a degree- $\ell_1$  extension of  $F$ . Allow the remaining points to all coincide. We turn to the situation where  $\ell_1 \leq m$ . If  $r = 3$  then we allow  $\ell_1$  points to coincide with  $[0, 1]$ ,  $\ell_2$  points to coincide with  $[1, 0]$ , and  $\ell_3$  points to coincide with  $[1, 1]$ . We may therefore assume that  $r \geq 4$  and work inductively on  $r$ . There exists two indices, say  $\ell_3$  and  $\ell_4$ , whose sum is less than  $m$ . Use this to “degenerate” to a new partition of  $2m + 1$ , refined by  $(\ell_1, \dots, \ell_r)$  but of length  $r - 1$ , all of whose entries are less than  $m$ . For example, we could take

$$(\ell_1, \ell_2, \ell_3 + \ell_4, \ell_5, \dots, \ell_r).$$

Continuing in this way, we generate a partition

$$\{1, 2, \dots, r\} = A \sqcup B \sqcup C$$

such that

$$\sum_{a \in A} \ell_a, \sum_{b \in B} \ell_b, \sum_{c \in C} \ell_c \leq m.$$

Let points coincide in three groups according to this coarsening of our original partition, the first group to  $[0, 1]$ , the second to  $[1, 0]$ , and the third to  $[1, 1]$ .

Assume  $p = 2$  and  $F$  is arbitrary. We continue to assume that  $\Lambda \subset V$  is a two-dimensional subspace that is stable in the sense of Geometric Invariant Theory. Let  $\mathbf{T}_{2m}$  denote the tangent space to the torus orbit at  $\Lambda$

$$\mathbf{T}_{2m} \subset \text{Hom}(\Lambda, V/\Lambda),$$

an  $2m$ -dimensional subspace of the tangent space to  $\text{Gr}(2, V)$  at  $\Lambda$ . We claim there exists a subspace

$$\Lambda \subset W \subset V,$$

where  $W$  has codimension  $m$  in  $V$ , such that the composition

$$\mathbf{T}_{2m} \subset \text{Hom}(\Lambda, V/\Lambda) \twoheadrightarrow \text{Hom}(\Lambda, V/W)$$

has full rank  $2m$ . Since the latter space is the normal directions to  $\text{Gr}(2, W)$  at  $\Lambda$ , this will yield transversality.

We record some basic geometry:

**Lemma 18.** *There is a distinguished orbit*

$$\mathbb{P}^1 \times \mathbb{P}^{m-2} \simeq \mathbb{P}(\Lambda^*) \times \mathbb{P}(V/\Lambda) \subset \mathbb{P}(\text{Hom}(\Lambda, V/\Lambda))$$

*invariant under automorphisms of  $\text{Gr}(2, V)$  fixing  $[\Lambda]$ .*

The subspace  $\mathbb{P}^{2m-1} \simeq \mathbb{P}(\mathbf{T}_{2m})$  cuts out the graph of a rational normal curve

$$\begin{aligned} \varrho : \mathbb{P}_{s_0, s_1}^1 &\hookrightarrow \mathbb{P}_{x_0, \dots, x_{2m-2}}^{2m-2} \\ [s_0, s_1] &\mapsto [s_0^{2m-2}, \dots, s_1^{2m-2}]. \end{aligned}$$

In these coordinates, the rational normal curve has equations

$$s_0 x_{i+1} = s_1 x_i, \quad i = 0, \dots, 2m-1.$$

Let  $\Gamma \subset \mathbb{P}^1$  denote the length- $(2m+1)$  subscheme that is the image of the eigenvectors for  $\mathbf{T}_{2m}$  under  $V^* \rightarrow \Lambda^*$ . Then  $\varrho$  realizes the Gale transform for  $\Gamma \subset \mathbb{P}^1$  as a subscheme of  $\mathbb{P}^{2m-2}$  contained in a rational normal curve.

The first assertion reflects the fact that the parabolic subgroup of  $\mathrm{PGL}_{2m+1}$  fixing  $[\Lambda]$  has semisimple part  $(\mathrm{GL}_2 \times \mathrm{GL}_{2m-1})/\mathbb{G}_m$ . Note that the unipotent part acts trivially on the tangent space. The second assertion is true for the generic codimension- $(2m-2)$  linear slice of  $\mathbb{P}^1 \times \mathbb{P}^{2m-1}$ . Of course, one has to show that this applies in our situation! This follows from the third assertion, a special case of [EP00, Corollary 3.2] – the first application following the statement. This completes the proof of the lemma.

Returning to the proof of the Proposition, we may take  $W$  as the subspace given by

$$\{x_{2j} = 0, j = 0, \dots, m-1\},$$

where we interpret  $x_j \in (V/\Lambda)^*a$ . It is clear that the products

$$\{s_i x_{2j}, i = 0, 1, j = 0, \dots, m-1\}$$

have the desired spanning property; the elements

$$s_0^{2m-1}, \dots, s_1^{2m-1}$$

are a basis for bilinear forms of degree  $2m-1$ . □

**Partitioning the points.** We start with a general construction: Let  $n \geq 3$  be an integer and  $n = \ell m$  a factorization in integers  $\ell, m > 1$ . Suppose that  $H \subset \mathfrak{S}_\ell, A \subset \mathfrak{S}_m$  are subgroups. The *wreath product*

$$A \wr H = A \wr_{1, \dots, \ell} H$$

is the semidirect product  $A^\ell \rtimes H$  where

$$(a_1, \dots, a_\ell) \cdot h = (a_{h^{-1}(1)}, \dots, a_{h^{-1}(\ell)}).$$

This comes with a natural embedding

$$\rho : A \wr H \hookrightarrow \mathfrak{S}_{\ell m}$$

as permutations of pairs

$$(i, j), \quad i \in \{1, \dots, m\}, j \in \{1, \dots, \ell\}.$$

Now assume that  $m \geq 3$ . Forgetting maps yield an equivariant morphism

$$\phi : {}^\rho \overline{\mathcal{M}}_{0, \ell m} \rightarrow \prod_H {}^\alpha \overline{\mathcal{M}}_{0, m},$$

where  $\alpha : A \hookrightarrow \mathfrak{S}_m$  and the twisted product denotes  $\ell$  copies of the moduli space with the associated  $H$ -action. The generic fiber of this morphism is irreducible of dimension

$$(\ell m - 3) - \ell(m - 3) = 3\ell - 3.$$

It is birational to the Hilbert scheme of multidegree- $(1, \dots, 1)$  curves in the  $H$ -twisted product  $\prod_H C_j$  of  $\ell$  genus-zero curves. Geometrically, this is a compactification of the homogeneous space

$$\underbrace{\mathrm{PGL}_2 \times \cdots \times \mathrm{PGL}_2}_{\ell \text{ times}} / \mathrm{PGL}_2$$

with the last  $\mathrm{PGL}_2$  embedded diagonally.

We record some observations on the generic fiber of  $\phi$ :

- Suppose  $\ell = 2$ . Geometrically,  $(1, 1)$  curves in  $\mathbb{P}^1 \times \mathbb{P}^1$  are parametrized by  $\mathbb{P}^3$  – the dual to the projective space containing the Segre embedding of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Over an arbitrary field the fiber is a Brauer-Severi threefold.
- Suppose that  $m$  is odd. Then the genus-zero curves  $C_j$  appearing in the twisted product are split and – over the extension/subgroup associated with  $A^\ell \subset A \wr H$  – isomorphic to  $\mathbb{P}^1$ 's. Here the twisted product  $\prod_H C_j$  is rational, as it is isomorphic to the restriction of scalars of  $\mathbb{P}^1$ .
- Now assume  $\ell = 2$  and  $m$  odd. Here the generic fiber of  $\phi$  is isomorphic to  $\mathbb{P}^3$  over the function field/linearizable for the full wreath product.

**Example 19.** Suppose  $n = 6$  and consider  $G = \mathfrak{S}_3 \wr \mathfrak{S}_2 \subset \mathfrak{S}_6$ , a subgroup of index 10 preserving an unordered partition

$$\{1, 2, 3, 4, 5, 6\} = \{i, j, k\} \sqcup \{a, b, c\}.$$

Then the associated  ${}^\rho\overline{\mathcal{M}}_{0,6}$  is rational/linearizable. These actions correspond to situations where the associated Segre threefold admits an invariant node (cf. Theorem 34 below).

**Theorem 20.** *Let  $n = 2m$ , with  $m \geq 3$  odd. Fix a subgroup  $A \subset \mathfrak{S}_m$  and the diagonal subgroup*

$$G := A \times \mathfrak{S}_2 \subset A \wr \mathfrak{S}_2 \subset \mathfrak{S}_{2m}.$$

- *For each Galois representation  $\rho : \Gamma \rightarrow G$  the twist  ${}^\rho\overline{\mathcal{M}}_{0,n}$  is rational over  $F$ .*
- *The  $G$  action on  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.*

*Proof.* We assume  $\mathfrak{S}_m$  permutes the points with odd and even indices respectively.

We focus first on the arithmetic case. Let  $L/F$  be the quadratic extension associated with  $A$ . Over  $L$ , the generic point of the twisted moduli space corresponds to  $\mathbb{P}^1$  equipped with reduced and disjoint zero-cycles  $Z_{\text{odd}}, Z_{\text{even}} \subset \mathbb{P}^1$  of length  $m$ . The parity of  $m$  ensures that the underlying curve is  $\mathbb{P}^1$ .

Note that the variety  ${}^\rho\overline{\mathcal{M}}_{0,n}$  is already stably rational over  $L$  by Proposition 5.

Consider forgetting the even and odd points

$$(\pi_{\text{odd}}, \pi_{\text{even}}) : ({}^\rho\overline{\mathcal{M}}_{0,n})_L \rightarrow {}^{\varpi_{\text{odd}}}\overline{\mathcal{M}}_{0,m} \times {}^{\varpi_{\text{even}}}\overline{\mathcal{M}}_{0,m}$$

where the Galois actions come via restriction to the even and odd points. These actions are conjugate for the quadratic extension  $L/F$ . Descent therefore gives a morphism over  $F$

$$\phi : {}^\rho\overline{\mathcal{M}}_{0,n} \rightarrow R_{L/F}({}^{\varpi_{\text{odd}}}\overline{\mathcal{M}}_{0,m}),$$

where the target is the restriction of scalars. The twists of  $\overline{\mathcal{M}}_{0,m}$  are rational over  $L$  by [FR18] and Corollary 16. The restriction of scalars of a rational variety is rational.

We claim that the generic fiber of  $\phi$  is rational over the function field of the base, which implies rationality for  ${}^\rho\overline{\mathcal{M}}_{0,n}$  over  $F$ . This follows from the analysis above for  $\ell = 2$  and odd  $m$ .

For the equivariant case, our geometric argument shows that the  $G$ -variety  $\overline{\mathcal{M}}_{0,n}$  is birationally the projectivization of an equivariant vector bundle over a stably linearizable variety (by Proposition 5). Note that restriction of scalars in the arithmetic situation corresponds to passing to an induced representation in the equivariant context; thus stable linearizability is clearly preserved. We conclude then that  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.  $\square$

**Corollary 21.** *Let  $C_{2m}$ , with  $m$  odd, be a cyclic group. Then twists of  $\overline{\mathcal{M}}_{0,n}$  by this group are rational (in the Galois case) and stably linearizable (in the equivariant situation).*

*Proof.* If the action has an odd orbit then this follows from Propositions 3 and 5. Otherwise, all the orbits are even and we may apply Theorem 20.  $\square$

**Remark 22.** Similar reasoning applies for a Galois action

$$\rho : \Gamma \rightarrow \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2} \subset \mathfrak{S}_{m_1+m_2}, \quad m_1, m_2 \geq 3 \text{ odd},$$

with restricted actions  $\varpi_1$  and  $\varpi_2$  on the first  $m_1$  points and last  $m_2$  points respectively. Proposition 5 already gives stable rationality in this case. The forgetting morphism

$$\phi : {}^\rho \overline{\mathcal{M}}_{0,m_1+m_2} \rightarrow {}^{\varpi_1} \overline{\mathcal{M}}_{0,m_1} \times {}^{\varpi_2} \overline{\mathcal{M}}_{0,m_2}$$

has generic fiber birational to  $\mathbb{P}^3$  by the reasoning above. Since the factors  ${}^{\varpi_i} \overline{\mathcal{M}}_{0,m_i}$  are rational,  ${}^\rho \overline{\mathcal{M}}_{0,m_1+m_2}$  is rational as well.

#### 4. STABLE LINEARIZABILITY VIA TORSORS

Let  $G$  be a finite group and  $\mathbb{T}$  a  $G$ -torus, i.e., a torus equipped with a representation of  $G$  on its character module  $\mathfrak{X}^*(\mathbb{T})$ . Recall that  $\mathbb{T}$  is stably linearizable if  $\mathfrak{X}^*(\mathbb{T})$  is stably permutation, see, e.g., [HT23, Proposition 2].

**Proposition 23.** *Let  $U$  be a smooth quasi-projective variety with  $G$ -action. Assume that we have a  $\mathbb{T}$ -torsor*

$$\mathcal{P} \rightarrow U,$$

*i.e., a  $\mathbb{T}$ -principal homogeneous space over  $U$ , in the category of  $G$ -varieties. Assume that*

- *the  $G$ -action on  $U$  is generically free,*
- *the characters  $\mathfrak{X}^*(\mathbb{T})$  are a stably permutation  $G$ -module,*
- *the  $G$ -action on  $\mathcal{P}$  is stably linearizable.*

*Then the  $G$ -action on  $U$  is stably linearizable.*

*Proof.* We claim there is a  $G$ -equivariant birational map,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\sim} & \mathbb{T} \times U \\ & \searrow & \swarrow \\ & U & \end{array}$$

which would follow if  $\mathcal{P} \rightarrow U$  admits a  $G$ -equivariant rational section. We clearly have such a section after discarding the  $G$ -action, by Hilbert's Theorem 90.

Since  $\mathbb{T}$  is stably permutation, a product  $\mathbb{T} \times \mathbb{T}_1$ , where  $\mathbb{T}_1$  is a permutation torus, is isomorphic to a permutation torus and may be realized as a dense open subset of affine space. It follows that we have an open embedding

$$\begin{array}{ccc} \mathcal{P} \times_U \mathbb{T}_1 & \hookrightarrow & \mathcal{V} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

where  $\mathcal{V} \rightarrow U$  is a vector bundle with  $G$ -action. The vector bundle admits a rational section (by the No-Name Lemma) thus  $\mathcal{P}$  does as well.

We assumed that  $\mathcal{P}$  is stably linearizable, i.e.  $\mathcal{P} \times \mathbb{G}_m^r$  is linearizable for some  $r$ . Thus  $U \times \mathbb{T} \times \mathbb{G}_m^r$  is as well. We observed that  $\mathbb{T}$  is stably linearizable because its character module is stably permutation, i.e.,  $\mathbb{T} \times \mathbb{T}_1$  is a permutation torus. Another application of the No-Name Lemma, using the assumption that the action on  $U$  is generically free, gives that  $U$  is stably linearizable.  $\square$

We recall the exact sequence (2.5)

$$0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0,$$

with  $M = \text{Pic}(\overline{\mathcal{M}}_{0,n})$ ,  $N$  an  $\mathfrak{S}_n$ -permutation module, and  $Q$  is an index-2 submodule of the permutation module  $\mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$ . We record:

- if  $H^1(G, Q) = 0$  for some  $G \subset \mathfrak{S}_n$ , then also  $H^1(G, M) = 0$ , by the long exact sequence in cohomology,
- if  $Q$  is a stably permutation  $G$ -module, then the sequence splits and  $\text{Pic}(\overline{\mathcal{M}}_{n,0})$  is a stably permutation module, by [CTS77, Lemma 1].

**Theorem 24.** *Let  $G \subseteq \mathfrak{S}_n$  be a subgroup such that  $Q$  is a stably permutation module. Then the  $G$ -action on  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable.*

*Let  $X$  be a form of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  such that the action of the absolute Galois group on  $Q$  gives rise to a stable permutation module. Then  $X$  is stably rational over  $F$ .*

*Proof.* For the equivariant statement, we apply Proposition 23. Here  $\mathbb{T}$ , with character module  $Q$  acts on  $\text{CGr}(2, n)$  (see Section 2). Let  $V \subset \text{CGr}(2, n)$  the open subset over which  $\mathbb{T}$  acts freely and  $U \subset X_n$

the corresponding locus in the quotient, i.e., remove all the strictly semistable points. We have a torsor

$$V \xrightarrow{\mathbf{T}} U.$$

By [HT23, Proposition 19], the  $\mathfrak{S}_n$ -action on  $\mathrm{Gr}(2, n)$  (and its cone) is stably linearizable. Assuming that  $Q = \mathfrak{X}^*(\mathbf{T})$  is a stable permutation module for  $G \subset \mathfrak{S}_n$ , and applying Proposition 23, we conclude that the  $G$ -action on  $U$ , and thus  $\overline{\mathcal{M}}_{0,n}$ , is stably linearizable as well.

The Galois-theoretic result is proven analogously, with [BCTSSD85, Prop. 3] playing the role of Proposition 23. This is an application of the torsor formalism of [CTS87].  $\square$

**Remark 25.** There exist linearizable  $G$ -actions on  $\overline{\mathcal{M}}_{0,n}$  such that the induced action on  $Q$  is not stably permutation. Consider  $n$  even and  $G = C_2$  generated by  $\sigma := (1, 2) \cdots (n-1, n)$ ; we have  $H^1(C_2, Q) \neq 0$  (see Remark 32) so  $Q$  is not stably permutation. This action is equivariantly birational – by Proposition 7 – to an action on a torus  $\mathbf{T} = \mathbb{G}_m^{n-3}$ . Its character module consists of the elements of  $\mathbb{Z}^{n-2}$  – the twisted permutation module on  $\{1, \dots, n-2\}$  – whose coordinates sum to zero (see Equation 2.1). The action of  $C_2$  on the twisted permutation module consists of  $(n-2)/2$  copies of  $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ . Hence  $\mathfrak{X}^*(\mathbf{T})$  decomposes as a sum of  $\frac{n}{2} - 2$  permutation modules and one invariant, a permutation module. We conclude  $\mathbf{T}$  is linearizable.

**Remark 26.** By [FR18, Remark 5.5], for *odd*  $n$ , every form of  $\overline{\mathcal{M}}_{0,n}$  over a nonclosed field  $F$  is an  $F$ -rational variety. *A priori*, this does *not* imply that  $\overline{\mathcal{M}}_{0,n}$  is (stably) linearizable for  $\mathfrak{S}_n$ . However, this does imply that  $M$  is a stable permutation module, for the  $\mathfrak{S}_n$ -action.

For  $n$  *odd*, we have

$$(4.1) \quad M \simeq N \oplus Q,$$

as  $\mathfrak{S}_n$ -modules, by Proposition 8. Since  $N$  is a permutation module, for all  $n$ , and  $M$  a stably permutation module, for odd  $n$ , we see that  $Q$  is also stably permutation, for odd  $n$ . Thus, the  $\mathfrak{S}_n$ -action on  $\overline{\mathcal{M}}_{0,n}$  is stably linearizable, by Theorem 24.

The splitting (4.1) can also be seen explicitly: Recall that under the Kapranov basis,  $Q = M/N$  is generated by the image of the classes

$$H, \quad E_i, \quad i = 1, \dots, n-1$$

in  $M$  under the projection modulo  $N$ . The  $\mathbb{Z}$ -linear map

$$s : Q \rightarrow M,$$

given on these generators by

$$H \mapsto H + \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} (|I| - 1) \cdot E_I, \quad E_i \mapsto E_i + \sum_{\substack{I \subset \{1, \dots, n-1\}, i \in I, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} E_I.$$

is a section of the exact sequence (2.5). We check that it is  $\mathfrak{S}_n$ -equivariant. Let  $\iota = (1, 2)$  and  $\gamma = (1, \dots, n)$ . In  $Q$ , one has

$$H = D_{12} + \sum_{i=3}^{n-1} E_i$$

and  $\iota(H) = H$ ,  $\iota(E_1) = E_2$ ,  $\iota(E_2) = E_1$  and  $\iota(E_i) = E_i$ . Note that  $s$  is  $\iota$ -equivariant by construction. Next, observe

$$\begin{aligned} s\gamma(H) &= s \left( \gamma \left( D_{12} + \sum_{i=3}^{n-1} E_i \right) \right) = s \left( (n-3)H - (n-4) \sum_{i=2}^{n-1} E_i \right) \\ &= (n-3)H - (n-4) \sum_{i=2}^{n-1} E_i - \sum_{\substack{I \subset \{1, \dots, n-1\}, 1 \notin I, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} (n - |I| - 3) \cdot E_I \\ &\quad + \sum_{\substack{I \subset \{1, \dots, n-1\}, 1 \in I, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} (|I| - 1) \cdot E_I. \\ \gamma s(H) &= \gamma \left( H + \sum_{\substack{I \subset \{1, \dots, n-1\} \\ |I| = \frac{n-1}{2}, \dots, n-4.}} (|I| - 1) \cdot E_I \right) \\ (4.2) \quad &= \gamma \left( D_{n-2, n-1} + \sum_{\substack{I \subset \{1, \dots, n-3\}, \\ |I| = 1, \dots, n-4.}} E_I + \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} (|I| - 1) \cdot E_I \right). \end{aligned}$$

Note that  $\gamma(D_{n-2, n-1}) = E_{n-1}$ . We compute

$$\gamma \left( \sum_{\substack{I \subset \{1, \dots, n-3\}, \\ |I| = 1, \dots, n-4.}} E_I \right) = \sum_{\substack{I \subset \{2, \dots, n-2\}, \\ |I| = 1, \dots, n-4.}} D_{\{1\} \cup I} = \sum_{\substack{I \subset \{2, \dots, n-2\}, \\ |I| = 1, \dots, n-4.}} D_{I \cup \{n-1, n\}}$$

$$\begin{aligned}
&= \left( \sum_{\substack{I \subset \{2, \dots, n-1\}, n-1 \in I, \\ |I|=2, \dots, n-4.}} E_I \right) + \left( \sum_{\substack{I \subset \{2, \dots, n-1\}, n-1 \in I, \\ |I|=n-3.}} E_I \right) \\
&= \left( \sum_{\substack{I \subset \{2, \dots, n-1\}, n-1 \in I, \\ |I|=2, \dots, n-4.}} E_I \right) + \left( \sum_{i=2}^{n-2} \left( H - \sum_{\substack{I \subset \{2, \dots, n-1\}, i \notin I, \\ |I|=1, \dots, n-4.}} E_I \right) \right) \\
(4.3)
\end{aligned}$$

$$= (n-3)H - E_{n-1} - (n-4) \sum_{i=2}^{n-2} E_i - \sum_{\substack{I \subset \{2, \dots, n-1\}, \\ |I|=2, \dots, n-4.}} (n-|I|-3) E_I.$$

Similarly, we have

$$\begin{aligned}
&\gamma \left( \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} (|I|-1) \cdot E_I \right) \\
&= \gamma \left( \sum_{\substack{I \subset \{1, \dots, n-1\}, n-1 \in I, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} (|I|-1) \cdot E_I + \sum_{\substack{I \subset \{1, \dots, n-1\}, n-1 \notin I, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} (|I|-1) \cdot E_I \right) \\
(4.4) \\
&= \sum_{\substack{I \subset \{1, \dots, n-1\}, 1 \in I, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} (|I|-1) \cdot E_I + \sum_{\substack{I \subset \{2, \dots, n-1\}, \\ |I|=2, \dots, \frac{n-1}{2}-1.}} (n-|I|-3) \cdot E_I.
\end{aligned}$$

Substituting (4.3) and (4.4) into (4.2), we find that  $\gamma s(H) = s\gamma(H)$ .

To check the actions on  $E_i$ , for  $i = 1, \dots, n-2$ , one computes

$$\begin{aligned}
s\gamma(E_i) &= s(H - \sum_{k=2, k \neq i+1}^{n-1} E_k) \\
&= H - \sum_{k=2, k \neq i+1}^{n-1} E_k - \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ 1, i+1 \notin I, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} E_I + \sum_{\substack{I \subset \{1, \dots, n-1\}, \\ 1, i+1 \in I, \\ |I|=\frac{n-1}{2}, \dots, n-4.}} E_I.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma s(E_i) &= \gamma(E_i + \sum_{\substack{I \subset \{1, \dots, n-1\}, i \in I, \\ |I| = \frac{n-1}{2}, \dots, n-4.}} E_I) \\
&= H - \sum_{\substack{I \subset \{2, \dots, n-1\}, i+1 \notin I, \\ |I| = 1, \dots, n-4.}} E_I + \sum_{\substack{I \subset \{2, \dots, n-1\}, i+1 \in I, \\ |I| = \frac{n-1}{2} - 1, \dots, n-5.}} E_{\{1\} \cup I} \\
&\quad + \sum_{\substack{I \subset \{2, \dots, n-1\}, i+1 \notin I, \\ |I| = 2, \dots, \frac{n-3}{2}.}} E_I.
\end{aligned}$$

Then one sees that  $\gamma s(E_i) = s\gamma(E_i)$  for  $i \neq n-1$ . Finally, one can verify that

$$s(\gamma(E_{n-1})) = s(E_1) = \gamma(s(E_{n-1})).$$

## 5. COMPUTING COHOMOLOGY

In this section, we study the  $G$ -module

$$M = \text{Pic}(\overline{\mathcal{M}}_{0,n}),$$

and the quotient  $Q = M/N$ , from (2.5), for various  $G \subset \mathfrak{S}_n$ .

**Cohomological criteria.** We focus on two properties, which are necessary for linearizability of a regular  $G$ -action on a smooth projective rational variety  $X$ , see, e.g., [BP13, Proposition 2.5]:

**(H1)** For all subgroups  $G' \subset G$  one has

$$H^1(G', \text{Pic}(X)) = H^1(G', \text{Pic}(X)^*) = 0.$$

**(SP)** The  $G$ -module  $\text{Pic}(X)$  is stably permutation.

Since  $H^1$  vanishes on permutation modules, **(SP)** implies **(H1)**, but the converse does not hold, in general. Computationally, it is easier to check **(H1)**.

**Example 27.** For  $n = 6$  and  $G \subseteq \mathfrak{S}_6$ , property **(H1)** for the action on  $M = \text{Pic}(\overline{\mathcal{M}}_{0,6})$  does not imply **(SP)**, e.g., for the action of

$$G \simeq C_2 \times C_4 := \langle (3, 4), (1, 2, 5, 6) \rangle,$$

and

$$G \simeq (C_2)^3 := \langle (1, 5)(2, 6), (3, 4), (1, 2)(5, 6) \rangle,$$

see the analysis in [CTZ23, Section 6], as well as [Kun87, Section 4]. Furthermore, there are  $G \subset \mathfrak{S}_6$  such that

- $Q$  fails **(H1)** but  $M$  satisfies it, e.g., for  $G = \langle (1, 2)(3, 4)(5, 6) \rangle$ , one has

$$H^1(G, M) = 0, \quad H^1(G, Q) = \mathbb{Z}/2.$$

Actually,  $M$  is a permutation module while  $Q$  is not. Indeed, under appropriate choices of basis,  $M$  is of the form

$$\mathbb{Z}^4 \oplus \mathbb{Z}[C_2]^6,$$

and  $Q$  is of the form

$$\mathbb{Z} \oplus \mathbb{Z}[C_2]^2 \oplus \mathbb{Z}[e],$$

where  $G$  acts on  $e$  via  $-1$ .

- Both  $Q$  and  $M$  fail **(H1)**: all groups containing  $G = C_2^2$  from Proposition 28, in these cases we have

$$H^1(G, M) = H^1(G, Q) = \mathbb{Z}/2.$$

### Statement of results.

**Proposition 28.** *For  $n_1, n_2, n_3 \in \mathbb{N}$  with  $2(n_1 + n_2 + n_3) = n$  let*

$$\sigma := (1, 2) \cdots (2(n_1 + n_2) - 1, 2(n_1 + n_2)),$$

$$\tau := (2n_1 + 1, 2n_1 + 2) \cdots (2(n_1 + n_2) - 1, 2(n_1 + n_2)) \cdots (n - 1, n),$$

*and put  $G := \langle \sigma, \tau \rangle$ . Then*

$$H^1(G, M) = \mathbb{Z}/2.$$

The first part of Theorem 1 follows:

**Corollary 29.** *For every even  $n > 5$  and every subgroup of  $\mathfrak{S}_n$  containing  $G$ , the induced action on  $\overline{\mathcal{M}}_{0,n}$  is not stably linearizable.*

For example, when  $n_1 = n_2 = n_3 = 1$

$$\sigma := (12)(34), \quad \tau := (34)(56),$$

and the corresponding action on  $\overline{\mathcal{M}}_{0,6}$ , which is  $\mathfrak{S}_6$ -equivariantly birational to the Segre cubic, is not stably linearizable.

We apply the results above to rationality questions over nonclosed fields, completing the proof of Theorem 1:

**Theorem 30.** *Let  $F$  be a field admitting a biquadratic extension. Then, for all even  $n \geq 6$  there exist forms of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  that are not retract rational, and thus not stably rational, over  $F$ .*

In particular, this yields nonrational forms over  $F = \mathbb{C}(t)$ , a field with trivial Brauer group.

*Proof.* Indeed, let  $G \simeq C_2^2$  be the group identified in Proposition 28, with  $H^1(G, \text{Pic}(\overline{\mathcal{M}}_{0,n})) = \mathbb{Z}/2$ . Let  $\Gamma = \text{Gal}(F'/F)$  be the Galois group of the biquadratic extension  $F'/F$ . We construct a form  $X$  of  $\overline{\mathcal{M}}_{0,n}$  over  $F$  such that  $\Gamma$  acts on  $\text{Pic}(\overline{X}) = \text{Pic}(\overline{\mathcal{M}}_{0,n})$  via  $G$ . This gives an **(H1)**-obstruction to retract rationality.  $\square$

**Proof of Proposition 28.** Put

$$\sigma := (1, 2) \cdots (2(n_1 + n_2) - 1, 2(n_1 + n_2)),$$

$$\tau := (2n_1 + 1, 2n_1 + 2) \cdots (n - 1, n),$$

so that  $G = \langle \sigma, \tau \rangle$ . We will repeatedly use the inflation-restriction exact sequence

$$(5.1) \quad 0 \rightarrow H^1(\langle \tau \rangle, A^\sigma) \rightarrow H^1(G, A) \rightarrow H^1(\langle \sigma \rangle, A)^\tau,$$

with the usual notation for invariants under the actions of  $\sigma, \tau$ .

*Step 1.* Observe that  $M$  admits a decomposition, as a  $G$ -module,

$$M = L \oplus P,$$

where  $L$  consists of  $\mathbb{Z}$ -linear combinations of  $H$  and  $E_I$ , with  $n - 1 \notin I$ , and  $P$  is generated, over  $\mathbb{Z}$ , by  $E_I$  with  $n - 1 \in I$ . We have

$$H^1(G, M) = H^1(G, L) \oplus H^1(G, P).$$

*Step 2.* The involution  $\sigma$  is contained in  $\mathfrak{S}_{n-1}$ , permuting  $(n - 1)$  points and therefore linearizable. Thus

$$H^1(\langle \sigma \rangle, M) = H^1(\langle \sigma \rangle, L) = H^1(\langle \sigma \rangle, P) = 0.$$

Moreover,  $P$  is a  $G$ -permutation module. Indeed, for  $I$  with  $n - 1 \in I$ ,  $\sigma E_I = E_{\sigma(I)} \in P$ , and  $\tau E_I = E_{(\tau \cdot (n-1, n))(I)} \in P$ . It follows that

$$H^1(G, P) = 0,$$

and

$$H^1(G, M) = H^1(G, L) = H^1(\langle \tau \rangle, L^\sigma).$$

**Remark 31.** Geometrically, cohomology is already contributed on the toric model  $\overline{L}_n$ , obtained by blowing up  $(n - 2)$  general points on  $\mathbb{P}^{n-3}$ .

*Step 3.* Let  $N \subset L$  be the submodule of  $\mathbb{Z}$ -linear combinations of  $E_I$  with  $|I| \geq 2$  and  $n - 1 \notin I$ . We have a short exact sequence

$$0 \rightarrow N \rightarrow L \rightarrow Q \rightarrow 0,$$

of  $G$ -modules, with  $Q$  generated by  $H, E_1, \dots, E_{n-2}$ , modulo  $N$ , and the corresponding long exact sequence of  $\langle \tau \rangle$ -modules:

$$0 \rightarrow N^\sigma \rightarrow L^\sigma \rightarrow Q^\sigma \rightarrow H^1(\langle \sigma \rangle, N) \rightarrow \dots$$

Since  $\sigma(E_I) = E_{\sigma(I)}$ , the  $\sigma$ -action on  $N$  yields naturally a permutation module, realized via permutation of indices of  $E_I$ . So

$$H^1(\langle \sigma \rangle, N) = 0.$$

The short exact sequence

$$0 \rightarrow N^\sigma \rightarrow L^\sigma \rightarrow Q^\sigma \rightarrow 0$$

gives rise to the long exact sequence

$$(5.2) \quad H^1(\langle \tau \rangle, N^\sigma) \rightarrow H^1(\langle \tau \rangle, L^\sigma) \rightarrow H^1(\langle \tau \rangle, Q^\sigma) \rightarrow H^2(\langle \tau \rangle, N^\sigma).$$

*Step 4.* The  $\langle \tau \rangle$ -module  $N^\sigma$  has the form:

$$N^\sigma = \mathbb{Z}[\langle \tau \rangle] \oplus \dots \oplus \mathbb{Z}[\langle \tau \rangle].$$

In particular,

$$H^1(\langle \tau \rangle, N^\sigma) = H^2(\langle \tau \rangle, N^\sigma) = 0.$$

Indeed, a  $\mathbb{Z}$ -basis of  $N^\sigma$  is given by

$$e_I := \begin{cases} E_I + E_{\sigma(I)} & \text{if } \sigma(I) \neq I, \\ E_I & \text{if } \sigma(I) = I, \end{cases}$$

for

$$I \subset \{1, 2, \dots, n-2\}, \quad 2 \leq |I| \leq n-4.$$

To show that  $N^\sigma$  is a direct sum of copies of  $\mathbb{Z}[\langle \tau \rangle]$ , it suffices to show that  $\tau(e_I) = e_{I'}$ , for some  $I' \neq I$  and  $e_I \neq e_{I'}$ . Observe that

$$\sigma(I)^c = \sigma(I^c), \quad I^c := \{1, \dots, n-2\} \setminus I.$$

There are four cases:

- If  $\sigma(I) = \tau(I) = I$ , then

$$\tau(e_I) = \tau(E_I) = D_{I \cup \{n-1\}} = E_{I^c} = e_{I^c}$$

and thus  $e_I \neq e_{I^c}$ .

- If  $\sigma(I) \neq I$  and  $\tau(I) = I$ , then

$$\begin{aligned} \tau(e_I) &= \tau(E_I) + \tau(E_{\sigma(I)}) = D_{I \cup \{n-1\}} + D_{\sigma(I) \cup \{n-1\}} \\ &= E_{I^c} + E_{\sigma(I)^c} = E_{I^c} + E_{\sigma(I^c)} = e_{I^c} \end{aligned}$$

where the last equality follows from the fact that  $I^c \neq \sigma(I^c)$  since  $\sigma(I) \neq I$ . Note that the indices  $2n_1 + 1$  and  $2n_1 + 2$  are

permuted by both  $\sigma$  and  $\tau$ . Then by the assumption  $\tau(I) = I$ , we know that either both of  $2n_1 + 1$  and  $2n_1 + 2$  are contained in  $I$  or none of them is. It follows that  $I^c \neq \sigma(I)$ . Then we conclude that  $e_I \neq e_{I^c}$  since it is clear that  $I \neq I^c$ .

- If  $\tau(I) \neq I$  and  $\sigma(I) \neq I$ , then  $\sigma(\tau(I)) \neq \tau(I)$ , and

$$\begin{aligned}\tau(e_I) &= E_{\tau(I)^c} + E_{(\tau\sigma(I))^c} = E_{\tau(I)^c} + E_{(\sigma\tau(I))^c} \\ &= E_{\tau(I)^c} + E_{\sigma(\tau(I)^c)} = e_{\tau(I)^c}\end{aligned}$$

where the last equality follows from

$$\sigma(\tau(I)^c) = \sigma(\tau(I))^c \neq \tau(I)^c.$$

Moreover, we have  $\tau(I)^c \neq I$  since 1 belongs to only one of the sets  $\tau(I)^c$  and  $I$ . Similarly,  $\tau(I)^c \neq \sigma(I)$  since  $2n_1 + 1$  belongs to only one of the sets  $\tau(I)^c$  and  $\sigma(I)$ . It follows that  $e_{\tau(I)^c} \neq e_I$ .

- If  $\tau(I) \neq I$  and  $\sigma(I) = I$ , then  $\sigma(\tau(I)) = \tau(I)$ , and

$$\tau(e_I) = \tau(E_I) = \tau(D_{I \cup \{n\}}) = E_{\tau(I)^c} = e_{\tau(I)^c}$$

where the last equality follows from

$$\sigma(\tau(I)^c) = \sigma(\tau(I))^c = \tau(I)^c.$$

Similarly as the previous case, one sees that  $e_{\tau(I)^c} \neq e_I$ .

In conclusion,  $\tau(e_I) \neq e_I$ , in all cases, and  $N^\sigma$  is as claimed, and thus has vanishing first and second cohomology. It follows that

$$H^1(\langle \tau \rangle, M^\sigma) = H^1(\langle \tau \rangle, L^\sigma) = H^1(\langle \tau \rangle, Q^\sigma).$$

*Step 5.* To show that  $H^1(\langle \tau \rangle, Q^\sigma) = \mathbb{Z}/2$ , let

$$\Sigma_i := \sum_{|I|=i} E_I,$$

where the sum is over  $I \subseteq \{1, 2, \dots, n-2\}$  with  $|I| = i$ . Put  $\Sigma := \Sigma_1$  and set

$$\begin{aligned}e_0 &:= H - \Sigma, \\ e_i &:= H - \Sigma + (E_{2i-1} + E_{2i}), & 1 \leq i \leq n_1 + n_2, \\ w_j &:= E_{2j-1}, & n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}, \\ v_j &:= H - \Sigma + E_{2j}, & n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}.\end{aligned}$$

Then

$$\{e_i, w_j, v_j\}$$

for  $0 \leq i \leq n_1 + n_2$  and  $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$  gives a  $\mathbb{Z}$ -basis of  $Q^\sigma$ . Moreover, for  $1 \leq i \leq n_1 + n_2$  and  $n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}$ , one has

$$\tau(e_0) = -e_0, \quad \tau(e_i) = e_i, \quad \text{and} \quad \tau(w_j) = v_j.$$

Indeed,  $Q^\sigma$  is generated, over  $\mathbb{Z}$ , by

$$H, (E_1 + E_2), \dots, (E_{2(n_1+n_2)-1} + E_{2(n_1+n_2)}), E_{2(n_1+n_2)+1}, \dots, E_{n-2}.$$

We now show that  $\{e_i, w_j, v_j\}$  gives another basis. First, observe that

$$H - \Sigma = D_{34\dots n} - (E_1 + E_2) + \underbrace{\sum_{\substack{1,2 \notin I, E_I \in N \\ \in N^\sigma}} E_I}.$$

Indeed, if  $1, 2 \notin I$  and  $E_I \in N$ ,  $1, 2 \notin \sigma(I)$  and  $E_{\sigma(I)}$  will also appear in the summand. Then  $\sigma(H - \Sigma) = H - \Sigma \pmod{N^\sigma}$  and

$$e_j, w_j, v_j \in Q^\sigma.$$

Moreover,  $\{e_j, w_j, v_j\}$  generates  $Q^\sigma$  since

$$E_{2i-1} + E_{2i} = e_i - e_0, \quad E_{2j} = v_j - e_0$$

and

$$H = \left(\frac{4-n}{2}\right)e_0 + \sum_{i=1}^{n_1+n_2} e_i + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} (w_j + v_j).$$

To compute the  $\tau$ -action on this basis, one can first compute

$$\begin{aligned} H - \Sigma &= D_{34\dots n} - (E_1 + E_2) \pmod{N^\sigma} \\ &\xrightarrow{\tau} D_{34\dots n} - D_{1,n-1} - D_{2,n-1} \\ &= D_{34\dots n} - 2H + 2\Sigma - (E_1 + E_2) \pmod{N^\sigma} \\ &= H - \Sigma + (E_1 + E_2) - 2H + 2\Sigma - (E_1 + E_2) \pmod{N^\sigma} \\ &= -H + \Sigma, \end{aligned}$$

i.e.,

$$\tau(e_0) = -e_0.$$

Then we have

$$\begin{aligned} H - \Sigma + E_{2i-1} + E_{2i} &\xrightarrow{\tau} -H + \Sigma + D_{2i-1,n-1} + D_{2i,n-1} \\ &= -H + \Sigma + 2H - 2\Sigma + (E_{2i-1} + E_{2i}) \pmod{N^\sigma} \\ &= H - \Sigma + (E_{2i-1} + E_{2i}) \pmod{N^\sigma}. \end{aligned}$$

Note that the equalities hold for all  $1 \leq i \leq \frac{n}{2}$ . In particular,

$$\tau(e_i) = e_i, \quad \text{for } 1 \leq i \leq n_1 + n_2.$$

Finally,

$$\begin{aligned}\tau(w_j) &= D_{2j,n-1} = H - \Sigma + E_{2j} - \sum_{\substack{2j \notin I \\ E_I \in N}} E_I \\ &= H - \Sigma + E_{2j} \pmod{N^\sigma},\end{aligned}$$

i.e.,

$$\tau(w_j) = v_j, \quad \text{for } n_1 + n_2 + 1 \leq j \leq \frac{n-2}{2}.$$

In conclusion,

$$Q^\sigma = \mathbb{Z}[e_0] + \sum_{i=1}^{n_1+n_2} \mathbb{Z}[e_i] + \sum_{j=n_1+n_2+1}^{\frac{n-2}{2}} \mathbb{Z}[w_j, v_j],$$

where  $\tau$  acts trivially on  $e_i$ , permutes  $w_j$  and  $v_j$ , and the unique  $(-1)$ -eigenvector  $e_0$  contributes to

$$H^1(\langle \tau \rangle, Q^\sigma) = \mathbb{Z}/2.$$

This completes the proof of Proposition 28.

**Remark 32.** Notice that when  $n_1 = n_2 = 0$ , the argument above shows

$$H^1(C_2, Q) = \mathbb{Z}/2,$$

where the  $C_2$  is generated by  $(1, 2)(3, 4) \dots (n-1, n)$ . Computational experiments suggest that

$$H^1(H, M) = 0,$$

for all cyclic subgroups  $H \subset \mathfrak{S}_n$ .

### Small dimensional examples.

**n = 6:** By Theorem 1 and the analysis in Section 6 of [CTZ23], we know that the  $G$ -action on  $\text{Pic}(\overline{\mathcal{M}}_{0,6})$  satisfies **(SP)** if and only if the  $G$ -action is linearizable, thus, nonlinearizable actions are not stably linearizable, as they fail **(SP)**.

**Remark 33.** This indicates an error in the application in [HT23, p. 295]: Proposition 21 there asserts that the standard and nonstandard actions of  $\mathfrak{A}_5$  are stably birational, contradicting our cohomology computation. The gap occurs in the sentence: ‘‘However, for any finite group  $G$  and automorphism  $a : G \rightarrow G$ , precomposing by  $a$  yields an action on  $G$ -modules; this respects permutation and stably permutation modules.’’

**n = 8:** There is a unique (conjugacy class of)  $G' = C_2^2 \subset \mathfrak{S}_8$  such that

$$H^1(G', \text{Pic}(\overline{\mathcal{M}}_{0,8})) = \mathbb{Z}/2,$$

and all  $G \subseteq \mathfrak{S}_8$  failing **(H1)** on  $M$  contain  $G'$ . With `magma`, we find:

- There are 66 (conjugacy classes of) groups containing this  $G'$ .
- Of the remaining 230 classes, 96 are contained in the (unique)  $\mathfrak{S}_7 \subset \mathfrak{S}_8$ , the corresponding actions are linearizable.
- After that, there are 56 contained in the (unique)  $\mathfrak{S}_6 \times C_2$  – these actions are birational to an action on a 5-dimensional torus; such actions have been analyzed, over nonclosed fields, in [HY17].
- We are left with 78 classes. Applying [HY17, Algorithm F4] to these classes, we found at least 37 classes of groups  $G \subset \mathfrak{S}_8$  having vanishing cohomology but with  $\text{Pic}(\overline{\mathcal{M}}_{0,8})$  failing the **(SP)** condition.
- Among the 41 remaining classes, 13 leave invariant an odd cycle. These actions are stably linearizable by Proposition 3.
- There are 28 remaining classes, including a minimal

$$C_2^2 = \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 3)(2, 4)(5, 7)(6, 8) \rangle,$$

which (up to conjugation) is contained in every remaining class. The action of this  $C_2^2$  on  $M$  yields a permutation module:

$$\mathbb{Z}[C_2^2]^{19} \oplus \mathbb{Z}[C_2^2/C_2]^3 \oplus \mathbb{Z}[C_2^2/C_2']^3 \oplus \mathbb{Z}[C_2^2/C_2'']^3 \oplus \mathbb{Z}^5.$$

However, on  $Q$ , this action fails **(H1)**, and Theorem 24 is not applicable to any of these cases.

**n = 10:** We find more minimal groups contributing cohomology:

$$H^1(G, \text{Pic}(\overline{\mathcal{M}}_{0,10})) = \mathbb{Z}/2$$

when

- $G = C_2^2 = \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 2)(9, 10) \rangle,$
- $G = C_2^2 = \langle (1, 2)(3, 4)(5, 6), (5, 6)(7, 8)(9, 10) \rangle,$
- $G = C_2 \times C_4 = \langle (3, 6)(8, 10), (1, 2)(5, 9), (1, 2)(3, 10, 6, 8)(4, 7) \rangle,$
- $G = \mathfrak{D}_4 = \langle (3, 6)(8, 10), (1, 2)(5, 9)(8, 10), (1, 2)(3, 10, 6, 8)(4, 7) \rangle.$

## 6. THREE-DIMENSIONAL CASE

Next, we give a criterion for rationality of the Segre cubic, exhibit forms failing stable rationality over arbitrary fields admitting a bi-quadratic extension, and establish stable rationality, provided  $Q$  is stably permutation, for the action of the absolute Galois group.

Recall that  $X_6$  denotes the symmetrically linearized GIT quotient with equivalent presentations:

- $(\mathbb{P}^1)^6$  under the diagonal action of  $\mathrm{SL}_2$ ; or
- $\mathrm{Gr}(2, 6)$  under the diagonal action of the torus  $\mathbb{T} \simeq \mathbb{G}_m^5$ .

These have ten isolated nodes, the images of the  $D_I, |I| = 3$  under the blow down  $\beta : \overline{\mathcal{M}}_{0,6} \rightarrow X_6$ . These are classically embedded  $X_6 \subset \mathbb{P}^4$  as cubic threefolds, known as Segre cubic threefolds [CTZ23]. The remaining boundary divisors  $D_I, |I| = 2$  correspond to planes passing through four nodes.

**Theorem 34.** *Let  $X$  be a form of the Segre cubic threefold over a nonclosed field  $F$  of characteristic zero, and  $\tilde{X}$  its standard resolution of singularities, a form of  $\overline{\mathcal{M}}_{0,6}$ . Then  $X$  is rational over  $F$  if and only if the Galois-module  $\mathrm{Pic}(\overline{\mathcal{M}}_{0,6})$  satisfies (SP).*

*Proof.* This is closely related to the linearizability result [CTZ23, Theorem 1]. The group-theoretic analysis there shows that the only cases where the Galois action on the Picard group is stably permutation are:

- when one of the ten nodes is Galois invariant;
- the Galois action is contained in an  $\mathfrak{S}_5$ -action associated with permutations of *five* of the marked points;
- the Galois group acts via  $C_2^2$ , leaving three planes invariant, and the set of nodes splits into a union of five Galois orbits of length two.

Note that the first two cases are easily shown to be rational: Projecting from a node gives a birational map to  $\mathbb{P}^3$ , cf. Example 19. And when the action factors through  $\mathfrak{S}_5$ , the moduli space arises via the Kapranov construction, i.e., is a blow-up of  $\mathbb{P}^3$ .

Recall that in the third case, the Galois action factors through  $\mathfrak{S}_2 \times \mathfrak{S}_4 \subset \mathfrak{S}_6$  corresponding to a partition of the six points conjugate to

$$\{1, 2, 3, 4, 5, 6\} = \{3, 4\} \cup \{1, 2, 5, 6\}.$$

Our  $C_2 \times C_2$  action is conjugate to

$$\langle (34), (15)(26) \rangle \subset \mathfrak{S}_6$$

This leaves the boundary divisors  $D_{34}$ ,  $D_{15}$ , and  $D_{26}$  invariant. Identifying singular points with the boundary divisors in  $\overline{\mathcal{M}}_{0,6}$ , the orbits are

$$\begin{aligned} \{D_{123} = D_{456}, D_{124} = D_{356}\}, & \quad \{D_{125} = D_{346}, D_{156} = D_{234}\}, \\ \{D_{126} = D_{345}, D_{256} = D_{134}\}, & \quad \{D_{135} = D_{246}, D_{145} = D_{236}\}, \\ \{D_{136} = D_{245}, D_{146} = D_{235}\}. \end{aligned}$$

We emphasize that the invariant divisor classes reflect boundary divisors defined over  $F$ . Indeed, our moduli space has  $F$ -rational smooth points so there is no obstruction to descending Galois-invariant divisors.

We claim this moduli space is birational over  $F$  to a toric threefold, i.e., an equivariant compactification of a nonsplit torus over  $F$ .

Consider the Losev-Manin moduli space associated to the partition above. Specifically, points 3 and 4 are not permitted to collide with other points but points from  $\{1, 2, 5, 6\}$  may collide with one another. This is toric by Proposition 7, i.e., the orbits of the homogeneous quartic forms vanishing along  $\{1, 2, 5, 6\}$  modulo the torus fixing  $\{3, 4\}$ . This geometric description is compatible with the Galois action.

Rationality of three-dimensional toric varieties has been settled in [Kun87, Theorem 2]: The variety is rational over  $F$  iff the Picard module is stably permutation for the Galois action.

Here is an alternative rationality construction: Pick one of the boundary divisors  $D_I$ ,  $|I| = 2$  invariant under the Galois action. With our choice of indexing this could be  $D_{34}$ ,  $D_{15}$ , or  $D_{26}$ ; we take  $D_{34}$ . This corresponds to a plane  $P \subset X$  containing four ordinary singularities, i.e., the images of  $D_{34j}$ ,  $j = 1, 2, 5, 6$ . We blow this plane up – inducing a small resolution of the four singularities – and then blow down the proper transform of the plane. This yields a complete intersection of two quadrics  $X_{2,2} \subset \mathbb{P}^5$  with six singularities, the images of the singularities of  $X$  *not* contained in  $P$ . Under the  $C_2 \times C_2$  action, we have three orbits each with two singular points. For each orbit, the line joining the singularities is contained in  $X_{2,2}$ . Projecting from that line gives

$$X_{2,2} \xrightarrow{\sim} \mathbb{P}^3;$$

the birationality is classical cf. [CTSSD87, Proposition 2.2].  $\square$

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