Correlation for Surfaces of General Type

Brendan Hassett*

January, 1996

1 Introduction

The purpose of this paper is to prove the following theorem:

Theorem 1.1 (Correlation Theorem for Surfaces) *Let $f : X \longrightarrow B$ be a proper morphism of integral varieties, whose general fiber is an integral surface of general type. Then for n sufficiently large, X_B^n admits a dominant rational map h to a variety W of general type such that the restriction of h to a general fiber of f^n is generically finite.*

This theorem has a number of geometric and number theoretic consequences which will be discussed in the final section of this paper. In particular, assuming Lang's conjecture on rational points of varieties of general type, we prove a uniform bound on the number of rational points on a surface of general type that are not contained in rational or elliptic curves.

The inspiration for this paper is the work of Caporaso, Harris, and Mazur [CHM], where the Correlation Theorem is proved for families of curves of genus $g \geq 2$. The same result is conjectured for families of varieties of general type of any dimension. The paper [CHM] contains many of the ideas needed for a proof of the general conjecture. However, at one point the argument hinges on the fact that the fibers of the map are curves: it invokes the existence of a 'nice' class of singular curves, the stable curves. For the purposes of this discussion, 'nice' means two things:

*This work was partially supported by a National Science Foundation Graduate Fellowship

1. Given any proper morphism $f : X \longrightarrow B$ whose generic fiber is a smooth curve of genus $g \geq 2$, there exists a generically finite base change $B' \longrightarrow B$ so that the dominating component $X' \subset X \times_B B'$ is birational to a family of stable curves over B'.

2. Let $X \to B$ be a family of stable curves, smooth over the generic point. Then the fiber products X^n_B have canonical singularities.

For the purpose of generalizing to higher dimensions, we make the following definitions:
Let $\mathcal{C}$ be a class of singular varieties.

> $\mathcal{C}$ is *inclusive* if for any proper morphism $f : X \to B$ whose generic fiber is a variety of general type, there is a generically finite base change $B' \to B$ so that the dominating component $X' \subset X \times_B B'$ is birational to a family $Y' \to B'$ with fibers in $\mathcal{C}$.
>
> $\mathcal{C}$ is *negligible* if for any family of varieties of general type $f : X \to B$ with singular fibers belonging to $\mathcal{C}$, the fiber products X^n_B have canonical singularities.

In a nutshell, the main obstruction to extending the results of [CHM] is to find a class of higher dimensional singular varieties that is both negligible and inclusive. In this paper, we identify such a class, the 'stable surfaces' at the boundary of a compactification of the moduli space of surfaces of general type.

In the second section of this paper, we describe these stable surfaces and motivate their definition. In the third section, we prove that these stable surfaces actually form an inclusive class of singularities. In the fourth section, we prove that stable surfaces are negligible. In the fifth section, we complete the proof of the Correlation Theorem for surfaces of general type outlined in [CHM]. Finally, we state some consequences of the Correlation Theorem, assuming various forms of the Lang conjectures. Throughout this paper, the base field will have characteristic zero.

I would like to thank Joe Harris for suggesting this problem and János Kollár for his helpful advice, especially concerning the results in section four. Dan Abramovich has also provided many useful comments and corrections.

2 Stable Surfaces

Here we describe the class of stable surfaces and their singularities, following [K-SB] and [K]. Stable surfaces are defined so that there is stable reduction theorem for surfaces, analogous to stable reduction for curves. Let $X \to \Delta$ be a family of varieties of general type over a disc Δ, and choose a semistable reduction $Y \to \tilde{\Delta}$ of this family [KKMS]. The fibers of $Y \to \tilde{\Delta}$ have very simple singularities (reduced normal crossings singularities), but Y is not unique. The stable reduction $\mathcal{Y} \to \tilde{\Delta}$ should be a modification of the semistable family with two salient properties: it is unique and the singularities of its fibers can be described explicitly. The natural candidate for $\mathcal{Y}$ is the canonical model of Y, and the existence of canonical models for semistable families of surfaces is proved in [Ka]. The fibers of the canonical model are called stable surfaces.

Now we introduce the formal definitions. A variety X is said to be *$\mathbb{Q}$-Gorenstein* if X is Cohen-Macaulay and $\omega_X^{[k]}$ is locally free for some k, where $\omega_X^{[k]}$ denotes the reflexive hull (i.e. the double dual) of the kth power of the dualizing sheaf. For a $\mathbb{Q}$-Gorenstein singularity, the smallest such k is called the index of the singularity. A surface T is *semi-smooth* if the only singularities of T are 2-fold normal-crossings singularities ($x^2 = y^2$) and pinch points ($x^2 = zy^2$). A *good semi-resolution* of a surface S is a proper map $g : T \longrightarrow S$ satisfying the following properties:

1. T is semi-smooth
2. g is an isomorphism in the complement of a codimension two subscheme of S
3. No component of the double curve of T is exceptional for g
4. The components of the double curve of T and the exceptional locus of g are smooth and meet transversally

A surface S is said to have *semi-log-canonical* singularities if

1. S is Cohen-Macaulay and $\mathbb{Q}$-Gorenstein with index k
2. S is semi-smooth in codimension one

3. The discrepancies of a good semi-resolution of S are all greater than or equal to -1 (i.e. $\omega_T^k = g^*\omega_S^{[k]}(ka_1E_1 + \cdots + ka_NE_N)$ where $a_i \geq -1$)

In [K-SB] a complete classification of semi-log-canonical singularities is given.

The following theorem explains why semi-log-canonical singularities are useful:

Theorem 2.1 *Let $f : X \to \Delta$ be a family of surfaces over the disc such that X is $\mathbb{Q}$-Gorenstein. Then the following are equivalent:*

1. *The general fiber has rational double points, and the central fiber has semi-log-canonical singularities.*

2. *For any base change $\tilde{\Delta} \to \Delta$, the base-changed family*

$$\tilde{f} : \tilde{X} \longrightarrow \tilde{\Delta}$$

has canonical singularities.

In fact, if $X \to \Delta$ has a semistable resolution of singularities, then X has canonical singularities if and only if the general fiber has rational double points and the central fiber has semi-log-canonical singularities.

In particular, this means that the 'bad' fibers in a stable reduction of surfaces have semi-log-canonical singularities. For the sake of this discussion, surfaces with only rational double point singularities are 'good' fibers, because the canonical model of a surface of general type should be 'good'.

A surface S is *stable* if S has semi-log-canonical singularities and $\omega_S^{[k]}$ is locally free and ample for some k. Note that a smooth surface of general type is not stable if it contains -1 or -2 curves, but its canonical model will be stable. A family of stable surfaces is a proper flat morphism $\mathcal{S} \to B$ whose fibers are stable surfaces, with the property that taking reflexive powers of the relative dualizing sheaf commutes with restricting to a fiber:

$$\omega_{\mathcal{S}/B}^{[k]}|\mathcal{S}_b = \omega_{\mathcal{S}_b}^{[k]}$$

In particular, the reflexive powers of the relative dualizing sheaf are flat. This additional condition is necessary to guarantee that the moduli space in the next section is separated.

Note that we can define

$$K_S^2 = \frac{1}{k^2}\#(\omega_S^{[k]}, \omega_S^{[k]})$$

for any stable surface S, and that this number is constant in families. The invariant $\chi_S = \chi(\mathcal{O}_S)$ is also constant in families. Finally, stable surfaces are analogous to stable curves in one more important sense:

Theorem 2.2 *A stable surface has a finite automorphism group.*

The essence of the proof is easy to grasp. Let S be stable, and let $\tilde{S}$ be its normalization. Let D be the double curve on $\tilde{S}$. The pair $(\tilde{S}, D)$, is log-canonical (see [K-SB]). Therefore, each component of $(\tilde{S}, D)$ is of log-general type, and has a finite automorphism group by [I].

3 Stable Surface Singularities are Inclusive

To prove that stable surfaces are inclusive, we use the moduli space of stable surfaces:

Theorem 3.1 *There exists a projective coarse moduli space $\overline{\mathcal{M}}_{\chi,K^2}$ for smoothable stable surfaces with invariants χ and K^2. There is a finite cover of the moduli space $\phi : \Omega \to \overline{\mathcal{M}}_{\chi,K^2}$ that admits a universal family $\mathcal{S} \to \Omega$:*

$$\begin{array}{ccc} \mathcal{S} & & \\ \downarrow & & \\ \Omega & \stackrel{\phi}{\to} & \overline{\mathcal{M}}_{\chi,K^2} \end{array}$$

By definition, a stable surface is *smoothable* if it is contained in a family of stable surfaces whose general member has only rational double points. The proof of Theorem 3.1 is scattered throughout the literature. The proof that the moduli space exists as a separated algebraic space is contained in [K-SB] §5. This relies on the properties of semi-log-canonical singularities and the finite automorphism theorem. The proof that the moduli space has a functorial semipositive polarization is contained in [K] §5. This paper also contains a general argument for the existence of a finite covering of the moduli space possessing a universal family (see Proposition 2.7). The proof that the moduli space is of finite type for a given pair of invariants (and thus proper and projective by [K]) is contained in [A].

Using this moduli space, we prove that the class of stable surface singularities is inclusive. The precise statement we need is the following:

Proposition 3.1 *Let $f : X \to B$ be a proper morphism of integral varieties. Assume that the general fiber of f is a surface of general type, and let Σ denote the image of B in the moduli space $\overline{\mathcal{M}}_{\chi,K^2}$. Then there exists a generically finite Galois base extension $B' \to B$ with Galois group G, a finite cover $\Sigma' \to \Sigma$, and a family of stable surfaces $T \to \Sigma'$ with the following properties:*

1. *G acts on Σ' and T, and there are G-equivariant maps $\nu : B' \to \Sigma'$ and $X' \dashrightarrow T$ so that diagram*

$$\begin{array}{ccc} X' & \dashrightarrow & T \\ \downarrow & & \downarrow \\ B' & \to & \Sigma' \end{array}$$

   commutes, where X' is a dominating component of $X \times_B B'$.

2. *The pull back family $Y' = T \times_{\Sigma'} B'$ is birational to X', and Y'/G is birational to X.*

We sketch the proof; see [CHM] and [K] for more details. This is an application of the properties of the moduli space described in Theorem 3.1. Let B_1 be the graph of the map $B \dashrightarrow \Sigma$, so there is a morphism $B_1 \to \Sigma$. Write $\Sigma_2 = \phi^{-1}(\Sigma)$, $T_2 = \mathcal{S} \times_\Omega \Sigma_2$, $B_2 = B_1 \times_\Sigma \Sigma_2$, $Y_2 = T_2 \times_{\Sigma_2} B_2$, and let X_2 be a dominating component of $X \times_B B_2$. If we denote the projection $\mu : B_2 \to \Sigma_2$, then for general $b \in B_2$ we have

$$(X_2)_b \sim_{\text{bir}} (T_2)_{\mu(b)} = (Y_2)_b$$

We might expect the families X_2 and Y_2 to be birational to each other, as their generic fibers are birational. This is not necessarily the case. However, since the automorphism group of any stable surface is finite, after another generically finite base change $B_3 \to B_2$ we find that X_3 and Y_3 are birational. The variety B' is the Galois normalization of B_3 over B, G the corresponding Galois group, Σ' the Stein factorization of $B' \to \Sigma$, and T the pull back of $\mathcal{S}$ to Σ'. In particular, X' is birational to the family of stable surfaces $Y' \to B'$. $\square$

4 Stable Surface Singularities are Negligible

In this section, we restrict our attention to families of stable surfaces $f : X \to B$, where B is of finite type over the base field. The locus $S \subset B$ corresponding to fibers with singularities worse than canonical singularities (i.e. rational double points) is Zariski closed. Here we assume that S is a proper subvariety of B. We will prove the following result on fiber products of these families:

Theorem 4.1 *Let $f : X \to B$ be a family of stable surfaces, and assume B has only canonical singularities. If the generic fiber has only canonical singularities, then the fiber products of this family*

$$f^n : X_B^n \longrightarrow B$$

also have canonical singularities.

We give the strategy of the proof. First, we factor f^n as a sequence of maps

$$X_B^n \to X_B^{n-1} \to \cdots \to X \to B$$

The theorem follows by applying the following proposition to each term of this sequence:

Proposition 4.1 *Let $f : X \to B$ be a family of stable surfaces, and assume B has only canonical singularities. If the generic fiber has only canonical singularities, then X also has canonical singularities.*

We shall prove this result by induction. The base case for the induction is essentially the fundamental property of stable surface singularities stated in Theorem 2.1. The inductive step uses a result of Stevens [St], which gives conditions for a fibration to have canonical singularities.

We will use the following preliminary lemma.

Lemma 4.1 *Let $f : X \to B$ a family of stable surfaces, such that the general fiber is normal. Then X is Cohen-Macaulay, normal, and $\mathbb{Q}$-Gorenstein. The relative canonical class $K_{X/B}$ is $\mathbb{Q}$-Cartier.*

By definition, stable surfaces are Cohen-Macaulay. Canonical singularities are rational [E], which implies they are also Cohen-Macaulay. In particular, the base and the fibers of the morphism $f : X \to B$ are both Cohen-Macaulay, so X itself is Cohen-Macaulay. Moreover, the singularities of X are in codimension ≥ 2, because the singularities of the base and the general fiber are in codimension two, and all the fibers are generically nonsingular (i.e. reduced). We conclude that X is normal by Serre's criterion.

The dualizing sheaves ω_X and ω_B are reflexive and determine Weil divisor classes K_X and K_B (see [H] for general properties of reflexive sheaves). The relative canonical class $K_{X/B}$ is defined by the formula

$$K_{X/B} = K_X - f^*K_B$$

K_X is $\mathbb{Q}$-Cartier if and only if $K_{X/B}$ is, because K_B is $\mathbb{Q}$-Cartier by hypothesis. We show that $K_{X/B}$ is $\mathbb{Q}$-Cartier. Since $X \to B$ is a family of stable surfaces, we have

$$\mathcal{O}(kK_{X/B})|_{X_b} = \omega_{X_b}^{[k]}$$

for all k. Stable surfaces are themselves $\mathbb{Q}$-Gorenstein, so for some sufficiently divisible k the right hand side is locally free for all $b \in B$. Consequently, $\mathcal{O}(kK_{X/B})$ is locally free and $kK_{X/B}$ is Cartier. □

Now we prove the base case of our induction, where the base B is one dimensional. A curve with only canonical singularities is actually smooth. Since our result is local on the base, we may assume $B = \Delta$. In this case, Proposition 4.1 is just the implication $1 \Rightarrow 2$ of Theorem 2.1. Canonical surface singularities are just rational double points, and the stable surface in the central fiber has semi-log-canonical singularities.

Now that we have established the base case for the inductive proof of Proposition 4.1, we turn to the inductive step. This inductive argument holds more generally than the context of stable surfaces, so we summarize it in the following proposition:

Proposition 4.2 *Let $f : X \to B$ be a flat family of varieties, where B has canonical singularities, and $K_{X/B}$ is $\mathbb{Q}$-Cartier. Assume that*

> *For any map $\phi : \Delta \to B$ of the disc into the base for which $X_{\phi(t)}$ is canonical for $t \neq 0$, the total space $X \times_B \Delta$ has canonical singularities.*

Then X has canonical singularities.

Note that this completes the proof of Proposition 4.1. By Lemma 4.1, the family $f : X \to B$ satisfies the hypotheses of Proposition 4.2. The assumption in the statement is the base case proved in the last paragraph.

Now we prove Proposition 4.2. First we reduce to the case where B is smooth. Let $\sigma : \tilde{B} \to B$ be a resolution of singularities, $\tilde{X} = X \times_B \tilde{B}$, and $\psi : \tilde{X} \to X$ the induced map. The family $\tilde{X} \to \tilde{B}$ still satisfies the hypotheses of the proposition. The base B has canonical singularities, so for some k we have

$$\omega^k_{\tilde{B}} = \sigma^* \omega^{[k]}_B (ka_1 E_1 + \cdots + ka_N E_N) \text{ where } a_i \geq 0$$

and

$$\omega^{[k]}_{\tilde{X}} = \psi^* \omega^{[k]}_X (ka_1 E'_1 \cdots + ka_N E'_N)$$

where the E_i are exceptional for σ and the E'_i are the corresponding divisors in $\tilde{X}$. In particular, ψ imposes no adjoint conditions on differentials and X has canonical singularities if $\tilde{X}$ does.

To complete the proof, we use the following result:

Theorem 4.2 (Stevens' Theorem [St]) *Let $h : X \to \Delta$ be a flat family of varieties. Assume:*

1. *X is a $\mathbb{Q}$-Gorenstein integral variety, and the fibers of h are integral varieties.*
2. *The general fibers $h^{-1}(s)$ have only canonical singularities.*
3. *The special fiber $h^{-1}(0)$ has log terminal singularities.*

Then X has canonical singularities.

We apply this theorem inductively to reduce the dimension of the base. Assume the proposition is proved if the dimension is less than $b = \dim(B)$. Choose a local analytic coordinate y on B, and consider the level surfaces

$$B_s = \{b \in B : y(b) = s\}$$

for $s \in \Delta$. Assume that the B_s are smooth and are not contained in the closed set S where the singularities of the fibers are not canonical. Let

$X_s = f^{-1}(B_s)$ and let $h : X \to \Delta$ be the induced fibration of X. The varieties X_s satisfy the hypotheses of Proposition 4.2, and so are canonical by the induction hypothesis. Since all the fibers of h are canonical and the total space X is $\mathbb{Q}$-Gorenstein, we may apply Stevens' theorem to conclude that X is also canonical. □

5 Proof of the Correlation Theorem

In this section, we prove the Correlation Theorem for surfaces of general type (Theorem 1.1). We first prove a special case where the family has maximal variation of moduli and the singularities are not too bad:

Theorem 5.1 (Correlation for Families with Maximal Variation) *Let $X \to B$ be a family of stable surfaces, with projective integral base and smooth general fiber. Assume that the associated map $\phi : B \to \overline{\mathcal{M}}$ is generically finite. Then there exists a positive integer n such that X^n_B is of general type.*

Being of general type is a birational property, so there is no loss of generality if we take the base B to be smooth. To show that X^n_B is of general type, we must verify two statements:

1. X^n_B has canonical singularities
2. $\omega_{X^n_B}$ is big

We proved the first statement in section four, so pluricanonical forms on X^n_B pull back to regular forms on any desingularization. The second statement says that there are enough pluricanonical differentials on X^n_B to guarantee it is of general type. To prove this we use the following theorem:

Theorem 5.2 *Let $f : X \to B$ be a family of surfaces, such that the general fiber is a surface of general type. Assume this family has maximal variation. Then for m sufficiently large, the sheaf $f_*\omega^m_{X/B}$ is big.*

This result is proven by Viehweg in [V] (and more generally for arbitrary dimensional fibers by Kollár in [K2]). Let $S^{[n]}$ denote the reflexive hull of the *nth* symmetric power of a sheaf. By definition, $f_*\omega^m_{X/B}$ is big if for any ample line bundle H on B there exists an integer n such that

$$S^{[n]}(f_*\omega^m_{X/B}) \otimes H^{-1}$$

is generically globally generated, i.e. the global sections of this sheaf generate over an open set of B. It is equivalent to say that this sheaf is generically globally generated for sufficiently large n. We need the following consequence of Viehweg's theorem:

Proposition 5.1 *Under the hypotheses of the Theorem 5.2, $\omega_{X^n_B}$ is big for n sufficiently large.*

We will show that Theorem 5.2 implies the proposition. We restrict ourselves to values of m for which $\omega^{[m]}_{X/B}$ is locally free and $f_*\omega^m_{X/B}$ is big. Let $T^{[n]}$ denote the reflexive hull of the nth tensor power of a sheaf. We claim that for sufficiently large n

$$T^{[n]}(f_*\omega^m_{X/B}) \otimes H^{-1}$$

is generically globally generated. Since quotients of generically globally generated sheaves are generically globally generated, this is a consequence of the following result from representation theory (see [H2]):

Proposition 5.2 *Let V be an r dimensional vector space over a field of characteristic zero, let $T^n(V)$ and $S^q(V)$ be the nth tensor power and qth symmetric power representations of $Gl(V)$ respectively, and write $t = r!$. Then each irreducible component of $T^n(V)$ is a quotient of a representation*

$$S^{q_1}(V) \otimes \cdots \otimes S^{q_k}(V)$$

where $q_i \geq \frac{n}{t+1}$.

Now we will prove that for some large n the dualizing sheaf $\omega_{X^n_B}$ is big, i.e. for large m we have

$$h^0(X^n_B, \omega^{[m]}_{X^n_B}) \approx m^{(b+2n)}$$

where $b = \dim(B)$. Generally, the relative dualizing sheaves of fibered products satisfy the equation

$$\omega_{X^n_B/B} = \pi_1^*\omega_{X/B} \otimes \cdots \otimes \pi_n^*\omega_{X/B}$$

where the π_j are the projections. Using the basic properties of reflexive sheaves we obtain:

$$\omega^{[m]}_{X^n_B} = \pi_1^*\omega^{[m]}_{X/B} \otimes \cdots \otimes \pi_n^*\omega^{[m]}_{X/B} \otimes f^{n*}\omega^m_B$$

Applying f^n_* to this gives

$$f^n_* \omega^{[m]}_{X^n_B} = T^n(f_* \omega^{[m]}_{X/B}) \otimes \omega^m_B$$

which is also a reflexive sheaf. The inclusion map $\omega^m_{X/B} \to \omega^{[m]}_{X/B}$ induces a map of reflexive sheaves

$$T^{[n]}(f_* \omega^m_{X/B}) \to T^n(f_* \omega^{[m]}_{X/B}) = f^n_* \omega^{[m]}_{X^n_B} \otimes \omega^{-m}_B$$

which is an isomorphism at the generic point of B.

Let H be an invertible sheaf on B so that $H \otimes \omega_B$ is very ample. We can choose n so that $T^{[n]}(f_* \omega^m_{X/B}) \otimes H^{-m}$ is generically globally generated for all admissible values of m. The computations of the last paragraph show that $f^n_* \omega^{[m]}_{X^n_B} \otimes (H \otimes \omega_B)^{-m}$ is also generically globally generated for these values of m. This sheaf has rank on the order of m^{2n}, so there at least this many global sections. By our assumption on H, we have that $(H \otimes \omega_B)^m$ has on the order of m^b sections varying horizontally along the base B. Tensoring, we obtain that $f^n_* \omega^{[m]}_{X^n_B}$ has on the order of m^{2n+b} global sections. Thus we conclude that

$$h^0(\omega^{[m]}_{X^n_B}) \approx m^{2n+b}$$

This completes the proof of Proposition 5.1 and the special case of the Correlation theorem. □

We use this special case to prove the general Correlation Theorem. Since stable surface singularities are inclusive, after a generically finite base extension $B' \to B$ every family of surfaces of general type dominates a family of stable surfaces $\psi : T \to \Sigma'$ with maximal variation:

$$\begin{array}{ccc} X' & \to & T \\ \downarrow & & \downarrow \\ B' & \to & \Sigma' \end{array} \quad (*)$$

By Theorem 5.1, the fiber products $T^n_{\Sigma'}$ are of general type for n sufficiently large. Let ${X'}^n_{B'}$ denote the component of the fiber product dominating B'. We obtain a diagram:

$$\begin{array}{ccc} {X'}^n_{B'} & \to & T^n_{\Sigma'} \\ \downarrow & & \downarrow \\ B' & \to & \Sigma' \end{array} \quad (**)$$

with ${X'}^{n}_{B'}$ dominating $T^{n}_{\Sigma'}$, a variety of general type.

We claim that the fiber products X^N_B also dominate varieties of general type, provided that N is large enough. The diagrams $(*)$ and $(**)$ are G-equvariant, so it suffices to prove the following claim:

> For very large N, enough forms on $T^N_{\Sigma'}$ descend to smooth forms on $W = (T^N_{\Sigma'})/G$ to guarantee that W is of general type.

By definition, smooth forms are pluricanonical forms that remain regular when pulled back to a desingularization. The claim implies that the map of quotients $({X'}^N_{B'})/G \to (T^N_{\Sigma'})/G$ gives a dominant rational map $h : X^N_B \to W$ to a variety of general type.

The proof of the claim is identical to the analogous proof for curves given in [CHM], so we only sketch the basic ideas. First let $D_0 \subset \Sigma'$ be a divisor containing the following points: (1) $s \in \Sigma'$ with nonminimal stabilizer in G, and (2) $s \in \Sigma'$ for which T_s is fixed pointwise by a nontrivial subgroup of G. Let D be the pull back of gD_0 to $T^N_{\Sigma'}$. Repeating the argument of Proposition 5.1 we show that for large N, $\omega_{T^N_{\Sigma'}}(-D)$ is big. Using a lemma from §4 of [CHM], the invariant sections of powers of this sheaf descend to smooth forms on W. Counting these smooth forms, we conclude that W is of general type. □

6 Consequences of the Correlation Theorem

We give some consequences of the Correlation Theorem. Many of these are stated in §6 of [CHM]. Recall the statement of the Geometric Lang Conjecture [L]

Conjecture 6.1 (Geometric Lang Conjecture) *If W is a variety of general type, the union of all irreducible, positive dimensional subvarieties of W not of general type is a proper, closed subvariety $\Xi_W \subset W$.*

We will call Ξ_W the *Lang exceptional locus* of W. The following theorem describes how the Lang exceptional locus varies in families, if the Geometric Lang Conjecture is true.

Theorem 6.1 *Assume the Geometric Lang Conjecture.*
Let $f : X \to B$ be a flat family of surfaces in projective space, such that the

general fiber is an integral surface of general type. Then there is a uniform bound on the degree of the Lang exceptional locus of fibers that are of general type i.e.

$$deg(\Xi_{X_b}) \leq D$$

We will not prove this here. The proof is sketched in the last section of [CHM], and is similar to the proof given below for Theorem 6.2. Theorem 6.1 gives many remarkable corollaries, for example:

Corollary 6.1 *Assume the Geometric Lang Conjecture. There exists a constant D such that the sum of the degrees of all the rational and elliptic curves on a smooth quintic surface in $\mathbb{P}^3$ is less than D. In particular, there is a uniform bound on the number of rational and elliptic curves on a quintic surface.*

Recently, Abramovich [AV] has found another proof of these results.

Now we shall discuss some number theoretic consequences of the Correlation Theorem. First, recall the Weak Lang Conjecture:

Conjecture 6.2 (Weak Lang Conjecture) *If W is a variety of general type defined over a number field K, then the K rational points of W are not Zariski dense in W.*

Assuming this conjecture, the Correlation Theorem implies the following:

Theorem 6.2 *Assume the Weak Lang Conjecture.*
Let $X \to B$ be a flat family of surfaces in projective space defined over a number field K, such that the general fiber is an integral surface of general type. For any $b \in B(K)$ for which X_b is of general type, let $N(b)$ be the sum of the degrees of the components of $\overline{X_b(K)}$. Then $N(b)$ is uniformly bounded; in particular, the number of K rational points not contained in the Lang locus is uniformly bounded.

The proof uses induction on the dimension of the base B. First, we shall show that the rational points of the fibers must lie on a proper subscheme of bounded degree. Choose an integer n so that there is a dominant rational map

$$\psi : X_B^n \to W$$

to a variety of general type W. Let Y be a proper subvariety of W that contains $W(K)$, and let Z be its preimage in X_B^n. All the K rational points of X_B^n are contained in Z. We use

$$\pi_j : X_B^j \rightarrow X_B^{j-1}$$

to denote the projection morphisms. Finally, let Z_j denote the maximal closed set in X_B^j whose preimage in X_B^n is Z, and let U_j be the complement to Z_j. Note that $\pi_j^{-1}(Z_{j-1}) \subset Z_j$ by definition and that for $u \in U_{j-1}$ we have that $\pi_j^{-1}(u) \cap Z_j$ is a proper subvariety of $\pi_j^{-1}(u)$. We will use d_j to denote the sum of the degrees of all the components of $Z_j \cap \pi_j^{-1}(u)$, regardless of their dimensions, and we set

$$N = \max_j(d_j)$$

If all the K rational points of B are concentrated along a closed subset, we are done by induction. Otherwise, pick a general K rational point $b \in B$. Let j be the smallest integer for which $U_j \cap X_b^j(K)$ is empty, and let $u \in U_{j-1} \cap X_b^{j-1}(K)$. We have that $X_b = \pi_j^{-1}(u)$ and our set-up guarantees that $X_b(K) \subset Z_j \cap \pi_j^{-1}(u)$. In particular, since we have chosen everything generically, we find that $X_b(K)$ is contained in a subscheme of degree N.

Now we complete the proof. We have shown that the rational points on each fiber are concentrated along a subscheme of degree N. The components of this subscheme consist of points, rational and elliptic curves, and curves of higher genus. The rational and elliptic curves are contained in the Lang locus, so we ignore them, and there are at most N components of dimension zero. Therefore, we just need the following lemma:

Lemma 6.1 *Assume the Weak Lang Conjecture. Let C be a (possibly singular) curve in projective space of degree N defined over a number field K. Assume C has no rational or elliptic components. Then there is a uniform bound on the number of K rational points on C.*

First, because the degree is bounded there are only finitely many possibilities for the geometric genera of the components of C. By the hypothesis, these genera are all at least two, so we can apply the uniform boundedness results for curves in [CHM]. This completes the proof of the theorem. □

In the corollary which follows, *quadratic points* are points defined over some degree two extension of the base field.

Corollary 6.2 *Assume the Weak Lang Conjecture. Fix a number field K, and an integer $g > 2$. Then there is a uniform bound on the number of quadratic points lying on a non-hyperelliptic, non-bielliptic curve C of genus g defined over K.*

Note that quadratic points on C correspond to K rational points on its symmetric square $\mathrm{Sym}^2(C)$. Moreover, a hyperelliptic (respectively bielliptic) system on C corresponds to a rational (respectively elliptic) curve on $\mathrm{Sym}^2(C)$ ([AH]). In particular, the curves allowed in the corollary are precisely those for which $\Xi_{\mathrm{Sym}^2(C)} = \emptyset$, so by the theorem $\# \mathrm{Sym}^2(C)(K)$ is finite and uniformly bounded. □

References

[A] V. Alexeev. *Boundedness and K^2 for log surfaces*, Internat. J. Math 5 (1994), 779-810.

[AH] D. Abramovich, J. Harris. *Abelian varieties and curves in $W_d(C)$*. Compositio Math. 78 (1991) 227-238.

[AV] D. Abramovich, J.F. Voloch. *Lang's conjectures, fibered powers, and uniformity*, New York J. Math. 2 (1996), 20-34.

[CHM] L. Caporaso, J. Harris, B. Mazur. *Uniformity of rational points.* Preprint to appear in J. Amer. Math. Soc.

[E] R. Elkik. *Rationalité des Singularités Canoniques.* Invent. math. 64 (1981) 1-6.

[H] R. Hartshorne. *Stable reflexive sheaves.* Math. Ann. 254 (1980) 121-176.

[H2] R. Hartshorne. *Ample vector bundles.* Publ. Math. IHES 29 (1966) 63-94.

[I] S. Iitaka. Birational Geometry for Open Varieties. Sém. Math. Sup. Montreal 76, Presses de l'Université de Montreal, Montreal 1981.

[Ka] Y. Kawamata. *Semistable minimal models of threefolds in positive or mixed characteristic.* J. Algebraic Geometry 3 (1994) 463-491.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint Donat. *Toroidal Embeddings.* Lect. Notes Math. 339 (1973).

[K] J. Kollár. *Projectivity of complete moduli.* J. Differential Geometry 32 (1990) 235-268.

[K2] J. Kollár. *Subadditivity of the kodaira dimension: fibers of general type.* In: Algebraic Geometry, Sendai, 1985, T. Oda ed., Adv. Stud. in Pure Math. 10, North Holland, New York 1987, 361-398.

[K-SB] J. Kollár, N. Shepherd-Barron. *Threefolds and deformations of surface singularities.* Invent. math. 91 (1988) 299-338.

[L] S. Lang. *Hyperbolic and diophantine analysis.* Bull. Amer. Math. Soc. 14, No. 2 (1986) 159-205.

[St] J. Stevens. *On canonical singularities as total spaces of deformations.* Abh. Math. Sem. Univ. Hamburg 58 (1988) 275-283..

[V] E. Viehweg *Weak positivity and the additivity of the Kodaira dimension, II:The local Torelli map.* In: Classification of Algebraic and Analytic Manifolds, Kenji Ueno ed., Progress in Mathematics 39, Birkhäuser, Boston 1983, 567-589.

Brendan Hassett
Department of Mathematics, Harvard University
1 Oxford Street, Cambridge, MA 02138
hassett@math.harvard.edu