Special Cubic Fourfolds

Brendan Hassett *
University of Chicago

September 18, 1998

Abstract

This paper is concerned with smooth cubic hypersurfaces of dimension four (cubic fourfolds) and the surfaces contained in them. A cubic fourfold is *special* if it contains a surface which is not homologous to a complete intersection. Special cubic fourfolds form a countably infinite union of irreducible families $\mathcal{C}_d$, where each $\mathcal{C}_d$ is a divisor in the moduli space $\mathcal{C}$ of cubic fourfolds. For an infinite number of these families, the Hodge structure on the nonspecial cohomology of the cubic fourfold is essentially the Hodge structure on the primitive cohomology of a K3 surface. We say that this K3 surface is *associated* to the special cubic fourfold. For any family $\mathcal{C}_d$ of special cubic fourfolds possessing associated K3 surfaces, we discuss how $\mathcal{C}_d$ is related to the moduli space $\mathcal{N}_d$ of degree d K3 surfaces. In particular, we prove that the moduli space of cubic fourfolds contains infinitely many moduli spaces of polarized K3 surfaces as closed subvarieties. In many cases, we construct a correspondence of rational curves on the special cubic fourfold parametrized by the K3 surface, which induces the isomorphism of Hodge structures. For infinitely many values of d, the Fano variety of lines on the cubic fourfold is isomorphic to the Hilbert scheme of length two subschemes of an associated K3 surface.

Keywords: K3 surfaces, Noether-Lefschetz loci, period domains
AMS Subject Classifications: 14C30, 14J28, 14J35

*This work was supported by Harvard University, a National Science Foundation Graduate Fellowship, and a Sloan Foundation Dissertation Fellowship. This paper was revised while the author was visiting the Institut Mittag-Leffler.

1 Introduction

In this paper, we systematically study special cubic fourfolds using the techniques of Hodge theory and classical algebraic geometry. In section two, we review the Hodge theory of cubic fourfolds. In the third section, we prove basic results about special cubic fourfolds and introduce the key concept of the *discriminant*. Let X be a special cubic fourfold, h the hyperplane class on X, and T the class of an algebraic surface in X not homologous to any multiple of h^2. The discriminant d is defined as the discriminant of the saturated lattice spanned by h^2 and T. Let $\mathcal{C}_d$ denote the special cubic fourfolds of discriminant d (see § 3.2 for a precise definition). Our first main theorem is

Theorem 1.0.1 (Classification of Special Cubic Fourfolds) *(see Theorems 3.2.3 and 4.3.1) $\mathcal{C}_d$ is an* irreducible *divisor in the moduli space of cubic fourfolds and is nonempty iff $d > 6$ and $d \equiv 0, 2 (mod\ 6)$.*

In the fourth section, we give concrete descriptions of special cubic fourfolds with small discriminants. Furthermore, we explain how some Hodge structures at the boundary of the period domain arise from singular cubic fourfolds.

In the fifth section, we turn to the connections between special cubic fourfolds and K3 surfaces. The *nonspecial cohomology* of a special cubic fourfold consists of the middle cohomology orthogonal to the distinguished classes h^2 and T. In many cases, this is essentially the primitive cohomology of a K3 surface of degree d; we say that the K3 surface is *associated* to the special cubic fourfold. Furthermore, the varieties $\mathcal{C}_d$ are often closely related to moduli spaces of polarized K3 surfaces. Let $\mathcal{C}_d^{\mathrm{mar}}$ denote the marked special cubic fourfolds of discriminant d (see § 5.3 for the precise definition). This is the normalization of $\mathcal{C}_d$ if $d \equiv 2 \pmod 6$ and is a double cover of the normalization otherwise. We prove the following theorem

Theorem 1.0.2 (Associated K3 Surfaces and Maps of Moduli Spaces) *(see Theorems 5.2.1 and 5.3.4) Special cubic fourfolds of discriminant d have associated K3 surfaces iff d is not divisible by four, nine, or any odd prime $p \equiv -1 (mod\ 3)$. In these cases, there is an open immersion of $\mathcal{C}_d^{\mathrm{mar}}$ into the moduli space of polarized K3 surfaces of degree d.*

In particular, an *infinite* number of moduli spaces of polarized K3 surfaces may be realized as moduli spaces of special cubic fourfolds.

In the sixth section, we explain geometrically the existence of certain associated K3 surfaces. By definition, the *Fano variety* of a cubic fourfold X parametrizes the lines contained in X. For certain special cubic fourfolds these Fano varieties are closely related to K3 surfaces:

Theorem 1.0.3 (Geometry of Fano Varieties) *(see Theorem 6.1.5) Assume that $d = 2(n^2+n+1)$ where n is an integer ≥ 2, and let X be a generic special cubic fourfold of discriminant d. Then the Fano variety of X is isomorphic to the Hilbert scheme of length two subschemes of a K3 surface associated to X.*

More concretely, each line on the cubic fourfold corresponds to an unordered pair of points, or a point and a tangent direction, on the K3 surface. We should point out that the hypothesis on d is stronger than necessary, but simplifies the proof considerably. Combining this with the results of section five, we obtain examples of distinct K3 surfaces with isomorphic Hilbert schemes of length two subschemes (Proposition 6.2.2.)

One motivation for this work is the rationality problem for cubic fourfolds: Which cubic fourfolds are birational to $\mathbb{P}^4$? The Hodge structures on special cubic fourfolds and their relevance to rationality questions have previously been studied by Zarhin [Za]. Izadi [Iz] has also studied Hodge structures on cubic hypersurfaces with a view toward rationality questions. All the examples of cubic fourfolds known to be rational ([Fa] [Tr1] [BD] [Tr2]) are special and have associated K3 surfaces. Indeed, a birational model of the K3 surface is blown-up in the birational map from $\mathbb{P}^4$ to the cubic fourfold. One wonders whether this is the case for all rational cubic fourfolds. In a subsequent paper [Ha2], we shall apply the methods of this paper to give new examples of rational cubic fourfolds. We show there is a countably infinite union of divisors in $\mathcal{C}_8$ parametrizing rational cubic fourfolds ($\mathcal{C}_8$ corresponds to the cubic fourfolds containing a plane).

Throughout this paper we work over $\mathbb{C}$. We use the term 'generic' to mean 'in the complement of some Zariski closed proper subset.' The term 'lattice' will denote a free abelian group equipped with a nondegenerate symmetric bilinear form.

I would like to acknowledge the help I received in the course of this project. I benefitted from conversations with David Eisenbud and Elham

Izadi, and from suggestions by Walter Baily and Robert Friedman. Barry Mazur provided important insights and comments, and Johan de Jong made many helpful observations and pointed out some errors in an early version of this paper. Finally, Joe Harris introduced me to this beautiful subject; his inspiration and advice have been invaluable.

2 Hodge Theory of Cubic Fourfolds

2.1 Cohomology and the Abel-Jacobi Map

Let X be a smooth cubic fourfold. The Hodge diamond of X has the form:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array}$$

We shall focus on the middle cohomology of X, which contains all the non-trivial Hodge theoretic information. We use L to denote the cohomology group $H^4(X, \mathbb{Z})$ and L^0 to denote the primitive cohomology $H^4(X, \mathbb{Z})^0$. The intersection form on L (and L^0) is a symmetric nondegenerate bilinear form denoted $\langle , \rangle$. We sometimes refer to L as the cohomology lattice and L^0 as the primitive cohomology lattice. If h is the hyperplane class on X then $h^2 \in L$ and L^0 is the orthogonal complement to h^2.

Our best tool for understanding the middle cohomology of X is the Abel-Jacobi mapping. Let F be the Fano variety of lines of X, the subvariety of the Grassmannian $\mathbb{G}(1, 5)$ parametrizing lines contained in X. This variety is a smooth fourfold ([AK] §1). Let $Z \subset F \times X$ be the 'universal line', the variety of pairs (ℓ, x) where $x \in \ell$; let p and q be the corresponding projections:

$$\begin{array}{ccc} Z & \xrightarrow{p} & F \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

The *Abel-Jacobi map* α is defined as the map of cohomology groups

$$\alpha = p_* q^* : H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z}).$$

Let $M = H^2(F, \mathbb{Z})$ and let g be the class of the hyperplane on F (induced from the Grassmannian). Recall that $\alpha(h^2)$ corresponds to the lines meeting a codimension-two subspace of $\mathbb{P}^5$, so $\alpha(h^2) = g$. Let $M^0 \subset M$ be the primitive cohomology of F. Following [B] and [BD], we define a symmetric bilinear form $(,)$ on M as follows. We assume that g and M^0 are orthogonal with respect to this form, and we set $(g, g) = 6$ and $(x, y) = \frac{1}{6}g^2xy$ for $x, y \in M^0$. Extending by linearity we obtain an integral form on all of M, which we shall call the *Beauville canonical form.* Beauville and Donagi prove that α preserves the bilinear forms on primitive cohomology:

Proposition 2.1.1 *([BD] Prop. 6) The Abel-Jacobi map induces an isomorphism between L^0 and M^0; moreover, for $x, y \in L^0$ we have $(\alpha(x), \alpha(y)) = -\langle x, y\rangle$.*

Indeed, we may interpret α is an isomorphism of Hodge structures

$$\alpha : H^4(X, \mathbb{C})^0 \to H^2(F, \mathbb{C})^0(-1).$$

The -1 means that the weight is shifted by two; this reverses the sign of the intersection form.

We apply this to compute explicitly the middle cohomology of the cubic fourfold.

Proposition 2.1.2 *The middle integral cohomology lattice of a cubic fourfold is*

$$L \cong (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}$$

i.e. the intersection form is diagonalizable over $\mathbb{Z}$ with entries ± 1 along the diagonal. The primitive cohomology is

$$L^0 \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus H \oplus H \oplus E_8 \oplus E_8$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic plane and E_8 is the positive definite quadratic form associated to the corresponding Dynkin diagram.

We shall sometimes use the shorthand

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We first prove the statement on the full cohomology. The bilinear form on L is unimodular by Poincaré duality. By the Riemann bilinear relations, we know that the signature of L is equal to $(21, 2)$. The class $h^2 \in L$ has self intersection $\langle h^2, h^2 \rangle = h^4 = 3$, so the lattice L is odd. Using the theory of indefinite quadratic forms (e.g. [Se]) we conclude the result.

Now we turn to the primitive cohomology L^0. This lattice is not unimodular so it is harder to understand. Using the Abel-Jacobi map, we know it is the same (up to sign) as the primitive cohomology lattice M^0 of the Fano variety. We first compute the lattice M, then the lattice M^0.

In [BD] Prop. 6, it is shown that F is a deformation of a variety $S^{[2]}$, where S is a degree fourteen K3 surface. Note that $S^{[2]}$ denotes the Hilbert scheme of length two zero-dimensional subschemes of S; this is sometimes called the *blown-up symmetric square* of S. Using the results of [B], we have the canonical orthogonal decomposition

$$H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta$$

where $\delta^2 = -2$ and the restriction of $(,)$ to $H^2(S, \mathbb{Z})$ is the intersection form. Geometrically, the divisor 2δ corresponds to the nonreduced length two subschemes of S. The cohomology lattice of a K3 surface is well-known (cf. [LP] Prop. 1.2)

$$H^2(S, \mathbb{Z}) \cong \Lambda := H^{\oplus 3} \oplus (-E_8)^{\oplus 2}.$$

We conclude that

$$M \cong H^{\oplus 3} \oplus (-E_8)^{\oplus 2} \oplus (-2).$$

To compute the lattice M^0, we must first identify the polarization class g. Following [BD], we obtain that

$$g = 2f - 5\delta$$

where $f \in H^2(S, \mathbb{Z})$ satisfies $(f, f) = 14$. Let v_1 and w_1 be a basis for the first summand H, such that $(v_1, v_1) = (w_1, w_1) = 0$ and $(v_1, w_1) = 1$. The automorphisms of $H^2(S, \mathbb{Z})$ act transitively on the primitive vectors of a given nonzero length ([LP] Theorem 2.4). Hence after applying an automorphism of $H^2(S, \mathbb{Z})$, we may assume that $f = v_1 + 7w_1$ and $g = 2v_1 + 14w_1 - 5\delta$. We use $v_1 + 3w_1 - 2\delta$ and $\delta - 5w_1$ as the first two elements of a basis of M^0.

Including the other summands, we obtain

$$M^0 \cong \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus H^{\oplus 2} \oplus (-E_8)^{\oplus 2}.$$

Since $L^0 = -M^0$, this completes the proof. □
Remark: Note that our computation shows that the lattice L^0 is even.

2.2 Hodge Theory and the Torelli Map

We review Hodge theory in the context of cubic fourfolds; a general introduction to Hodge theory is [GrS]. Recall that a complete marking of a polarized cubic fourfold is an isomorphism

$$\phi : H^4(X, \mathbb{Z}) \to L$$

mapping the square of the hyperplane class to $h^2 \in L$. If we are given a complete marking ϕ, we may identify $H^4(X, \mathbb{C})^0$ with $L^0_{\mathbb{C}} = L^0 \otimes_{\mathbb{Z}} \mathbb{C}$. The complex structure on X determines a distinguished subspace $F^3(X) = H^{3,1}(X, \mathbb{C}) \subset L^0_{\mathbb{C}}$ satisfying the following properties:

1. $F^3(X)$ is isotropic with respect to the intersection form $<,>$.

2. The Hermitian form $H(u, v) = -\langle u, \bar{v} \rangle$ on $F^3(X)$ is positive.

Let $Q \subset \mathbb{P}(L^0_{\mathbb{C}})$ be the quadric hypersurface defined by the intersection form, and let $U \subset Q$ be the topologically open subset where the positivity condition holds. The open manifold U is a homogeneous space for the real Lie group $\mathrm{SO}(L^0_{\mathbb{R}}) = \mathrm{SO}(20, 2)$. This group has two components; one of them reverses the orientation on the negative definite part of $L^0_{\mathbb{R}}$. Note that the negative definite part of $L^0_{\mathbb{R}}$ coincides with $(F^3 \oplus \overline{F}^3) \cap L^0_{\mathbb{R}}$; changing the orientation corresponds to exchanging F^3 and $\overline{F}^3$ (see §6 of the appendix to [Sa] for details). Hence the two connected components of U parametrize the subspaces F^3 and $\overline{F}^3 = H^{1,3}(X)$ respectively; we denote them $\mathcal{D}'$ and $\overline{\mathcal{D}'}$. The component $\mathcal{D}'$ is a twenty-dimensional open complex manifold, called the *local period domain* for cubic fourfolds with complete marking. It is a classifying space for the polarized Hodge structures arising from these manifolds.

Let Γ denote the group of automorphisms of L preserving the intersection form and the distinguished class h^2. We let $\Gamma^+ \subset \Gamma$ denote the subgroup stabilizing $\mathcal{D}'$. This is the index-two subgroup of Γ which preserves the orientation on the negative definite part of $L^0_{\mathbb{R}}$. Γ^+ acts holomorphically on $\mathcal{D}'$ from the left; for a point in $\mathcal{D}'$ corresponding to the marked cubic fourfold (X, ϕ) the action is $\gamma(X, \phi) = (X, \gamma \circ \phi)$. The orbit space $\mathcal{D} = \Gamma^+ \backslash \mathcal{D}'$ exists as an analytic space and is called the *global period domain*. Indeed, we have the following result:

Proposition 2.2.1 *The global period domain $\mathcal{D}$ for cubic fourfolds is a quasi-projective variety of dimension twenty.*

In §6 of the appendix to [Sa], it is shown that the manifold $\mathcal{D}'$ is a bounded symmetric domain of type IV. The group Γ^+ is arithmetically defined and acts holomorphically on $\mathcal{D}'$. In this situation we may introduce the Borel-Baily compactification ([BB] §10): there exists a compactification of $\mathcal{D}'$, compatible with the action of Γ^+, so that the quotient is projective. Moreover, $\mathcal{D} = \Gamma^+ \backslash \mathcal{D}'$ is a Zariski open subvariety of this quotient. □

Let $\mathcal{C}$ denote the coarse moduli space for smooth cubic fourfolds. We digress to explain the construction of $\mathcal{C}$. The smooth cubic fourfolds form a Zariski open subset V of the projective space $\mathbb{P}^{55}$ of all cubic fourfolds. Two cubic fourfolds are isomorphic if and only if they are congruent under the action of SL_6. Using the results of [MFK] ch.4 §2, we see that the smooth cubic fourfolds are properly stable in the sense of Geometric Invariant Theory. Consequently, the quotient $\mathcal{C} := V // \mathrm{SL}_6$ exists as a quasi-projective variety and is a coarse moduli space for cubic fourfolds ([MFK] ch.1 §4). Counting parameters, we see that $\mathcal{C}$ has dimension twenty.

Each cubic fourfold determines a point in $\mathcal{D}$, and the corresponding map

$$\tau : \mathcal{C} \longrightarrow \mathcal{D}$$

is called the *period map*. By general results of Hodge theory, this is a holomorphic map of twenty-dimensional analytic spaces. For cubic fourfolds, we can say much more. First of all, we have the following result due to Voisin:

Theorem 2.2.2 *(Torelli Theorem for Cubic Fourfolds[V]) The period map for cubic fourfolds $\tau : \mathcal{C} \to \mathcal{D}$ is an open immersion of analytic spaces.*

In particular, if X_1 and X_2 are cubic fourfolds and there exists an isomorphism of Hodge structures $\psi : H^4(X_1, \mathbb{C}) \to H^4(X_2, \mathbb{C})$, then X_1 and X_2 are isomorphic.

Using the fact that $\mathcal{D}$ is the quotient of a bounded symmetric domain, we prove the following further result:

Proposition 2.2.3 *The period map for cubic fourfolds $\tau : \mathcal{C} \to \mathcal{D}$ is an algebraic map.*

To prove the claim, we use the following consequence of A. Borel's extension theorem [Bo]:

Let D' be a bounded symmetric domain, and G an arithmetically defined torsion-free group of automorphisms. Let $D = G\backslash D'$ be the quasi-projective quotient space, and Z an algebraic variety. Then any holomorphic map $Z \to D$ is algebraically defined.

We would like to apply this for $D = \mathcal{D}$, $G = \Gamma^+$, and $Z = \mathcal{C}$. Unfortunately, the group Γ^+ is not torsion-free. However, Γ^+ contains a normal subgroup H of finite index that is torsion-free ([Sa] IV Lemma 7.2). Let $\Gamma^+(N)$ denote the subgroup of Γ^+ acting trivially on L/NL. For some large N, Γ^+/H acts faithfully on L/NL so $\Gamma^+(N) \subset H$ and is torsion-free. Let $\mathcal{C}(N)$ denote the moduli space of cubic fourfolds with marked $\mathbb{Z}/N\mathbb{Z}$ cohomology, i.e. cubic fourfolds along with an isomorphism

$$\phi_n : H^4(X, \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/N\mathbb{Z}) \to L/NL$$

preserving h^2. This is a finite (and perhaps disconnected) cover of $\mathcal{C}$; we use $\mathcal{C}^0(N)$ to denote a connected component. This cover is ramified only over cubic fourfolds X possessing an automorphism which acts trivially on $H^4(X, \mathbb{Z}/N\mathbb{Z})$. Let $\mathcal{D}(N)$ be the corresponding global period domain, the quotient $\Gamma^+(N)\backslash\mathcal{D}'$; this is also finite over $\mathcal{D}$. The period map lifts to a map

$$\tau_N : \mathcal{C}^0(N) \to \mathcal{D}(N).$$

We may apply Borel's theorem to conclude τ_N is an algebraic map. By a descent argument it follows that τ itself is algebraic. This completes the proof of the proposition. □

We also obtain the following useful corollary:

Corollary 2.2.4 *The moduli space $\mathcal{C}$ of cubic fourfolds is a Zariski open subset of the global period domain $\mathcal{D}$. In particular, the complement to $\mathcal{C}$ in $\mathcal{D}$ is defined by algebraic equations.*

3 Special Cubic Fourfolds

3.1 Basic Definitions

Definition 3.1.1 *A cubic fourfold X is* special *if it contains an algebraic surface T which is not homologous to a complete intersection.*

Let X be a cubic fourfold and let $A(X)$ denote the lattice $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$. This lattice is positive definite by the Riemann bilinear relations. The Hodge conjecture is true for cubic fourfolds [Zu], so $A(X)$ is generated (over $\mathbb{Q}$) by the classes of algebraic cycles and is called the *saturated lattice of algebraic classes*. X is special if and only if the rank of $A(X)$ is at least two. Equivalently, X is special if and only if $A(X)^0 := A(X) \cap H^4(X, \mathbb{Z})^0 \neq 0$.

Definition 3.1.2 *Let $(K, \langle , \rangle)$ be a positive definite rank-two lattice containing a distinguished element h^2 with $\langle h^2, h^2 \rangle = 3$. A* marked *special cubic fourfold is a special cubic fourfold X with the data of a primitive imbedding of lattices $K \hookrightarrow A(X)$ preserving the class h^2. A* labelled *special cubic fourfold is a special cubic fourfold with the data of the image of a marking, i.e. a saturated rank-two lattice of algebraic classes containing h^2. A special cubic fourfold is* typical *if it has a unique labelling.*

We now describe the structure of special cubic fourfolds and prove that 'most' cubic fourfolds are not special. Actually, we shall prove a much more precise statement which may be found at the end of the proof.

Proposition 3.1.3 *The special cubic fourfolds form a countably infinite union of irreducible divisors in $\mathcal{C}$.*

To prove this, we need to translate the definition of 'special' into Hodge theory and use the properties of the period domain and the Torelli map. The lattice $A(X)$ is equal to $H^4(X, \mathbb{Z}) \cap (H^{3,1} \oplus H^{1,3})^{\perp}$. An integral class orthogonal to $H^{3,1}$ is automatically orthogonal to $H^{1,3} = \overline{H}^{3,1}$. Hence $A(X)$ is equal to $H^4(X, \mathbb{Z}) \cap H^{3,1}(X)^{\perp} = L \cap (F^3)^{\perp}$. In particular, X is special if and only if the rank of $L \cap F^3(X)^{\perp}$ is at least two (or if $L^0 \cap F^3(X)^{\perp}$ is nonzero).

We now characterize the points $x \in \mathcal{D}'$ corresponding to special cubic fourfolds. We say that such a Hodge structure is *special* if $L^0 \cap F^3(x)^{\perp}$ is nonzero. Let K be a rank-two positive definite saturated sublattice of L containing the class h^2 and write $K^0 = K \cap L^0$. The set of such sublattices form a countably infinite set. Recall that $\mathcal{D}'$ is an open subset of a quadric hypersurface in $\mathbb{P}(L^0)$. Let $\mathcal{D}'_K$ be the $x \in \mathcal{D}'$ such that $K^0 \subset x^\perp$; this is a hyperplane section of $\mathcal{D}'$. Each special Hodge structure in $\mathcal{D}'$ is contained in some $\mathcal{D}'_K$, so the special Hodge structures form a countably infinite union of divisors in $\mathcal{D}'$.

We now consider the geometry of the divisors $\mathcal{D}'_K$ in more detail. Let $K^\perp$ denote the orthogonal complement to K in L. We see that $\mathcal{D}'_K$ is a topologically open subset of a quadric hypersurface in $\mathbb{P}(K^\perp_{\mathbb{C}})$ and has dimension nineteen. We may think of this manifold as a classifying space for Hodge structures on the lattice $K^\perp$. As in the previous section, we can prove that $\mathcal{D}'_K$ is a bounded symmetric domain of type IV (see §6 of the appendix to [Sa]). Let Γ^+_K denote the subgroup of Γ^+ stabilizing K, i.e. $\gamma \in \Gamma^+$ such that $\gamma(K) \subset K$. This group is arithmetic and acts holomorphically on $\mathcal{D}'_K$. Again we may use the Borel-Baily compactification [BB] to show that the quotient $\Gamma^+_K \backslash \mathcal{D}'_K$ is quasi-projective. Furthermore, the induced holomorphic map $\Gamma^+_K \backslash \mathcal{D}'_K \to \Gamma^+ \backslash \mathcal{D}' = \mathcal{D}$ is algebraically defined, so $\Gamma^+_K \backslash \mathcal{D}'_K$ is mapped to an irreducible algebraic divisor in the global period domain.

We enumerate the corresponding divisors of special cubic fourfolds in the global period domain. Of course, each of these corresponds to $\Gamma^+_K \backslash \mathcal{D}'_K$ for some $K \subset L$ as above, but this K is not uniquely determined. Two such lattices K_1 and K_2 give rise to the same divisor in $\mathcal{D}$ if and only if $K_1 = \gamma(K_2)$ for some $\gamma \in \Gamma^+$. In other words, the stabilizing subgroups $\Gamma^+_{K_1}$ and $\Gamma^+_{K_2}$ must be conjugate in Γ^+. We use $[K]$ to denote the orbit of K under the action of Γ^+. We let $\mathcal{D}_{[K]}$ denote the corresponding irreducible divisor in $\mathcal{D}$.

We now complete the proof of the proposition. From the previous results on the Torelli map, we may regard $\mathcal{C}$ as a Zariski open subset of the period domain $\mathcal{D}$. By the Hodge conjecture, the locus of special cubic fourfolds corresponds to the intersection of this open set with the divisors $\mathcal{D}_{[K]}$. However, since $\mathcal{C}$ is *Zariski* open in $\mathcal{D}$, we have that $\mathcal{C}_{[K]} = \mathcal{C} \cap \mathcal{D}_{[K]}$ is an irreducible (possibly empty) algebraic divisor in $\mathcal{C}$. Hence the special cubic fourfolds are a countable union of divisors in $\mathcal{C}$. $\square$

We have actually proved the following more precise statement:

Theorem 3.1.4 (Structure of Special Cubic Fourfolds) *Let $K \subset L$ be a positive definite rank-two saturated sublattice containing h^2 and let $[K]$*

be the Γ^+ orbit of K. Let $\mathcal{C}_{[K]}$ be the special cubic fourfolds X such that $A(X) \supset K'$ for some $K' \in [K]$. Then $\mathcal{C}_{[K]}$ is an irreducible (possibly empty) algebraic divisor of $\mathcal{C}$. Every special cubic fourfold is contained in some such $\mathcal{C}_{[K]}$.

Remark: We shall use $\mathcal{D}^{\mathrm{lab}}_{[K]}$ to denote Hodge structures $x \in \mathcal{D}'$ with $K \subset H^{2,2}(x) \cap L$, modulo elements of Γ^+_K. This is called the *labelled special Hodge structures of type* $[K]$ and coincides with the quotient $\Gamma^+_K \backslash \mathcal{D}'_K$. In particular, $\mathcal{D}^{\mathrm{lab}}_{[K]}$ is a normal quasi-projective variety of dimension nineteen [BB]. The morphism $\mathcal{D}^{\mathrm{lab}}_{[K]} \to \mathcal{D}$ maps $\mathcal{D}^{\mathrm{lab}}_{[K]}$ birationally onto $\mathcal{D}_{[K]}$; a general point in $\mathcal{D}_{[K]}$ has a unique labelling. Indeed, $\mathcal{D}^{\mathrm{lab}}_{[K]}$ is the normalization of $\mathcal{D}_{[K]}$. The fiber product $\mathcal{D}^{\mathrm{lab}}_{[K]} \times_{\mathcal{D}} \mathcal{C}$ will be called the *moduli space of labelled special cubic fourfolds of type* $[K]$, denoted $\mathcal{C}^{\mathrm{lab}}_{[K]}$. The points of this variety correspond to special cubic fourfolds with the data of a rank-two saturated sublattice of $A(X)$ congruent to K.

3.2 Discriminants and Special Cubic Fourfolds

We now refine the classification worked out in the previous section by working out the orbits of the rank-two sublattices $K \subset L$ under the action of Γ^+. The following definition is the key to this computation:

Definition 3.2.1 *Let (X, K) be a labelled special cubic fourfold. The* discriminant *of the pair (X, K) is the determinant of the intersection matrix of K.*

Proposition 3.2.2 *Let (X, K) be a labelled special cubic fourfold of discriminant d and let v be a generator of $K^0 = K \cap L^0$.*

1. *$d > 0$ and $d \equiv 0, -1$ (mod 3)*
2. *$d' := \langle v, v \rangle = \begin{cases} 3d & \text{if } d \equiv -1 \text{ (mod 3)} \\ \frac{d}{3} & \text{if } d \equiv 0 \text{ (mod 3)} \end{cases}$*
3. *$\langle v, L^0 \rangle = \begin{cases} 3\mathbb{Z} & \text{if } d \equiv -1 \text{ (mod 3)} \\ \mathbb{Z} & \text{if } d \equiv 0 \text{ (mod 3)} \end{cases}$*

4. d is even

In particular, $d > 0$ *and* $d \equiv 0, 2$ *(mod 6).*

The discriminant is positive because K is positive definite. Choose a class $T \in K$ so that h^2 and T generate K over $\mathbb{Z}$. We have the formulas

$$\begin{aligned} d &= \langle h^2, h^2 \rangle \langle T, T \rangle - \langle h^2, T \rangle^2 \\ &= 3 \langle T, T \rangle - \langle h^2, T \rangle^2 \\ &\equiv -\langle h^2, T \rangle^2 \pmod 3 . \end{aligned}$$

This implies that $-d$ is a square modulo three, so d is congruent to 0 or -1 modulo three.

For the second statement, we treat the two cases separately. If d is a multiple of three then $\langle h^2, T \rangle$ is also a multiple of three. In this case, we may write the generator of K^0 as

$$v = \frac{1}{3} \langle h^2, T \rangle h^2 - T$$

so $\langle v, v \rangle = -\frac{1}{3} \langle h^2, T \rangle^2 + \langle T, T \rangle = \frac{d}{3}$. If d is congruent to -1 modulo three then we may write

$$v = \langle h^2, T \rangle h^2 - 3T$$

so $\langle v, v \rangle = -3 \langle h^2, T \rangle^2 + 9 \langle T, T \rangle = 3d$.

Now we turn to the third statement. For every primitive element $v \in L^0$ we have $\langle v, L^0 \rangle = \mathbb{Z}$ or $3\mathbb{Z}$. This is because the discriminant group $d(L^0)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (this terminology is explained in an addendum to this section). Moreover, $\langle v, L^0 \rangle = 3\mathbb{Z}$ if and only if the sublattice spanned by h^2 and v is not saturated. From the calculations above, if d is divisible by three then the sublattice spanned by h^2 and v is K, which is saturated. On the other hand, if $d \equiv -1 \pmod 3$ then the span of h^2 and v is equal to the span of h^2 and $3T$, which cannot be saturated.

Now we prove the last statement. We have computed the lattice L^0 and found it is even (Proposition 2.1.2.) Consequently, $\langle v, v \rangle$ is even and so d is also even. □

The main result of this section is:

Theorem 3.2.3 (Irreducibility Theorem) *The special cubic fourfolds possessing a labelling of discriminant d form an irreducible (possibly empty) algebraic divisor $\mathcal{C}_d \subset \mathcal{C}$.*

In Theorem 4.3.1 we determine which $\mathcal{C}_d$ are nonempty. The elements of $\mathcal{C}_d$ are sometimes called the *special cubic fourfolds of discriminant d*. The corresponding rank-two lattice of discriminant d is denoted K_d. We write $\mathcal{D}_d$ for $\mathcal{D}_{[K_d]}$, $\mathcal{D}_d^{\text{lab}}$ for $\mathcal{D}_{[K_d]}^{\text{lab}}$, $\mathcal{C}_d$ for $\mathcal{C}_{[K_d]}$, and $\mathcal{C}_d^{\text{lab}}$ for $\mathcal{C}_{[K_d]}^{\text{lab}}$.

The proof of the Irreducibility Theorem hinges on Theorem 3.1.4, which reduces it to a computation with lattices. Let K and K' be saturated rank-two sublattices of L containing h^2. When are K and K' in the same orbit under the action of Γ^+? Clearly it is necessary that K' and K' have the same discriminant. We shall prove this is also a sufficient condition:

Proposition 3.2.4 *Let K and K' be saturated rank-two nondegenerate sublattices of L containing h^2. Then $K = \gamma(K')$ for some $\gamma \in \Gamma^+$ if and only if K and K' have the same discriminant.*

To establish the proposition we prove a more precise result. By Proposition 2.1.2

$$\begin{aligned} L^0 &\cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus H \oplus H \oplus E_8 \oplus E_8 \\ &= B \oplus H \oplus H \oplus E_8 \oplus E_8. \end{aligned}$$

Let a and b be generators for B and v_1, w_1, v_2, w_2 generators for the hyperbolic planes so that

$$\langle a, a\rangle = \langle b, b\rangle = 2 \quad \langle a, b\rangle = 1 \qquad \langle v_i, v_i\rangle = \langle w_i, w_i\rangle = 0 \quad \langle v_i, w_i\rangle = 1$$

where $i = 1, 2$.

Proposition 3.2.5 (Structure of $\Gamma^+ \backslash L^0$) *Every primitive element $v \in L^0$ with $\langle v, v\rangle \neq 0$ is congruent under the action of Γ^+ to one of the following:*

1. $v_1 + nw_1$ *with* $n \neq 0$
2. $\pm(a + b) + 3(v_1 + nw_1)$

We first show how Proposition 3.2.4 follows from this. Let K and K' be lattices of the same discriminant satisfying the hypotheses of Proposition 3.2.4; let v and v' be generators of K^0 and $(K')^0$. These elements are both congruent (up to sign) to some element of the list above. Indeed, the elements $v_1 + nw_1$ correspond to discriminants $d \equiv 0 \pmod 6$ and the elements $\pm(a+b) + 3(v_1 + nw_1)$ correspond to discriminants $d \equiv 2 \pmod 6$. Hence there exists some $\gamma \in \Gamma^+$ such that $v = \pm\gamma(v')$ and $K = \gamma(K')$.

We claim that if the proposition holds for the group Γ then it also holds for the subgroup Γ^+. To prove this, it suffices to find some $g \in \Gamma \backslash \Gamma^+$ stabilizing all the elements of our list. Take g to be the identity except on the second hyperbolic plane in the orthogonal decomposition for L^0; on this component set g equal to multiplication by -1.

We should point out that the elements $a+b$ and $-(a+b)$ are not equivalent under the action of Γ. The elements of Γ fix h^2, so they act trivially on the discriminant group $d(\mathbb{Z}h^2)$. Since L^0 is the orthogonal complement of $\mathbb{Z}h^2$ we have a canonical isomorphism $d(\mathbb{Z}h^2) = d(L^0)$ [Ni] §1.5. Hence the elements of Γ act trivially on $d(L^0)$. However, $\frac{1}{3}(a+b)$ and $-\frac{1}{3}(a+b)$ yield two distinct elements of $d(L^0) \cong \mathbb{Z}/3\mathbb{Z}$, so they cannot be exchanged by elements of Γ. Note that any automorphism of L^0 that acts trivially on $d(L^0)$ extends to give an element of Γ [Ni] 1.5.1, so we may identify Γ with the group of such automorphisms of L^0.

Let K^0 denote the rank-one lattice generated by an element v satisfying $\langle v, v \rangle = d'$. Let q_{K^0} denote the quadratic form on $d(K^0)$ and let q denote the quadratic form on $d(L^0)$. The lattice L^0 is the unique even lattice of signature $(20, 2)$ with discriminant quadratic form q [Ni] 1.14.3. Furthemore, any saturated codimension-one sublattice $K^\perp \subset L^0$ is the orthogonal complement of a rank-two sublattice $K \subset L$. Hence there is a canonical isomorphism of discriminant groups $d(K^\perp) = d(K)$, and the discriminant group of $K^\perp$ is generated by at most two elements. This implies the isomorphism class of $K^\perp$ is determined by its signature and discriminant form, and any isomorphism of $d(K^\perp)$ preserving the discriminant quadratic form is induced by an automorphism of $K^\perp$ [Ni] 1.14.3.

Our goal is to classify the primitive imbeddings of K^0 into L^0, up to automorphisms of L^0 acting trivially on $d(L^0)$. Two imbeddings differing only by such an automorphism are said to be *congruent*. Applying the results of [Ni] §1.15 in our situation, we obtain a characterization of the congruence classes of primitive imbeddings of $i : K^0 \to L^0$:

Primitive imbeddings $i : K^0 \to L^0$ correspond to the following data:
1)A subgroup $H_q \subset d(L^0)$.
2)A subgroup $H_{K^0} \subset d(K^0)$.
3)An isomorphism $\phi : H_{K^0} \to H_q$ preserving the restrictions of the quadratic forms to these subgroups, with graph $\Gamma_\phi = \{(h, \phi(h)) : h \in H_{K^0}\} \subset d(K^0) \oplus d(L^0)$.
4)An even lattice $K^\perp$ with complementary signature and discriminant form $q_{K^\perp}$, and an isomorphism $\phi_{K^\perp} : q_{K^\perp} \to -\delta$, where $\delta = ((q_{K^0} \oplus -q)|\Gamma_\phi^\perp)/\Gamma_\phi$ (and $\Gamma_\phi^\perp$ is the orthogonal complement to Γ_ϕ with respect to $q_{K^0} \oplus q$).

Consider another imbedding with data $(H'_q, H'_{K^0}, \phi', (K')^\perp, \phi_{(K')^\perp})$. These two imbeddings are congruent if and only if $H_{K^0} = H'_{K^0}$ and $\phi = \phi'$.

Our proof now divides into cases.

Case I: $H_q = \{0\}$
This condition is equivalent to stipulating that $\langle i(K^0), L^0)\rangle = \mathbb{Z}$. Using the characterization above, we see that under these conditions all imbeddings of K^0 are congruent. In particular, any primitive $v \in L^0$ with $\langle v, L^0\rangle = \mathbb{Z}$ is congruent to $v_1 + \frac{d'}{2} w_1$.

Case II: $H_q = d(L^0) \cong \mathbb{Z}/3\mathbb{Z}$
This is equivalent to stipulating that $\langle i(K^0), L^0)\rangle = 3\mathbb{Z}$. In this case, $d(K^0)$ has a subgroup H_{K^0} of order three and $3|d'$. There are two possible isomorphisms between $d(L^0)$ and H_{K^0}, thus two classes of imbeddings of K^0 into L^0. In particular, any primitive $v \in L^0$ with $\langle v, L^0\rangle = 3\mathbb{Z}$ is congruent to $\pm(a + b) + 3(v_1 + nw_1)$. □

At this point, it is convenient to compute the discriminant quadratic forms of the lattices $K_d^\perp$, the orthogonal complements to K_d in L.

Proposition 3.2.6 *If $d \equiv 0$ (mod 6) then $d(K_d^\perp) \cong \mathbb{Z}/\frac{d}{3}\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, which is cyclic unless nine divides d. Furthermore, we may choose this isomorphism so that $q_{K_d^\perp}(0, 1) = \frac{2}{3}$ (mod $2\mathbb{Z}$) and $q_{K_d^\perp}(1, 0) = -\frac{3}{d}$ (mod $2\mathbb{Z}$).*

If $d \equiv 2$ (mod 6) then $d(K_d^\perp) \cong \mathbb{Z}/d\mathbb{Z}$. Furthermore, we may choose a generator u so that $q_{K_d^\perp}(u) = \frac{2d-1}{3d}$ (mod $2\mathbb{Z}$).

We retain the notation used in the previous proof so in particular $K^0 \cong K_d^0$. First, $d(L^0) \cong \mathbb{Z}/3\mathbb{Z}$ and q takes the value $\frac{2}{3}$ (mod $2\mathbb{Z}$) on any generator. By

Proposition 3.2.2, $d(K_d^0) \equiv \mathbb{Z}\frac{v}{d'}/\mathbb{Z}v \cong \mathbb{Z}/d'\mathbb{Z}$ where

$$d' = \begin{cases} 3d & \text{if } d \equiv 2 \pmod 6 \\ \frac{d}{3} & \text{if } d \equiv 0 \pmod 6 \end{cases};$$

the quadratic form $q_{K_d^0}$ takes the value $\frac{1}{d'} \pmod{2\mathbb{Z}}$ on the generator.

The first part of the proposition corresponds to the first case of the previous proof. In particular, $d = 3d'$ and Γ_ϕ is trivial. Applying the characterization of imbeddings cited above, we find that $d(K_d^\perp) = d(K_d^0) \oplus d(L^0) \cong \mathbb{Z}/d'\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. By the Chinese remainder theorem, this is cyclic of order d unless d' is divisible by three. Furthermore, $q_{K_d^\perp} = -q_{K_d^0} \oplus q$ so we may compute the values given above.

The second part of the proposition corresponds to the case where $H_q = d(L^0) \cong \mathbb{Z}/3\mathbb{Z}$, so in particular $d' = 3d$. We identify $d(K_d^0) \oplus d(L^0) \cong \mathbb{Z}/d'\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ so that Γ_ϕ is generated by $(\frac{d'}{3}, 1)$. We then have that $\Gamma_\phi^\perp$ is generated by $u = (1, -1)$ and so $d(K_d^\perp) = \Gamma_\phi^\perp / \Gamma_\phi \cong \mathbb{Z}/\frac{d'}{3}\mathbb{Z}$. Since $q_{K_d^\perp} = -q_{K_d^0} \oplus q$ we find that $q_{K_d^\perp}(u) = \frac{2}{3} - \frac{1}{d'} = \frac{2d-1}{3d} \pmod{2\mathbb{Z}}$. $\square$

Addendum: Discriminant Groups

Here we give an informal introduction to discriminant forms, to acquaint the reader with the basic ideas and notation. See [Ni] or [Do] for more details and proofs. The precise results we use are stated as they are applied.

Let $\mathcal{L}$ be a lattice with bilinear form $\langle , \rangle$. We use $\mathcal{L}^*$ to denote the group $\mathrm{Hom}(\mathcal{L}, \mathbb{Z})$. Assume $\mathcal{L}$ is nondegenerate, so the bilinear form $\langle , \rangle$ induces an inclusion $i : \mathcal{L} \hookrightarrow \mathcal{L}^*$. By definition, $\mathcal{L}$ is unimodular if and only if i is an isomorphism. More generally the cokernel of i is a finite abelian group, called the *discriminant group* of $\mathcal{L}$ and denoted $d(\mathcal{L})$.

The discriminant group $d(\mathcal{L})$ comes with some additional structure. The bilinear form $\langle , \rangle$ extends by linearity to a $\mathbb{Q}$-valued bilinear form on $\mathcal{L}^*$. Let $b_\mathcal{L}$ be the reduction of the bilinear form modulo $\mathbb{Z}$, i.e. $b_\mathcal{L} \equiv \langle , \rangle \pmod{\mathbb{Z}}$. Then $\mathcal{L}$ is isotropic for $b_\mathcal{L}$ and $b_\mathcal{L}$ gives a well defined $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on $d(\mathcal{L})$. Now assume further that $\mathcal{L}$ is even and let $q_\mathcal{L}$ be the reduction of the *quadratic form* modulo $2\mathbb{Z}$. Again $\mathcal{L}$ is isotropic for $q_\mathcal{L}$ and so $q_\mathcal{L}$ gives a well-defined $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form on $d(\mathcal{L})$, called the *discriminant quadratic form.*

4 Examples

4.1 Special Cubic Fourfolds with Small Discriminants

In this subsection, we give some examples of special cubic fourfolds. Specifically, for small values of d we describe surfaces contained in a general cubic fourfold of $\mathcal{C}_d$. In general, it is hard to prove that the closure of the cubic fourfolds containing a given type of surface form a divisor; the $d = 12$ case indicates some of the difficulties which arise.

First, we make some preliminary computations with Chern classes. Let X be a cubic fourfold containing a surface T and assume

$$K_d = \mathbb{Z}h^2 + \mathbb{Z}T.$$

To compute the discriminant d in practice, we must first compute the self-intersection $\langle T, T\rangle$ on X. We interpret $\langle T, T\rangle$ as $c_2(\mathcal{N}_{T/X})$, the highest Chern class of the normal bundle to T in X. Letting $h = h|_T$ we find that $c_t(\mathcal{T}_X|_T) = 1 + 3ht + 6h^2t^2$ and

$$c_t(\mathcal{T}_T) = 1 - K_T t + \chi_T t^2$$

where χ_T denotes the topological Euler characteristic of T. Using the exact sequence

$$0 \to \mathcal{T}_T \to \mathcal{T}_X|_T \to \mathcal{N}_{T/X} \to 0$$

we conclude that

$$\langle T, T\rangle = c_2(\mathcal{N}_{T/X}) = 6h^2 + 3hK_T + K_T^2 - \chi_T.$$

4.1.1 d=8: Cubic Fourfolds Containing a Plane

Let X be a smooth cubic fourfold containing the plane P. These cubic fourfolds are used by Voisin in her proof of the Torelli theorem [V]. By the results of the previous paragraph $\langle P, P\rangle = 3$ and X is special with marking

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}$$

which has discriminant eight. In fact, every cubic fourfold in $\mathcal{C}_8$ (cf. Theorem 3.2.3) contains a plane [V] §3. The cubic fourfolds in $\mathcal{C}_8$ contain many other familiar surfaces:

1. Quadric surfaces: the intersection of X with two hyperplanes containing P consists of the union of P and a quadric surface Q.

2. Quartic del Pezzo surfaces: the intersection of X with a quadric *three-fold* containing Q consists of the union of Q and a quartic del Pezzo surface W.

3. Octic K3 Surfaces

One can continue in this way to obtain many different surfaces linked to the plane P.

4.1.2 d=12: Cubic Fourfolds Containing a Cubic Scroll

Let X be a smooth cubic fourfold containing a rational normal cubic scroll T. We have $\langle T, T\rangle = 7$ and X is special with marking

$$K_{12} = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 3 \\ T & 3 & 7 \end{array}$$

which has discriminant twelve.

Lemma 4.1.1 *Let X be a general cubic fourfold containing a rational normal cubic scroll T. Then the scrolls in X rationally equivalent to T form a two parameter family.*

First, we construct a two parameter family of cubic scrolls rationally equivalent to T in X. Let $H \subset X$ denote the hyperplane section containing T. The ideal of T in $\mathbb{P}^4$ is generated by a net of quadrics. If Q is one of these quadrics then

$$Q \cap H = T \cup T'$$

where T' is residual to T. For general Q, T' is again a cubic scroll cut out by a net of quadrics Q'_s. For each $s \in \mathbb{P}^2$, we have

$$Q'_s \cap H = T' \cup T_s$$

where T_s is a cubic scroll rationally equivalent to T in X. This gives a two parameter family of deformations of T, all of which are contained in H.

We prove that all deformations of T are of this form. If $\tilde{T}$ is an arbitrary deformation of T then $\tilde{T} \cap T' = -1$, so T' and $\tilde{T}$ meet in a nondegenerate curve $C \subset H$. Since $\tilde{T}$ must also be a cubic scroll, we conclude that $\tilde{T}$ is contained in H. We claim that the scrolls contained in H form a union of two parameter families. This is because the equation of a cubic *threefold* containing a cubic scroll is the determinant of a 3×3 matrix. Such an equation generally has six double points (where the matrix has rank one), and a general cubic threefold with six double points can be represented in this way. The Hilbert scheme of cubic scrolls in $\mathbb{P}^4$ has dimension 18, and each scroll is contained in a twelve-dimensional system of cubic threefolds. However, the locus of cubic threefolds with six double points has dimension 28, so generally such a cubic threefold contains a two parameter family of cubic scrolls. This proves the claim and the lemma. □

Now we count the dimension of the cubic fourfolds containing cubic scrolls. Consider pairs of scrolls and cubic fourfolds containing them

$$\mathcal{W} = \{(X, T) : T \subset X\}.$$

The Hilbert scheme of cubic scrolls in $\mathbb{P}^5$ is irreducible of dimension $5 + 24 - 6 = 23$. Any given scroll is contained in a projective space of cubic hypersurfaces of dimension $55 - 22 = 33$, so $\mathcal{W}$ is irreducible of dimension 56. However, each cubic fourfold containing a cubic scroll contains a two parameter family of such scrolls, so the cubic fourfolds containing a cubic scroll form a divisor in $\mathcal{C}$. This coincides with $\mathcal{C}_{12}$.

4.1.3 d=14: Cubic Fourfolds Containing a Quartic Scroll/Pfaffian Cubic Fourfolds

These have been extensively studied by many people, including Fano [Fa], Morin [Mo], Tregub [Tr1], and Beauville-Donagi [BD]. The degenerate case of cubic fourfolds containing two disjoint planes is discussed in § 6.1.

Let X be a cubic fourfold containing a rational normal quartic scroll T. We have $\langle T, T\rangle = 10$ so our marking is

$$K_{14} = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 4 \\ T & 4 & 10 \end{array}$$

which has discriminant fourteen. We show that X also contains quintic del Pezzo surfaces and quintic rational scrolls. Let $\Sigma = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ denote a Segre threefold; this is imbedded by the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1)$. Divisors of type $(1,2)$ in Σ are generally quintic del Pezzo surfaces and divisors of type $(3,1)$ are quintic rational normal scrolls. Furthermore, divisors of type $(2,1)$ and $(0,2)$ are quartic rational normal scrolls. Hence there exist Segre threefolds $\Sigma_1, \Sigma_2 \supset T$ such that

$$\begin{array}{rcll} \Sigma_1 \cap X & = & T \cup W & \text{a quintic del Pezzo surface} \\ \Sigma_2 \cap X & = & T \cup T_5 & \text{a quintic rational scroll} \end{array}$$

One can show that the quartic scrolls, quintic scrolls, and quintic del Pezzos on X generally form families of dimensions two, two, and five respectively. A dimension count shows that the cubic fourfolds containing any of these surfaces forms a divisor $\mathcal{C}_{14} \subset \mathcal{C}$. (Note that Morin uses a spurious parameter count to conclude that the quartic scrolls form a *one*-dimensional family. From this, he concludes incorrectly that *every* cubic fourfold contains a quartic scroll.)

We should point out another description of an open subset of $\mathcal{C}_{14}$: the Pfaffian cubic fourfolds constructed by Beauville and Donagi [BD]. The dimension counts in the last paragraph follow from their results. Beauville and Donagi also show that the Pfaffian cubic fourfolds are rational. From our point of view this is not hard to see. Let $W \subset X$ be a quintic del Pezzo surface and consider the linear system of quadric hypersurfaces containing W. This linear system induces a birational morphism $f : \mathrm{Bl}_W(X) \to \mathbb{P}^4$. The inverse f^{-1} is obtained by the linear system of quartics passing though a degree-nine surface $\tilde{S} \subset \mathbb{P}^4$. The surface $\tilde{S}$ is the projection of a degree-fourteen K3 from five collinear points.

4.1.4 d=20: Cubic Fourfolds Containing a Veronese

Let X contain a Veronese surface V, which is isomorphic to $\mathbb{P}^2$ imbedded by the line bundle $\mathcal{O}_{\mathbb{P}^2}(+2)$. We have that $\langle V, V \rangle = 12$ so the marking is

$$K_{20} = \begin{array}{c|cc} & h^2 & V \\ \hline h^2 & 3 & 4 \\ V & 4 & 12 \end{array}$$

which has discriminant twenty. A general cubic fourfold containing a Veronese contains only one Veronese, so a dimension count implies that the cubic fourfolds containing a Veronese form a divisor in $\mathcal{C}$. This is the divisor $\mathcal{C}_{20}$.

4.2 d=6: Cubic Fourfolds with an Ordinary Double Point

A double point is *ordinary* if its projectivized tangent cone is a smooth quadric. Cubic hypersurfaces in $\mathbb{P}^5$ with an ordinary double point are stable in the sense of Geometric Invariant Theory. This is proved using Mumford's numerical criterion for stability (cf [MFK] §2.1) and the methods of ([MFK] §4.2). We conclude there exists a quasi-projective variety parametrizing cubic fourfolds with (at worst) a single ordinary double point. This is denoted $\tilde{\mathcal{C}} \supset \mathcal{C}$.

Let X_0 be a cubic fourfold with a single ordinary double point p. Then projection from the point p gives a birational map

$$\pi_p : X_0 \dashrightarrow \mathbb{P}^4.$$

This map can be factored

$$\begin{array}{ccc} \overline{X_0} = \mathrm{Bl}_S(\mathbb{P}^4) & \xrightarrow{q_1} & X_0 \\ \downarrow{\scriptstyle q_2} & & \\ \mathbb{P}^4 & & \end{array}$$

The map q_1 is the blow-up of the double point p and q_2 is the blow-down of the lines contained in X_0 passing through p. These lines are parametrized by a surface S, which we describe explicitly. Choose coordinates $x_1, \ldots, x_5$ so that $p = (0,0,0,0,0)$. Then the equation for X_0 takes the form

$$f = f_2(x_1, \ldots, x_5) + f_3(x_1, \ldots, x_5) = 0$$

where f_2 and f_3 are homogeneous of degrees two and three. Note that f_2 is nonsingular because p is ordinary. The lines through p correspond to points of the complete intersection $f_2 = f_3 = 0$ in $\mathbb{P}^4$. This complete intersection must be smooth, because p is the only singularity of X_0.

We see that the lines through the double point p are parametrized by the smooth complete intersection of a smooth quadric and a cubic in $\mathbb{P}^4$. This surface is a sextic K3 surface. The inverse map π_p^{-1} is given by the linear system of cubic polynomials through this K3 surface. Indeed, for any such K3 surface the image of $\mathbb{P}^4$ under this linear system is a cubic fourfold with an ordinary double point. The map q_2 is the blow-up of S and q_1 is the blow-down of the quadric hypersurface containing S. Thus there is a one-to-one correspondence between:

1. points of $\tilde{\mathcal{C}}\backslash\mathcal{C}$, i.e. cubic fourfolds with a single ordinary double point
2. smooth complete intersections of a smooth quadric and a cubic in $\mathbb{P}^4$, modulo automorphisms of $\mathbb{P}^4$.

Almost all smooth sextic K3 surfaces in $\mathbb{P}^4$ can be represented in this way. The only exceptions are the sextic K3 surfaces containing a cubic plane curve; they are contained in a singular quadric hypersurface. Note that these correspond to cubic fourfolds for which the double point has singular tangent cone.

This construction suggests that we associate a sextic K3 surface to any cubic fourfold in $\tilde{\mathcal{C}}\backslash\mathcal{C}$. The following proposition explains how this can be done:

Proposition 4.2.1 *The Torelli map extends to an open immersion*

$$\tilde{\tau} : \tilde{\mathcal{C}} \to \mathcal{D}.$$

The closed set $\tilde{\mathcal{C}}_6 := \tilde{\mathcal{C}}\backslash\mathcal{C}$ *is mapped into* $\mathcal{D}_6$.

In § 5.3 we shall show that $\mathcal{D}_6$ coincides with the period domain for sextic K3 surfaces. A detailed proof of the proposition is given in §4 of [V], so we merely explain some details needed for our calculations. (This proposition also follows from the delicate analysis of singular cubic fourfolds in §6.3.) Let X_0 be a cubic fourfold with an ordinary double point and let S be the associated K3 surface. Smoothings of ordinary double points of even codimension have monodromy satisfying $T^2 = I$. Thus any smoothing of X_0 yields a limiting mixed Hodge structure H^4_{lim} which is actually pure. The corresponding point of the period domain is denoted $\tilde{\tau}(X_0)$. The limiting Hodge structure may be computed with the Clemens-Schmid exact sequence [Cl]. We make some observations about this Hodge structure. The desingularization of X_0 is obtained by blowing-up the K3 surface S. This induces a natural imbedding

$$H^2(S, \mathbb{C})^0(-1) \hookrightarrow H^4_{\text{lim}}$$

where the left hand side denotes the primitive cohomology of S with the sign of the intersection form reversed. The orthogonal complement to the image of this map consists of a rank-two lattice of integral $(2,2)$ classes

$$K_6 = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 0 \\ T & 0 & 2 \end{array}$$

which implies that $\tilde{\tau}(X_0) \in \mathcal{D}_6$.

4.3 Existence of Special Cubic Fourfolds of Discriminant d

In this section, we determine the values of d for which $\mathcal{C}_d$ is nonempty. The divisor $\mathcal{D}_d \subset \mathcal{D}$ is nonempty if and only if d is positive and congruent to $0, 2 \pmod 6$, so we restrict to these values of d. Here we prove the following

Theorem 4.3.1 (Existence of Special Cubic Fourfolds) *Let $d > 6$ be an integer with $d \equiv 0, 2$ (mod 6) . Then the divisor $\mathcal{C}_d$ is nonempty.*

In other words, there are labelled special cubic fourfolds with all possible discriminants besides two and six. We saw in the last section why there are no cubic fourfolds of discriminant six: $\mathcal{D}_6$ corresponds to the limiting Hodge structures arising from cubic fourfolds with double points. In the next section we shall explain why there are no cubic fourfolds of discriminant two: $\mathcal{D}_2$ corresponds to the limiting Hodge structures arising from another class of singular cubic fourfolds. We can speculate a bit on the complement of the moduli space $\mathcal{C}$ in the period domain $\mathcal{D}$. The theorem and examples above, along with some experimental evidence, suggests the following guess:

> Is the complement of the moduli space $\mathcal{C}$ in the period domain $\mathcal{D}$ equal to the union of the divisors $\mathcal{D}_2$ and $\mathcal{D}_6$?

Now we turn to the proof of the theorem; we use a deformation argument. Fix an integer d satisfying the conditions of the theorem. First, we describe singular cubics X_0 in $\tilde{\mathcal{C}}_6$ such that

$$\tilde{\tau}(X_0) \in \mathcal{D}_d.$$

Then we construct a smoothing $\phi : \mathcal{X} \to \Delta$, where X_t is smooth for $t \neq 0$ and $\tau(X_t) \in \mathcal{D}_d$. In other words, X_0 can be smoothed to a special cubic fourfold with discriminant d. In particular, this proves that $\mathcal{C}_d$ is not empty.

We need the following lemma:

Lemma 4.3.2 *Let P be an indefinite even rank-two lattice representing six. Assume that P is not isomorphic to any of the following:*

$$\begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} \quad \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}.$$

Then there exists a smooth sextic K3 surface S lying on a smooth quadric with $\mathrm{Pic}(S) \cong P$.

In proving this, we shall use the following more general lemma:

Lemma 4.3.3 *Let P be a rank-two indefinite even lattice, $f \in P$ a primitive element with $d := f^2 > 0$, and assume there is no $E \in P$ with $E^2 = -2$ and $fE = 0$. Then there exists a K3 surface S with* $\mathrm{Pic}(S) = P$ *and f a polarization on S. Moreover, f is very ample unless there exists an elliptic curve C on S with $C^2 = 0$ and $fC = 1$ or 2.*

Recall that Λ denotes the lattice isomorphic to the middle cohomology of a K3 surface. Using the results of §2 of [LP], there exists an imbedding $P \hookrightarrow \Lambda$. So for some elements of the period domain P is contained in the lattice of $(1,1)$-classes. The surjectivity of the period map for K3 surfaces implies the existence of a K3 surface S with Picard group P so that f contained in the Kähler cone of S (see pp. 127 of [B2]). This implies f is a polarization of S.

To complete the proof, we apply results for linear systems on K3 surfaces proved in [SD]; the references below are to this paper. We analyze the linear system $|f|$. First we prove that $|f|$ is base point free. If $|f|$ has base points, then they are necessarily contained in some fixed component of f (Theorem 3.1). This fixed component is supported on a -2 curve E and f is necessarily of the form $\frac{d+2}{2}C + E$, where C is an elliptic curve such that $C^2 = 0$ and $EC = 1$ (§2.7). But then $fC = 1$, which is excluded by the hypotheses. Now we consider the morphism obtained from the linear series $|f|$. This morphism is an isomorphism provided it is birational (Theorem 6.1). It fails to be birational only if all the curves in $|f|$ are hyperelliptic. This happens only if there exists an elliptic curve C with $fC = 2$ (Theorem 5.2). This again is excluded by the hypotheses. □

To complete the proof of the main lemma, we note that the image under $|f|$ is not contained in a singular quadric because $P \ncong \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$. □

We use this lemma to construct singular cubics $X_0 \in \tilde{\mathcal{C}}_6$ such that $\tilde{\tau}(X_0) \in \mathcal{D}_d$. Let S be one of the K3 surfaces constructed in the lemma, and let X_0 be the corresponding singular cubic fourfold. Let $v \in P = \mathrm{Pic}(S)$ be primitive with respect to the degree six polarization, i.e. $v \in H^2(S,\mathbb{Z})^0 \cap H^{1,1}(S)$. Recall that $H^2(S,\mathbb{C})^0(-1)$ is naturally imbedded into the limiting Hodge structure $H^4_{\lim}$ arising from X_0. The image of v under this map is an integral class of type $(2,2)$ in $H^4_{\lim}$, denoted v'. We use v' to relabel $H^4_{\lim}$ by letting K_d denote the saturation of the lattice $\mathbb{Z}h^2 + \mathbb{Z}v'$. Using Proposition 3.2.2, we may compute

$$\begin{aligned} d &= \begin{cases} 1/3\,\langle v',v'\rangle & \text{if } < v', H^4(X,\mathbb{Z})^0 >= 3\mathbb{Z} \\ 3\,\langle v',v'\rangle & \text{if } < v', H^4(X,\mathbb{Z})^0 >= \mathbb{Z} \end{cases} \\ &= \begin{cases} -1/3\,(v,v) & \text{if } (v, H^2(S,\mathbb{Z})^0) = 3\mathbb{Z} \\ -3\,(v,v) & \text{if } (v, H^2(S,\mathbb{Z})^0) = \mathbb{Z} \end{cases} \\ &= -\frac{1}{2}\,\mathrm{disc}(P). \end{aligned}$$

For each $d > 6$ with $d \equiv 0, 2 \pmod 6$, there exist lattices P satisfying the conditions of the lemma with discriminant $-2d$. If $d = 6n$ we may take

$$P = \begin{pmatrix} 6 & 0 \\ 0 & -2n \end{pmatrix}$$

and if $d = 6n + 2$ we may take

$$P = \begin{pmatrix} 6 & 2 \\ 2 & -2n \end{pmatrix}.$$

This completes the construction.

We assume that X_0 is the singular cubic constructed in the previous paragraph, $x_0 = \tilde{\tau}(X_0)$, and $x_0 \in \mathcal{D}_6 \cap \mathcal{D}_d$. To complete the proof, we construct a smoothing $\phi : \mathcal{X} \to \Delta'$ where X_t is smooth for $t \neq 0$, and $\tau(X_t) \in \mathcal{D}_d$. Let

$\gamma : \Delta \to \mathcal{D}$ be a holomorphic map such that $\gamma(0) = x_0$ and $\gamma(u) \in \mathcal{D}_d \backslash \mathcal{D}_6$ for $u \neq 0$. The existence of such a curve follows from the construction of $\mathcal{D}$ as the quotient $\Gamma^+ \backslash \mathcal{D}'$. Because $\tilde{\tau}$ is an open immersion, we may shrink Δ so that γ lifts through $\tilde{\tau}$, giving a map $\mu : \Delta \to \tilde{\mathcal{C}}$. Consequently, there exists a ramified base change $b : \Delta' \to \Delta$ and a family $\mathcal{X} \to \Delta'$ so that $X_t = \mu(b(t))$. By construction we have

$$X_t \in \mathcal{C} \cap \tau^{-1}(\mathcal{D}_d) = \mathcal{C}_d$$

for $t \neq 0$. In particular, $\mathcal{C}_d$ is nonempty. □

4.4 d=2: The Determinantal Cubic Fourfold

The results of the previous sections allow us to exhibit (possibly singular) cubic fourfolds of all discriminants greater that two. In this section, we address the remaining case where the discriminant is equal to two. We shall not exhibit cubic fourfolds of discriminant two; indeed, there are no such smooth cubic fourfolds. However, we can explain how the Hodge structures parametrized by the divisor $\mathcal{D}_2 \subset \mathcal{D}$ arise as limiting Hodge structures of smooth cubic fourfolds.

To this end, we introduce the *determinantal cubic fourfold* X_0, defined by the homogeneous equation:

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} = 0.$$

The determinantal cubic fourfold is singular along the locus where the 2×2 minors of the determinant are simultaneously zero. These minors are precisely the equations cutting out a Veronese surface $V \subset \mathbb{P}^5$, the image of $\mathbb{P}^2$ under the linear system $|\mathcal{O}_{\mathbb{P}^2}(+2)|$. We shall consider deformations $\mathcal{X} \to \Delta$ of X_0 with equations

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} + tG = 0$$

where G is the equation of a smooth cubic. We assume that the curve $C \subset V$ defined by the equation $G|_V = 0$ is also smooth. The curve C is a sextic plane curve, and we let S denote the double cover of V branched over the curve C, a degree-two K3 surface. We have the following theorem:

Theorem 4.4.1 *Let $\mathcal{X} \to \Delta$ be a deformation of the determinantal cubic fourfold X_0, satisfying the conditions above. Then the limiting mixed Hodge structure H^4_{lim} of this family is pure and special of discriminant two. The primitive Hodge structure $H^2(S,\mathbb{C})^0(-1)$ imbeds into the orthogonal complement of the rank-two lattice K_2.*

Geometrically, the determinantal cubic is contained in the indeterminacy locus of the Torelli map, but after blowing up this point the Torelli map is well-defined, at least at the generic point of the exceptional divisor. Moreover, this exceptional divisor maps birationally to the divisor $\mathcal{D}_2 \subset \mathcal{D}$.

The proof of the theorem is essentially a calculation with the Clemens-Schmid exact sequence. We begin by computing the semistable reduction for $\mathcal{X}$:

Lemma 4.4.2 *A semistable reduction $\mathcal{X}' \to \Delta'$ of $\mathcal{X} \to \Delta$ is obtained from the following operations:*

1. *Take the base change $\Delta' \to \Delta$ given by the formula $t = u^2$.*
2. *Blow-up the subvariety $V \subset \mathcal{X} \times_\Delta \Delta'$.*

The equation of the base-changed family $\mathcal{X} \times_\Delta \Delta' \subset \mathbb{P}^5_{\Delta'}$ is

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} + u^2 G = 0.$$

The total space of $\mathcal{X} \times_\Delta \Delta'$ is now singular along the Veronese V in the central fiber. We let $\mathcal{X}'$ be the blow-up of $\mathcal{X} \times_\Delta \Delta'$ along V, E the exceptional divisor, and $\overline{X_0}$ the proper transform of X_0. We claim that $\mathcal{X}'$ is now smooth, and that E and $\overline{X_0}$ intersect transversally along the smooth threefold $E_0 = \overline{X_0} \cap E$.

The variety $\mathcal{X} \times_\Delta \Delta'$ is double along V, a codimension-three subvariety. Using the equation above we find that for $x \in V \backslash C$, the projectivized tangent cone to $\mathcal{X} \times_\Delta \Delta'$ at x is a cone over a smooth quadric surface. At points $x \in C$ this quadric surface acquires an ordinary double point. Codimension-three double points with these tangent cones are resolved by a single blow-up, so blowing-up V resolves the singularities of $\mathcal{X} \times_\Delta \Delta'$. The exceptional divisor

$q : E \to V$ has the structure of a quadric surface bundle, and the fibers over C have ordinary double points. Thus E is smooth and reduced in the central fiber of $\mathcal{X}'$.

Now we analyze the proper transform of X_0. The fourfold X_0 is double along the codimension-two subvariety V, which parametrizes the matrices of rank one. The proper transform $\overline{X_0}$ is the blow-up of X_0 along V, with exceptional divisor E_0. The fibers of $E_0 \to V$ have a natural interpretation: for each $M \in V$, the fiber over M is the projectivization of the kernel of the corresponding rank-one matrix. This implies that E_0 is a $\mathbb{P}^1$ bundle over V. We may interpret $\overline{X_0}$ as the set of pairs (M, x), where M is a symmetric 3×3 matrix and $x \in \mathbb{P}(\ker(M))$. In particular, $\overline{X_0}$ is smooth and meets E along the smooth subvariety E_0. It is not hard to see that this intersection is transverse.□

The next lemma gives a more precise description of the relationship between E and E_0:

Lemma 4.4.3

1. *The quadric surface bundle $q : E \to V$ can be imbedded into a $\mathbb{P}^3$ bundle*

$$\begin{array}{ccc} E & \hookrightarrow & \mathbb{P}(N) \\ & q \searrow & \downarrow \pi \\ & & V \end{array}$$

   where N is the normal bundle to V in $\mathbb{P}^5_{\Delta'}$.

2. *The class of E in* $\mathrm{Pic}(\mathbb{P}(N))$ *is* $2\xi + 6\eta$, *where* $\xi = c_1(\mathcal{O}_{\mathbb{P}(N)}(+1))$ *and* $\eta = c_1(\pi^*\mathcal{O}_{\mathbb{P}^2}(+1))$.

3. *The class of E_0 in* $\mathrm{Pic}(E)$ *is just* $\xi|_E$. *In other words, E_0 is a hyperplane section to E in $\mathbb{P}(N)$.*

Let $\mathcal{Y}$ denote the blow-up of $\mathbb{P}^5_{\Delta'}$ along V, and let $\mathcal{E}$ denote the exceptional divisor. Since $\mathcal{Y}$ is a blow-up along a smooth center, we have that $\mathcal{E} = \mathbb{P}(N)$ is a $\mathbb{P}^3$ bundle over V. We interpret $\mathcal{X}'$ as the proper transform of $\mathcal{X} \times_\Delta \Delta'$ in $\mathcal{Y}$. The exceptional divisor $E \subset \mathcal{X}'$ imbeds into the exceptional divisor $\mathcal{E} \subset \mathcal{Y}$, giving the diagram in the lemma.

To prove the other statements we compute in $\mathrm{Pic}(\mathcal{Y})$. Let h denote the pullback of the hyperplane class from $\mathbb{P}^5_{\Delta'}$. The class of $\mathcal{X}'$ in $\mathrm{Pic}(\mathcal{Y})$ is

$3h - 2\mathcal{E}$. Since $E = \mathcal{X}' \cap \mathcal{E}$, we find that the class of E in $\mathrm{Pic}(\mathcal{E})$ is just the restriction of $3h - 2\mathcal{E}$ to $\mathcal{E} = \mathbb{P}(N)$. Since $h|_{\mathbb{P}(N)} = 2\eta$ and $-\mathcal{E}|_{\mathbb{P}(N)} = \xi$, we conclude the second statement of the lemma. The subvariety $E_0 \subset E$ is obtained by intersecting E with the proper transform in $\mathcal{Y}$ of the central fiber of $\mathbb{P}^5_{\Delta'}$. The class of this proper transform is equal to $-\mathcal{E}$, which restricts to the class ξ on $\mathbb{P}(N)$. This proves the third statement. □

To apply the Clemens-Schmid exact sequence to $\mathcal{X}'$, we must know the cohomology of the components of the central fiber. In fact, all the interesting cohomology comes from the exceptional divisor E. Let ξ and η denote the divisor classes on E induced from $\mathbb{P}(N)$, and let $T \subset H^4(E, \mathbb{Z})$ denote the sublattice generated by $\xi^2, \xi\eta, \eta^2$. We use W to denote the orthogonal complement to T in $H^4(E, \mathbb{Z})$ and W_E to denote the corresponding Hodge structure, which is called the *nonspecial cohomology* of E.

Lemma 4.4.4 *The nonspecial cohomology of E is isomorphic to the primitive cohomology of the K3 surface S:*

$$W_E(+1) = H^2(S, \mathbb{C})^0.$$

To expedite our proof, we use the following result of Laszlo on quadric bundles over $\mathbb{P}^2$:

Proposition 4.4.5 *([La] §II 1,2) The cohomology group $H^4(E, \mathbb{Z})$ is torsion free. There exists a morphism of Hodge structures*

$$\alpha : W_E \to H^2(S, \mathbb{C})^0$$

satisfying the cup product condition

$$\langle x, y \rangle_E = -\left(\alpha(x), \alpha(y)\right)_S .$$

The image $\alpha(W) \subset H^2(S, \mathbb{Z})^0$ is a sublattice of finite index.

This map is analogous to the Abel-Jacobi map, and is constructed by using a correspondence between E and S. The proposition implies that α is an isomorphism if and only if

$$|\operatorname{disc}(W)| = |\operatorname{disc}(H^2(S, \mathbb{Z})^0)| = 2.$$

Since W is the orthogonal complement to T, it suffices to prove that the saturation of T has discriminant two.

We begin by computing the discriminant of T itself. The lattice T has intersection matrix

	ξ^2	$\xi\eta$	η^2
ξ^2	48	-12	2
$\xi\eta$	-12	2	0
η^2	2	0	0

and thus has discriminant eight. To prove the lemma, we must show that T is an index two sublattice of its saturation.

We shall produce a class ν in $H^4(E,\mathbb{Z})$ such that $2\nu \in T$ but $\nu \notin T$. We construct this class as the push-forward of a divisor on E_0 by $j : E_0 \hookrightarrow E$. Recall that in the proof of the first lemma, we found that $E_0 \to V$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^2$. ¿From the second lemma, we know that ξ restricts to a class ξ_0 meeting each of the fibers in two points, and η restricts to a class η_0 that does not meet the fibers. In particular, this implies that ξ_0 and η_0 do not generate the Picard group of E_0, but an index two subgroup. Let ν_0 be an element of $\mathrm{Pic}(E_0)$ intersecting the fibers in a single point. Since the class of E_0 in $\mathrm{Pic}(E)$ is ξ, we see that $j_*(\xi_0) = \xi^2$ and $j_*(\eta_0) = \xi\eta$. The class $\nu = j_*\nu_0$ has the properties we seek. □

The following lemma completes our description of the relevant parts of the cohomology of $\overline{X_0}$, E, and E_0:

Lemma 4.4.6

1. *We have*

$$\mathrm{Pic}(E) \otimes \mathbb{Q} = H^2(E,\mathbb{Q}) = \mathbb{Q}\eta \oplus \mathbb{Q}\xi$$
$$\mathrm{Pic}(E_0) \otimes \mathbb{Q} = \mathbb{Q}\eta_0 \oplus \mathbb{Q}\xi_0$$
$$\mathrm{Pic}(\overline{X_0}) \otimes \mathbb{Q} = \mathbb{Q}h \oplus \mathbb{Q}e_0$$

   where h is the restriction of the hyperplane class and e_0 is the class of the exceptional divisor E_0. If $i : E_0 \hookrightarrow \overline{X_0}$ is the inclusion, then we have $i^*(e_0) = -\xi_0$ and $i^*(h) = 2\eta_0$.

2. *The cohomology of the varieties E_0 and $\overline{X_0}$ are generated by their Picard groups.*

3. *The cohomology groups* $H^4(\overline{X_0}, \mathbb{Z})$ *and* $H^4(E_0, \mathbb{Z})$ *have ranks three and two respectively. The restriction map* $i^* : H^4(\overline{X_0}, \mathbb{Z}) \to H^4(E_0, \mathbb{Z})$ *has finite cokernel.*

For the first statement, observe that ξ and η give independent elements of the Picard group of E. However, $H^2(E, \mathbb{Q})$ has rank two (see [La] §I.1), so these two elements generate.

For the statements concerning E_0, recall from the proof of the first lemma that E_0 is a $\mathbb{P}^1$ bundle over $V = \mathbb{P}^2$. This implies that $\mathrm{Pic}(E_0)$ generates the cohomology of E_0 and $H^4(E_0, \mathbb{Z})$ has rank three. Moreover, ξ_0 and η_0 generate an index two subgroup of $\mathrm{Pic}(E_0)$.

Now recall the geometric description of $\overline{X_0}$ from the proof of the first lemma: $\overline{X_0}$ is the set of pairs (M, x) where M is a symmetric 3×3 matrix and $x \in \mathbb{P}(\ker(M))$. If we consider $\overline{X_0}$ as a subvariety of $X_0 \times \mathbb{P}^2$, the second projection ϕ gives $\overline{X_0}$ the structure of a $\mathbb{P}^2$ bundle over $\mathbb{P}^2$. For $x \in \mathbb{P}^2$, $\phi^{-1}(x)$ is the set of 3×3 symmetric matrices M with kernel containing x. Consequently, the cohomology of $\overline{X_0}$ is generated by its Picard group, which has rank two. This also implies that the middle cohomology of $\overline{X_0}$ has rank three. Moreover, the classes h and e_0 are independent in the Picard group and generate over $\mathbb{Q}$. By the proof of the second lemma $h|_E = 2\eta$ and $e_0 = \mathcal{E}|_{\overline{X_0}}$, so $i^*(h) = h|_{E_0} = 2\eta_0$ and $i^*(e_0) = \mathcal{E}|_{E_0} = -\xi|_{E_0} = -\xi_0$. The last statement of the lemma follows. □

To complete the proof of the theorem, we apply the Clemens-Schmid exact sequence [Cl] (or [GrS]):

$$0 \to H^2_{lim} \to H_6(X_0', \mathbb{C}) \xrightarrow{\psi} H^4(X_0', \mathbb{C}) \xrightarrow{\rho} H^4_{lim} \xrightarrow{N} H^4_{lim}. \quad (*)$$

Here X_0' denotes the central fiber of $\mathcal{X}'$, so we have

$$X_0' = \overline{X_0} \cup_{E_0} E.$$

Our goal is to prove that $\rho : H_4(X_0', \mathbb{C}) \to H^4_{lim}$ is surjective; this implies that $N = 0$ and the limiting mixed Hodge structure H^4_{lim} is actually pure. We shall also prove that ρ maps the nonspecial cohomology of E into H^4_{lim}, so the third lemma shows that $H^4_{lim} \in \mathcal{D}_2$.

We begin by computing the terms of the exact sequence $(*)$. The first term $H^2_{lim} = \mathbb{C}h$, because the monodromy action on H^2 is trivial. Since

E_0 has no odd-dimensional homology, the spectral sequence computing the second term degenerates to

$$0 \to H_6(E_0) \to H_6(\overline{X_0}) \oplus H_6(E) \to H_6(X_0') \to 0.$$

The left arrow is the map $(i_*, -j_*)$, where i and j are the corresponding inclusions. Applying the fourth lemma and Poicaré duality, we see that $H_6(X_0', \mathbb{C})$ is three-dimensional. The third term of $(*)$ can be computed from

$$0 \to H^4(X_0', \mathbb{C}) \to H^4(\overline{X_0}, \mathbb{C}) \oplus H^4(E, \mathbb{C}) \to H^4(E_0, \mathbb{C}) \to 0$$

where the right arrow is the difference of the restriction maps i^* and j^*. Applying the third and fourth lemmas, we conclude that $H^4(X_0', \mathbb{C})$ is twenty-five-dimensional. This implies that the image of ρ is twenty-three-dimensional, so ρ is surjective and $N = 0$.

Recall that W_E was defined as the orthogonal complement to the lattice T generated by $\xi^2, \xi\eta, \eta^2$. The restrictions of these classes generate $H^4(E_0, \mathbb{C})$, so j^* sends the nonspecial cohomology to zero and we may consider $0 \oplus W_E \subset H^4(X_0', \mathbb{C})$. Because ψ is a morphism of Hodge structures, $\psi(H_6(X_0', \mathbb{C})) \subset H^4(X_0', \mathbb{C})$ is spanned by integral $(2, 2)$ classes. If S is a typical degree-two K3 surface, the only integral $(2, 2)$ classes in $H^4(X_0', \mathbb{C})$ are mapped into the saturation of $H^4(\overline{X_0}, \mathbb{Z}) \oplus T$, and so are orthogonal to $0 \oplus W_E$. In particular, the image of ψ is orthogonal to $0 \oplus W_E$. Consequently ρ maps W_E isomorphically onto its image in H^4_{lim}. This completes the proof of the theorem. □

5 Associated K3 Surfaces

5.1 A Motivating Example

Let (X, K_d) be a labelled special cubic fourfold with discriminant d. The results of section three imply that d completely determines the lattice structure on $K_d^\perp$. We shall use the following terminology for this lattice:

Definition 5.1.1 *Let (X, K_d) denote a labelled special cubic fourfold with discriminant d. The orthogonal complement to K_d*

$$K_d^\perp \subset L^0$$

will be called the nonspecial cohomology lattice *of* (X, K_d). *Let* W_{X,K_d} *denote the polarized Hodge structure on* $K_d^{\perp}$ *induced from the Hodge structure on* $H^4(X, \mathbb{C})^0$. *This is called the* nonspecial cohomology *of* (X, K_d).

We illustrate with an example how this notion is related to the rationality of certain cubic fourfolds. Let X be a generic Pfaffian cubic fourfold (cf. [BD] or § 4.1.3). We have a birational map from $\mathbb{P}^4$ to X which blows up a surface $\tilde{S}$ birational to a degree fourteen K3 surface S. Consequently $\tilde{S}$ parametrizes a correspondence of rational curves on X, which induces an imbedding of the primitive cohomology $H^2(S, \mathbb{Z})^0$ into $H^4(X, \mathbb{Z})^0$. The following proposition states this more precisely:

Proposition 5.1.2 *Let* (X, K_{14}) *be a generic Pfaffian cubic fourfold. Then there exists a degree fourteen K3 surface* S *and an isomorphism of Hodge structures:*

$$W_{X,K_{14}} = H^2(S, \mathbb{C})^0(-1).$$

Note that the weight is shifted by two, so the sign of the intersection form is reversed.

5.2 Computation of Nonspecial Cohomology

In this section, we determine the special cubic fourfolds for which the nonspecial cohomology is isomorphic to the primitive cohomology of a polarized K3 surface. More generally, we shall consider pairs (S, f) where S is a K3 surface and f is a pseudo-polarization on S with $f^2 = d$. Recall that a *pseudo-polarization* is a divisor contained in the closure of the Kähler cone. We use $H^2(S, \mathbb{C})^0$ to denote the orthogonal complement of f in $H^2(S, \mathbb{C})$. Our goal is the following theorem:

Theorem 5.2.1 (Existence of Associated K3 Surfaces) *Let* (X, K_d) *be a labelled special cubic fourfold of discriminant d, with nonspecial cohomology* W_{X,K_d}. *There exists a polarized K3 surface* (S, f) *such that*

$$W_{X,K_d} \cong H^2(S, \mathbb{C})^0(-1)$$

if and only if the following conditions are satisfied:

1. $4 \not| d$ *and* $9 \not| d$

2. $p \not| d$ if p is an odd prime, $p \equiv -1 (mod\ 3)$

We say that the pair (S, f) is associated *to (X, K_d).*

We first show how the proof of the theorem boils down to a computation of lattices. Let Λ_d^0 be a lattice isomorphic to the primitive middle cohomology of a degree d K3 surface and let $K_d^{\perp}$ denote the orthogonal complement of K_d. The isomorphism asserted in the theorem implies an isomorphism of lattices $K_d^{\perp} \cong -\Lambda_d^0$. On the other hand, assume we are given a labelled special cubic fourfold (X, K_d) and an isomorphism of lattices $K_d^{\perp} \cong -\Lambda_d^0$. Then $W_{X,K_d}(+1)$ has the form of the primitive cohomology of a pseudo-polarized K3 surface. Indeed, since the Torelli map for K3 surfaces is surjective [B2] [Si], there exists a pseudo-polarized K3 surface (S, f) such that $H^2(S, \mathbb{C})^0(-1) \cong W_{X,K_d}$. Moreover, X is smooth so $H^4(X, \mathbb{Z})^0 \cap H^{2,2}(X)$ does not contain any classes with self-intersection $+2$ ([V] §4 Prop. 1). Therefore there are no -2 curves on S orthogonal to f, so f is actually a polarization.

The arguments of the previous paragraph reduce the theorem to the following proposition:

Proposition 5.2.2 *Let Λ_d^0 be the cohomology lattice of a degree d K3 surface and let $K_d^{\perp}$ be the nonspecial cohomology lattice of a labelled special cubic fourfold of discriminant d. Then $K_d^{\perp} \cong -\Lambda_d^0$ if and only if the following conditions are satisfied:*

1. $4 \not| d$ and $9 \not| d$

2. $p \not| d$ if p is an odd prime, $p \equiv -1 (mod\ 3)$

In order to prove this result, we need to compute the structure of the lattice Λ_d^0. By the properties of $\Lambda = H^2(S, \mathbb{Z})$ stated in § 2.1, we may assume that the polarization $f = v_1 + \frac{d}{2} w_1$. Consequently, we obtain the isomorphism

$$\Lambda_d^0 \cong (-d) \oplus H^{\oplus 2} \oplus (-E_8)^{\oplus 2}.$$

The term $(-d)$ represents the sublattice generated by $y = v_1 - \frac{d}{2} w_1$. The discriminant group $d(\Lambda_d^0)$ is equal to $\mathbb{Z}(\frac{y}{d})/\mathbb{Z}y$. We use $q_{\Lambda_d^0} : d(\Lambda_d^0) \to \mathbb{Q}/2\mathbb{Z}$ to denote the discriminant quadratic form. Note that $q_{\Lambda_d^0}(\frac{y}{d}) = \frac{-1}{d} (\bmod 2\mathbb{Z})$, which completely determines $q_{\Lambda_d^0}$.

We now determine when $d(K_d^\perp)$ and $d(-\Lambda_d^0)$ are isomorphic as groups with a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form. We first consider the case $d \equiv 2 \pmod 6$. In this case both discriminant groups are isomorphic to $\mathbb{Z}/d\mathbb{Z}$, so we just need to check when the quadratic forms are conjugate by an automorphism of $\mathbb{Z}/d\mathbb{Z}$. Let u and w be generators of $d(K_d^\perp)$ and $d(-\Lambda_d^0)$ such that $q_{K_d^\perp}(u) = \frac{2d-1}{3d} \pmod{2\mathbb{Z}}$ and $q_{-\Lambda_d^0}(w) = \frac{1}{d} \pmod{2\mathbb{Z}}$ (see Proposition 3.2.6). The quadratic forms are conjugate if and only if the integer $\frac{2d-1}{3}$ is a square modulo $2d$. This is equivalent to saying that -3 is a square modulo $2d$. By quadratic reciprocity this is the case if and only if d is not divisible by four and any odd prime $p|d$ satisfies $p \not\equiv -1 \pmod 3$. We conclude that the discriminant forms agree if and only if the conditions of the proposition are satisfied.

We consider the case $d \equiv 0 \pmod 6$ and we write $d = 6n$. The group $d(K_d^\perp)$ is cyclic if and only if nine does not divide d, so we restrict to this case. Let u and w be generators of $d(K_d^\perp)$ and $d(-\Lambda_d^0)$ such that $q_{K_d^\perp}(u) = \frac{2}{3} - \frac{3}{d} \pmod{2\mathbb{Z}}$ and $q_{-\Lambda_d^0}(w) = \frac{1}{d} \pmod{2\mathbb{Z}}$. The quadratic forms are conjugate if and only if the integer $\frac{2}{3}d - 3$ is a square modulo $2d$. Equivalently $4n - 3$ must be a square modulo $12n$, or -3 is a square mod $4n$ and $4n$ is a square mod 3. By quadratic reciprocity, this is the case if and only if n is odd and any prime $p|n$ satisfies $p \equiv 1 \pmod 3$. Again, we conclude the discriminant forms agree if and only if the conditions of the proposition are satisfied.

This argument immediately implies that the conditions on d are necessary for $K_d^\perp$ to be isomorphic to $-\Lambda_d^0$. On the other hand, $K_d^\perp$ is the *unique* even lattice of signature $(19, 2)$ with discriminant form $d(K_d^\perp)$ [Ni] 1.14.3. Hence if the discriminant forms of $K_d^\perp$ and $-\Lambda_d^0$ agree, we may conclude that $K_d^\perp \cong -\Lambda_d^0$. This concludes the proof. □

5.3 Isomorphisms of Period Domains

In this section, we elaborate the connection between certain special cubic fourfolds and K3 surfaces. We begin with a brief description of the period domains for polarized K3 surfaces; we retain the notation introduced in § 2.1 and § 5.2. Let Σ denote the automorphisms of Λ, and Σ_d the automorphisms fixing some primitive $v = v_1 + \frac{d}{2}w_1 \in \Lambda$. The elements of Σ_d yield automorphisms of $\Lambda_d^0 = v^\perp$. The intersection form determines a quadric hypersurface $Q \subset \mathbb{P}(\Lambda_d^0 \otimes \mathbb{C})$. $\Lambda_d^0 \otimes \mathbb{C}$ is equipped with a Hermitian form given by the formula $(u, \bar{v})$. The locus in Q where this form is positive is a topological open subset with two connected components, denoted $\mathcal{N}'_d$ and $\overline{\mathcal{N}}'_d$ respectively. These two components correspond to the two orientations on the positive definite part of $\Lambda^0_d \otimes \mathbb{R}$. $\mathcal{N}'_d$ is called the *local period domain* for degree d K3 surfaces; it is an open nineteen-dimensional complex manifold. Let $\Sigma^+_d \subset \Sigma_d$ denote the subgroup stabilizing $\mathcal{N}'_d$. Again applying the results from §6 of the appendix of [Sa], we find that $\mathcal{N}'_d$ is a bounded symmetric domain of type IV and Σ^+_d is an arithmetic group acting holomorphically on $\mathcal{N}'_d$. The quotient $\mathcal{N}_d := \Sigma^+_d / \mathcal{N}'_d$ is therefore a quasi-projective variety, called the *global period domain* for degree d K3 surfaces.

We introduce a bit more notation for special cubic fourfolds as well. Recall that a marked special Hodge structure is a point $x \in \mathcal{D}'$ along with the data of an imbedding $K_d \hookrightarrow H^{2,2}(x) \cap L$ preserving the class h^2. Let $G^+_d \subset \Gamma^+_d$ be the subgroup acting trivially on K_d. We use $\mathcal{D}^{\mathrm{mar}}_d$ to denote the marked special Hodge structures of discriminant d, modulo the action of G^+_d. The fiber product $\mathcal{D}^{\mathrm{mar}}_d \times_{\mathcal{D}} \mathcal{C}$ is written $\mathcal{C}^{\mathrm{mar}}_d$, the marked special cubic fourfolds of discriminant d. We have natural forgetting maps

$$\mathcal{D}^{\mathrm{mar}}_d \to \mathcal{D}^{\mathrm{lab}}_d \qquad \mathcal{C}^{\mathrm{mar}}_d \to \mathcal{C}^{\mathrm{lab}}_d.$$

The following proposition describes the relationship among $\mathcal{D}'_d, G^+_d, \mathcal{D}^{\mathrm{mar}}_d$, and $\mathcal{D}^{\mathrm{lab}}_d$:

Proposition 5.3.1 *$G^+_d = \Gamma^+_d$ if $d \equiv 2$ (mod 6) and $G^+_d \subset \Gamma^+_d$ is an index-two subgroup if $d \equiv 0$ (mod 6). The natural map $\mathcal{D}^{\mathrm{mar}}_d \to \mathcal{D}^{\mathrm{lab}}_d$ is an isomorphism if $d \equiv 2$ (mod 6) and a double cover if $d \equiv 0$ (mod 6). Furthermore, $\mathcal{D}^{\mathrm{mar}}_d = G^+_d \backslash \mathcal{D}'_d$ and thus is connected for all $d \neq 6$.*

We begin with the first statement. The lattice K_d has no automorphisms preserving h^2 if $d \equiv 2$ (mod 6). This implies that $G^+_d = \Gamma^+_d$ for these values of d. If $d \equiv 0$ (mod 6) then K_d has an involution, which acts on K^0_d as multiplication by -1. We claim this involution extends to an element $\gamma \in \Gamma^+_d$. By the classification of the orbits $\Gamma^+ \backslash L^0$ in Proposition 3.2.5, we may assume $v = v_1 + \frac{d}{6} w_1$. Choose γ equal to multiplication by -1 on both hyperbolic summands of L^0 and equal to the identity elsewhere. We have that $\gamma \in \Gamma^+_d$ but $\gamma \notin G^+_d$, so G^+_d is a proper subgroup of Γ^+_d.

Now we turn to the second statement. For $d \equiv 2$ (mod 6) we have seen that each labelling has a unique marking, so the forgetting map is an isomorphism. In the case $d \equiv 0$ (mod 6) we saw that each labelling has two markings permuted by the action of Γ_d^+. This implies that the forgetting map is a double cover.

For the third statement, we recall that $\mathcal{D}_d^{\text{lab}} = \Gamma_d^+ \backslash \mathcal{D}_d'$. Hence for $d \equiv 2 \pmod 6$ the result is immediate. For $d \equiv 0 \pmod 6$, we must check that any $\gamma \in \Gamma_d^+$ acting nontrivially on K_d also acts nontrivially on $\mathcal{D}_d'$. For $d \neq 6$, if γ acts nontrivially on K_d then the induced action on $d(K_d)$ is not equal to ± 1. However, the groups $d(K_d)$ and $d(K_d^\perp)$ are isomorphic, so the induced action on $d(K_d^\perp)$ is not ± 1. Now $\mathcal{D}_d'$ is a topologically open subset of a quadric hypersurface in $\mathbb{P}(K_d^\perp \otimes \mathbb{C})$, so only scalar multiplications act trivially on $\mathcal{D}_d'$. In particular, γ necessarily acts nontrivially. $\square$

Remark: There exists a nontrivial element $\gamma \in \Gamma_6^+ \backslash G_6^+$ which acts trivially on $K_6^\perp$. It follows that $\mathcal{D}_6^{\text{mar}} \neq G_6^+ \backslash \mathcal{D}_6'$ but rather that $\mathcal{D}_6^{\text{lab}} = G_6^+ \backslash \mathcal{D}_6'$.

We now restrict to values of d for which there exists an isomorphism $j_d : K_d^\perp \to -\Lambda_d^0$ (see Proposition 5.2.2). We choose orientations on the negative definite parts of $K_d^\perp$ and $-\Lambda_d^0$ compatible with j_d. Then there is an induced isomorphism between the local period domains $\mathcal{D}_d'$ and $\mathcal{N}_d'$. We would like to compare the corresponding global period domains $\mathcal{D}_d^{\text{lab}}$ and $\mathcal{N}_d$. We shall prove the following results:

Theorem 5.3.2 *Let d be a positive integer such that there exists an isomorphism $j_d : K_d^\perp \to -\Lambda_d^0$. If $d \neq 6$ then there is an induced isomorphism $i_d : \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d$. Furthermore, we have that $\mathcal{D}_6^{\text{lab}} \cong \mathcal{N}_6$.*

This isomorphism of period domains depends on the choice of j_d. Each isomorphism j_d induces an isomorphism of discriminant groups $j_d' : d(K_d^\perp) \to d(-\Lambda_d^0)$ preserving the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic forms on these groups [Ni] §1.3. We shall denote the set of such isomorphisms $\text{Isom}(d(K_d^\perp), d(-\Lambda_d^0))$. Note that the group $\{n \in \mathbb{Z}/d\mathbb{Z} : n^2 = 1\}$ acts faithfully and transitively on this set.

Theorem 5.3.3 *For $d \neq 6$, the various isomorphisms $i_d : \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d$ correspond to elements of* $\text{Isom}(d(K_d^\perp), d(-\Lambda_d^0))/(\pm 1)$. *The isomorphism $i_6 : \mathcal{D}_6^{\text{lab}} \to \mathcal{N}_6$ is unique.*

These two theorems have the following corollary:

Corollary 5.3.4 (Immersions into Moduli Spaces of K3 Surfaces) *Let $d \neq 6$ be a positive integer such that there exists an isomorphism $j_d : K_d^\perp \to$*

$-\Lambda_d^0$. *Then there is an imbedding* $i_d : \mathcal{C}_d^{\mathrm{mar}} \hookrightarrow \mathcal{N}_d$ *of the marked special cubic fourfolds of discriminant* d *into the global period domain for degree* d *K3 surfaces, which is unique up to the choice of an element of*

$$\mathrm{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1).$$

Moreover, there is a unique imbedding $i_6 : \tilde{\mathcal{C}}_6^{\mathrm{lab}} \hookrightarrow \mathcal{N}_6$.

As we shall see in §6, geometrical considerations will sometimes mandate specific choices of i_d (e.g. in the case $d = 14$).

We prove the first theorem. First, we compare the action of Σ_d^+ on Λ_d^0 to the action of G_d^+ on $K_d^{\perp}$. We claim that Σ_d^+ is the group of automorphisms of Λ_d^0 preserving the orientation on the positive definite part of $\Lambda_d^0 \otimes \mathbb{R}$ and acting trivially on the discriminant group $d(\Lambda_d^0)$. This follows from the results of [Ni] §1.4, which imply that any such automorphism extends uniquely to an element of Σ_d^+. By the same argument, G_d^+ is the group of automorphisms of $K_d^{\perp}$ preserving the orientation on the negative definite part of $K_d^{\perp} \otimes \mathbb{R}$ and acting trivially on the discriminant group $d(K_d^{\perp})$.

Now suppose we are given an isomorphism $j_d : K_d^{\perp} \rightarrow -\Lambda_d^0$. This induces an isomorphism $\mathcal{D}'_d \rightarrow \mathcal{N}'_d$. By the arguments of the previous paragraph, it also induces an isomorphism $G_d^+ \rightarrow \Sigma_d^+$. Consequently, we obtain an isomorphism $i_d : G_d^+ \backslash \mathcal{D}'_d \rightarrow \Sigma_d^+ \backslash \mathcal{N}'_d$. Applying Proposition 5.3.1, this translates into an isomorphism $i_d : \mathcal{D}_d^{\mathrm{mar}} \rightarrow \mathcal{N}_d$ for $d \neq 6$. The remark after the proposition also yields an isomorphism $i_6 : \mathcal{D}_6^{\mathrm{lab}} \rightarrow \mathcal{N}_6$. This completes the proof of the first theorem. $\square$

We turn to the proof of the second theorem. We must determine when two different isomorphisms $j_d^1 : K_d^{\perp} \rightarrow -\Lambda_d^0$ and $j_d^2 : K_d^{\perp} \rightarrow -\Lambda_d^0$ induce the same isomorphism $i_d : G_d^+ \backslash \mathcal{D}'_d \rightarrow \Sigma_d^+ \backslash \mathcal{N}'_d$. If $j_d^2 = \sigma \circ j_d^1$ for some $\sigma \in \Sigma_d^+$ then j_d^1 and j_d^2 induce the same isomorphisms of period domains. Also, if $j_d^1 = -j_d^2$ then j_d^1 and j_d^2 induce the same isomorphism between $\mathcal{D}'_d$ and $\mathcal{N}'_d$, because these manifolds lie in the projective spaces $\mathbb{P}(K_d^{\perp} \otimes \mathbb{C})$ and $\mathbb{P}(\Lambda_d^0 \otimes \mathbb{C})$.

On the other hand, assume that j_d^1 and j_d^2 induce the same isomorphism between $G_d^+ \backslash \mathcal{D}'_d$ and $\Sigma_d^+ \backslash \mathcal{N}'_d$. Then there exist $\gamma \in G_d^+$ and $\sigma \in \Sigma_d^+$ such that $j_d^1 \circ \gamma$ and $\sigma \circ j_d^2$ induce the same isomorphism between $\mathcal{D}'_d$ and $\mathcal{N}'_d$. But then $j_d^1 \circ \gamma = \pm \sigma \circ j_d^2$. Combining the arguments of these two paragraphs, we conclude that the isomorphisms between $G_d^+ \backslash \mathcal{D}'_d$ and $\Sigma_d^+ \backslash \mathcal{N}'_d$ correspond to certain elements of $\mathrm{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1)$.

It remains to check that each element of $\mathrm{Isom}(d(K_d^\perp), d(-\Lambda_d^0))/(\pm 1)$ actually arises from an isomorphism between $K_d^\perp$ and $-\Lambda_d^0$ respecting the orientations on the negative definite parts. Now $K_d^\perp$ has an automorphism g reversing the orientation on the negative part and acting trivially on $d(K_d)$. Take g to be the identity except on a hyperbolic summand of the orthogonal decomposition for $K_d^\perp$; on the hyperbolic summand set g equal to multiplication by -1. Hence it suffices to show that the automorphisms of $K_d^\perp$ induce all the automorphisms of $d(K_d^\perp)$, which is proved in [Ni], Theorem 1.14.2 and Remark 1.14.3. This completes the proof of the second theorem. □

6 Fano Varieties of Special Cubic Fourfolds

6.1 Introduction and Necessary Conditions

In this section, we provide a geometric explanation for the K3 surfaces associated to some special cubic fourfolds. The general philosophy underlying our approach is due to Mukai [Mu1],[Mu2],[Mu3]. Let S be a polarized K3 surface and let $\mathcal{M}_S$ be a moduli space of simple sheaves on S. Quite generally, $\mathcal{M}_S$ is smooth and possesses a natural nondegenerate holomorphic two-form ([Mu1] Theorem 0.1). Furthermore, the Chern classes of the 'quasi-universal sheaf' on $S \times \mathcal{M}_S$ induce correspondences between S and $\mathcal{M}_S$. If $\mathcal{M}_S$ is compact of dimension two then it is a K3 surface isogenous to S; the Hodge structure of $\mathcal{M}_S$ can be read off from the Hodge structure of S and the numerical invariants of the sheaves ([Mu2] Theorem 1.5). Conversely, given a variety F with a nondegenerate holomorphic two-form and an isogeny $H^2(S,\mathbb{Q}) \to H^2(F,\mathbb{Q})$, one can try to interpret F as a moduli space of sheaves on S. In the case where F is a K3 surface, we often have such interpretations ([Mu2] Theorem 1.9).

The paradigm for our discussion is the discriminant fourteen case. We have seen that a labelled special cubic fourfold of discriminant fourteen has an associated degree fourteen K3 surface. For a generic such cubic fourfold this may be explained as follows. Let X be a generic special cubic fourfold of discriminant fourteen, F the Fano variety of X, and S the degree fourteen K3 surface associated to S. Then F is isomorphic to $S^{[2]}$, the Hilbert scheme of length two subschemes of S [BD]. Note that the ideal sheaves defining elements of $S^{[2]}$ can be interpretted as simple sheaves on S.

There exist $X \in \mathcal{C}_{14}$ for which the isomorphism between F and $S^{[2]}$ constructed in [BD] breaks down. We give an example. Let X be a smooth cubic fourfold containing two disjoint planes π_1 and π_2. If we put $K_{14} =< h^2, \pi_1 + \pi_2 >$ then (X, K_{14}) is a labelled special cubic fourfold of discriminant fourteen. The associated K3 surface S can be represented as the complete intersection of forms of types $(1, 2)$ and $(2, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. The classes $(1, 0)$ and $(0, 1)$ restrict to classes C_1 and C_2 on S with intersections $C_1^2 = 2, C_2^2 = 2$, and $C_1C_2 = 5$. The class $f = C_1 + C_2$ gives the degree fourteen polarization on S. Note that the curves in $|C_1|$ and $|C_2|$ are genus two and degree seven on S. Assume that F isomorphic to $S^{[2]}$ with hyperplane class $g = 2f - 5\delta$. A general $C \in |C_1|$ is smooth and hyperelliptic, and the g^1_2 on C corresponds to a rational curve C' on $S^{[2]}$. The degree of g on C' is

$$g|C' = 2f|C' - 5\delta|C' = 14 - 15 = -1$$

which is absurd. This proves that the construction of Beauville and Donagi does not give an isomorphism between F and $S^{[2]}$.

Let X be a cubic fourfold and assume we have an isomorphism between its Fano variety F and $S^{[2]}$ for some K3 surface S. In this situation we have an isomorphism (§ 2.1)

$$H^2(F, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta$$

and the hyperplane class of F may be written $g = af - b\delta$ where f is some polarization of S. We set $d = (f, f)$. The Abel-Jacobi map $\alpha : H^4(X, \mathbb{C})^0 \to H^2(F, \mathbb{C})^0(-1)$ induces an isomorphism between $H^2(S, \mathbb{C})^0(-1)$ and a codimension-two subspace of $H^4(X, \mathbb{C})$, which we take as the nonspecial cohomology of X. Our labelling K_d is then the orthogonal complement to $\alpha^{-1}(H^2(S, \mathbb{Z})^0)$ in $H^4(X, \mathbb{Z})$. We also obtain an isomorphism between $K_d^\perp$ and $-\Lambda_d^0$, the primitive cohomology lattice of S with respect to f.

This discussion is summarized in the following proposition:

Proposition 6.1.1 *Let X be a cubic fourfold with Fano variety F. Assume there is an isomorphism between F and $S^{[2]}$ for some K3 surface S. Then X has a labelling K_d such that S is associated to (X, K_d). Moreover, the map*

$$i_d : \mathcal{C}_d^{\mathrm{mar}} \hookrightarrow \mathcal{N}_d$$

may be chosen so that (X, K_d) is mapped to S.

Only the last statement requires any explanation. By Theorem 5.3.2 i_d is determined by the choice of an isomorphism between $K_d^{\perp}$ and $-\Lambda_d^0$.

We can explain the map i_d geometrically, at least for generic cubic fourfolds along $\mathcal{C}_d$. First, we need the following theorem on the deformation spaces of the varieties $S^{[2]}$:

Theorem 6.1.2 (Deformation Spaces of the Varieties $S^{[2]}$) *Let S be a K3 surface, and let 2δ denote the elements of $S^{[2]}$ which are supported at a single point. The deformation space of $S^{[2]}$ is smooth and has dimension twenty-one. Deformations of the surface S correspond to a divisor in this space which may be characterized as the deformations for which δ remains a divisor.*

This is one of a number of results on the deformation theory of $S^{[2]}$ which are contained in [B]. We retain the assumptions and notation of the last proposition. By construction, the variety $\mathcal{C}_d$ corresponds to the deformations of F for which δ remains algebraic. Applying the theorem quoted above, there is some small analytic neighborhood in $\mathcal{C}_d$ such that the deformations of F in this neighborhood correspond to deformations of the polarized K3 surface S. The Fano varieties in this neighborhood are all isomorphic to $S_1^{[2]}$ for some deformation S_1 of S. Furthermore, it is not hard to see that the isomorphism between the Fano varieties and the blown-up symmetric squares remains valid in an open étale neighborhood of X in $\mathcal{C}_d$. In particular, a generic cubic fourfold in $\mathcal{C}_d$ has Fano variety isomorphic to the blown-up symmetric square of a K3 surface. From this argument, we may conclude the following proposition:

Proposition 6.1.3 *Retain the notation and assumptions of the previous proposition. Let (X_1, K_d) be a generic cubic fourfold of $\mathcal{C}_d^{\mathrm{mar}}$ and let $S_1 = i_d(X_1, K_d)$. Then the Fano variety of X_1 is isomorphic to $S_1^{[2]}$.*

As we have seen, the cubic fourfolds with Fano varieties isomorphic to blown-up symmetric squares of K3 surfaces are particularly nice. So we ask the following question:

> Let (X, K_d) be a generic special cubic fourfold of discriminant d and let F be the Fano variety of X. For which values of d is F isomorphic to $S^{[2]}$ for some K3 surface S?

Theorem 5.2.1 gives sufficient conditions for the existence of a K3 surface associated to (X, K_d). The next proposition shows that these are not sufficient to guarantee that the Fano varieties are isomorphic to blown-up symmetric squares of K3 surfaces. For instance, Fano varieties of special cubic fourfolds of discriminant 74 are not generally of the form $S^{[2]}$, because the equation

$$74a^2 = 2(n^2 + n + 1)$$

has no integral solutions (see [G] or the tables in [CS]). However, these special cubic fourfolds do have associated $K3$ surfaces.

Proposition 6.1.4 *Assume that for a generic special cubic fourfold of discriminant d, the Fano variety is isomorphic to $S^{[2]}$ for some K3 surface S. Then there exist positive integers n and a such that*

$$d = 2\frac{n^2 + n + 1}{a^2}.$$

Let (S, f) be a polarized K3 surface of degree d such that $\mathrm{Pic}(S) = \mathbb{Z}f$. Assume that $S^{[2]}$ is the Fano variety of a cubic fourfold. Then $S^{[2]}$ admits a very ample line bundle g of degree 108, such that the primitive cohomology $g^\perp$ is isomorphic to M^0. We write $g = bf - m\delta$. Computing in the cohomology ring of $S^{[2]}$ we find

$$g^4 = 3(db^2 - 2m^2)^2$$

which implies that

$$db^2 - 2m^2 = 6.$$

Furthermore, if the orthogonal complement $g^\perp$ is isomorphic to M^0 then $\mathrm{disc}(g^\perp) = 3$. For this to be the case, it is necessary that $(M, g) = 2\mathbb{Z}$ and so b is even. Writing $b = 2a$ and substituting, we find that the equation

$$2da^2 - m^2 = 3$$

has a solution. We also see that m is necessarily odd, so writing $m = 2n + 1$ we obtain

$$da^2 = 2(n^2 + n + 1)$$

which is what we seek. □

We can produce infinitely many examples of special cubic fourfolds with Fano variety isomorphic to the symmetric square of a K3 surface:

Theorem 6.1.5 *Assume that d can be written*

$$d = 2(n^2 + n + 1) \quad n \geq 2$$

where n is an integer. If X is a generic special cubic fourfold of discriminant d then the Fano variety F is isomorphic to $S^{[2]}$, where S is a K3 surface associated to (X, K_d).

This is proved in the next two sections. Note that the condition on d corresponds to setting $a = 1$ in previous proposition. This is a significant restriction, e.g. $d = 38$ satisfies the hypotheses of the proposition but not the hypotheses of the theorem. The ideas behind the proof strongly suggest that the conditions stated in the proposition are the correct sufficient conditions. Proving the theorem in this generality turns out to be technically very awkward.

6.2 The Beauville/Debarre Construction and Ambiguous Symplectic Varieties

We begin by giving a special case of a construction of Beauville and Debarre described in [D]. Let S be a smooth quartic surface in $\mathbb{P}^3$. Let $p_1 + p_2$ be a generic point in $S^{[2]}$, and let $\ell(p_1 + p_2)$ be the line in $\mathbb{P}^3$ containing p_1 and p_2. By Bezout's theorem

$$\ell(p_1 + p_2) \cap S = p_1 + p_2 + q_1 + q_2.$$

Setting $j(p_1 + p_2) = q_1 + q_2$ for each $p_1 + p_2$, we obtain a birational involution

$$j : S^{[2]} \dashrightarrow S^{[2]}.$$

If S contains no lines, then j is well-defined everywhere and extends to a biregular morphism. If S contains a line ℓ, then j is a birational map with indeterminacy along the plane $\pi \subset S^{[2]}$ consisting of subschemes contained in ℓ. The map j induces an isomorphism

$$j^* : H^2(S^{[2]}, \mathbb{Z}) \to H^2(S^{[2]}, \mathbb{Z}).$$

Recall the direct sum decomposition

$$H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$$

and let f_4 be the degree four polarization on S and the corresponding class on $S^{[2]}$. Following [D], one may compute

$$j^*(x) = -x + (x, f_4 - \delta)\,(f_4 - \delta).$$

We now use F to denote the variety $S^{[2]}$. One interpretation of our construction is that F is the Hilbert scheme of length two subschemes of S in two different ways. There are two distinct isomorphisms

$$r_1, r_2 : F \to S^{[2]}$$

where $r_2 = j \circ r_1$. Recall that the divisor $E_1 = 2\delta$ corresponds to the subschemes of S supported at a single point and is isomorphic to $\mathbb{P}(T_S)$, the projectivization of the tangent bundle to S. In this case, the divisor $E_2 = j^*(E_1) = 2(2f_4 - 3\delta)$ also has this structure. To see this geometrically, note that j maps the subschemes supported at $p \in S$ into the g^1_2 of the hyperplane section of S tangent at p. This example suggests the following definition:

Definition 6.2.1 *Let F be an irreducible symplectic Kähler manifold, and assume that there exist K3 surfaces S_1 and S_2 and isomorphisms*

$$r_1 : F \to S_1^{[2]}$$

$$r_2 : F \to S_2^{[2]}$$

such that $r_1^*\delta_1 \neq r_2^*\delta_2$. Then we say that F is ambiguous.

We digress to give a beautiful example of ambiguous varieties:

Proposition 6.2.2 *Assume that $3|d$ and that the Fano variety F of a generic cubic fourfold X in $\mathcal{C}_d$ is isomorphic to $S_1^{[2]}$ for some K3 surface S_1. Then F is ambiguous, so there exists a second K3 surface S_2 such that F is also isomorphic to $S_2^{[2]}$. In particular, $\mathcal{C}_d^{\mathrm{lab}}$ corresponds to an open subset of a $\mathbb{Z}/2\mathbb{Z}$-quotient of the moduli space of degree d K3 surfaces.*

By Proposition 6.1.1 we obtain

$$i_d : \mathcal{C}_d^{\mathrm{mar}} \to \mathcal{N}_d$$

which associates to a generic marked cubic fourfold the K3 surface S such that the Fano variety is isomorphic to $S^{[2]}$. Assume now that d is divisible by three. By Proposition 5.3.1, the forgetting map

$$\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d^{\mathrm{lab}}$$

has degree two. Hence for each labelled cubic fourfold of discriminant d, there are two different K3 surfaces S_1 and S_2, with $S_1^{[2]}$ and $S_2^{[2]}$ both isomorphic to the Fano variety F. Furthermore, i_d descends to a map from $\mathcal{C}_d^{\mathrm{lab}}$ to a $\mathbb{Z}/2\mathbb{Z}$ quotient of $\mathcal{N}_d$. $\square$

We may interpret this phenomenon in terms of monodromy. The decomposition of the cohomology of these Fano varieties

$$H^2(F, \mathbb{Z}) \cong H^2(S_1, \mathbb{Z}) \oplus \mathbb{Z}\delta$$

is not respected by the monodromy group.

Recall the description of the deformation spaces of the varieties $S^{[2]}$ in Theorem 6.1.2. We apply this to an ambiguous variety F, to show that the deformations of F isomorphic to some $S'^{[2]}$ correspond to two transverse divisors in the deformation space of F. In the case where S is a quartic K3, these are just the deformations for which δ or $2f_4 - 3\delta$ remain algebraic.

We sketch the construction of the examples of special cubic fourfolds with Fano variety isomorphic to $S^{[2]}$, where S is a K3 surface. This will be made precise in the following section. Assume one already knows that for the generic cubic fourfold in $\mathcal{C}_d$, the Fano variety F isomorphic to $S^{[2]}$ for some K3 surface S. Specialize so that S contains an irreducible curve f_4 such that the linear system $|f_4|$ imbeds S as a smooth quartic K3 surface that does not contain a line. The Fano variety F is then ambiguous, so the deformation space contains two divisors parametrizing varieties of the form $S'^{[2]}$. One of these divisors corresponds to deformations contained in $\mathcal{C}_d$, and we claim that the other corresponds to deformations contained in $\mathcal{C}_{d'}$ for $d' \neq d$. This implies that $\mathcal{C}_{d'}$ contains typical cubic fourfolds X' for which F' is isomorphic to $S'^{[2]}$. We conclude that for a generic cubic $X' \in \mathcal{C}_{d'}$ the Fano variety F' is isomorphic to $S'^{[2]}$ for some K3 surface.

6.3 Construction of the Examples

This section makes precise the argument of the previous section. We will construct our ambiguous varieties along the divisor $\tilde{\mathcal{C}}_6$. The first step is to describe the Fano varieties of these singular cubics and natural desingularizations of them.

We begin by setting our notation. Let X_0 be a singular cubic fourfold with a single ordinary double point p and let F_0 be its Fano variety of lines. Let S be the sextic K3 surface associated to X_0 (see § 4.2). Let $\phi : \mathcal{X} \to \Delta$ be a family in $\tilde{\mathcal{C}}$ with central fiber X_0 and X_t smooth for $t \neq 0$. Let $\mathcal{F} \to \Delta$ be the corresponding family of Fano varieties, and let $\mathcal{X}' \to \Delta'$ be a semistable reduction of $\mathcal{X} \to \Delta$. For simplicity, we assume that the central fiber of the semistable family is of the form

$$X_0' = \overline{X_0} \cup Q$$

where $\overline{X_0} = \mathrm{Bl}_S(\mathbb{P}^4)$ is the desingularization of X_0, Q is a smooth quadric fourfold, and $Q_0 = \overline{X_0} \cap Q$ is the smooth quadric in $\mathbb{P}^4$ containing S. Our assumptions on the form of the semistable reduction are valid if ϕ is a sufficiently generic smoothing of X_0, which is enough for our application.

We first prove the following lemma describing the singularities of F_0:

Lemma 6.3.1 *F_0 is singular along the lines passing through the double point, which are parametrized by the K3 surface S. These singularities are ordinary codimension-two double points and the blow-up*

$$\sigma : \mathrm{Bl}_S F_0 \to F_0$$

desingularizes F_0. If S_0 does not contain a line then $\mathrm{Bl}_S F_0 \cong S^{[2]}$.

The first part of the proposition follows from § 4.2 and [AK] 1.10. For the second part, we realize σ by blowing-up the Grassmannian $\mathbb{G}(1,5)$ along the locus $L(p)$ of lines containing p. The fiber square

$$\begin{array}{ccc} S & \to & F_0 \\ \downarrow & & \downarrow \\ L(p) & \to & \mathbb{G}(1,5) \end{array}$$

gives a natural closed imbedding of normal cones

$$C_S F_0 \hookrightarrow C_{L(p)}\mathbb{G}(1,5)|S.$$

The projectivized normal cone $\mathbb{P}(C_{L(p)}\mathbb{G}(1,5))$ corresponds to $\mathbb{P}(\mathbb{C}^6/\mathcal{S})$, where $\mathcal{S}$ is the restriction of the universal subbundle. Each line $\ell \in L(p)$ may be regarded as a point $\ell \in \mathbb{P}^4$; the corresponding fiber of the normal cone is parametrized by the lines in $\mathbb{P}^4$ containing ℓ. Now let ℓ be a singular point of F_0. The fiber of the projectivized normal cone $\mathbb{P}(C_S F_0)$ at ℓ corresponds to the lines λ containing ℓ and contained in Q_0. Since Q_0 is a smooth quadric threefold, these are parametrized by a smooth conic curve. We conclude that F_0 has codimension-two ordinary double points along S and $\mathrm{Bl}_S F_0$ is smooth.

This description implies that we can regard $\mathrm{Bl}_S F_0$ as a parameter space for certain curves on $\overline{X_0}$. These curves are of the following types:

1. Lines on X_0 disjoint from p.
2. Unions of proper transforms of lines through p and lines contained in $Q_0 \subset \overline{X_0}$.

These in turn may be identified with:

1. Two-secants λ to $S \subset \mathbb{P}^4$.
2. Three-secants λ with a distinguished point $s \in \lambda \cap S$.

We emphasize that each line meeting S in more than two points is contained in Q_0 but not in S (by hypothesis). Consequently, such lines are three-secants to S. We claim elements of $S^{[2]}$ naturally correspond to curves of this type. For each ideal sheaf I of colength two there is a unique line λ containing the corresponding subscheme. Either λ is a two-secant, or λ is a three-secant and s is the support of $I/I_{\lambda\cap S}$. $\square$

Applying the same argument to the family of Fano varieties gives the following lemma:

Lemma 6.3.2 *Retain the notation and assumptions introduced above. The family of Fano varieties $\mathcal{F}\times_\Delta \Delta'$ has ordinary codimension-three double points along the surface S. The variety $\mathcal{F}' = \mathrm{Bl}_S(\mathcal{F}\times_\Delta \Delta')$ is smooth, and the exceptional divisor $E \subset F'_0$ is a smooth quadric surface bundle over S. The component of F'_0 dominating F_0 is isomorphic to $S^{[2]}$.*

We now prove our key result:

Proposition 6.3.3 *Retain the notation and assumptions introduced above. Then there is a smooth family*

$$\overline{\mathcal{F}} \to \Delta'$$

birational to $\mathcal{F} \times \Delta'$*, such that* $\overline{F}_u = F_u$ *and* $\overline{F}_0 = S^{[2]}$*.*

We start with the family $\mathcal{F}'$ described in the previous lemma. The fibers of $E \to S$ are all smooth quadric surfaces, so the variety parametrizing rulings of E is an étale double cover of S. However, the K3 surface S has no nontrivial étale coverings, so we may choose a ruling of E. We may blow down E in the direction of this ruling to obtain a smooth family $\overline{\mathcal{F}}$. This map induces an isomorphism from the proper transform of F_0 in F_0' to the central fiber of $\overline{\mathcal{F}}$. The proper transform to F_0 in F_0' is isomorphic to $S^{[2]}$, so $\overline{\mathcal{F}}$ satisfies the conditions stipulated in the proposition. $\square$

Now we complete the construction of the examples. Let S be an algebraic K3 surface with Picard group generated by classes f_4 and f_6 with the intersection form

$$P = \begin{array}{c|cc} & f_6 & f_4 \\ \hline f_6 & 6 & n+5 \\ f_4 & n+5 & 4 \end{array}$$

and $n \geq 2$. In § 4.3 we showed that such a surface exists and that we may assume that $|f_6|$ imbeds it as a smooth sextic surface. The divisor f_4 is effective because it has positive degree with respect to f_6. We claim that f_4 is very ample. If f_4 were not ample, then there would exist a -2 curve E with $f_4 E \leq 0$. This follows from the structure of the Kähler cone of S ([LP] §1,§10). Note that $f_4 E \neq 0$ because P does not contain a rank-two sublattice of discriminant -8. Recall that the Picard-Lefschetz reflection associated to E is given by the equation $r_E(x) = x + (E, x)E$. Applying this to the class f_4, we find that $r_E(f_4)^2 = 4$ and $(f_6, r_E(f_4)) < (f_6, f_4)$. Hence that f_6 and $r(f_4)$ span a sublattice with discriminant smaller than that of P, which is impossible. Finally, applying Lemma 4.3.3 we see that the linear system $|f_4|$ imbeds S as a smooth quartic surface.

Our hypothesis on P implies that the image of S under $|f_6|$ lies on a smooth quadric hypersurface and does not contain a line, and that the image of S under $|f_4|$ also does not contain a line. In particular, S corresponds to a singular cubic fourfold $X_0 \in \tilde{\mathcal{C}}_6$. Furthermore $S^{[2]}$ is ambiguous and there is an involution

$$j : S^{[2]} \to S^{[2]}$$

so that

$$\delta_2 := j^*\delta = 2f_4 - 3\delta.$$

Using the previous proposition and the arguments of § 4.3, there exists a smoothing $\phi : \mathcal{X} \to \Delta$ of X_0, such that after base change the corresponding family of smooth symplectic varieties

$$\overline{\mathcal{F}} \to \Delta'$$

is a deformation of $S^{[2]}$ for which δ_2 remains algebraic. By Theorem 6.1.2 the Fano variety $\overline{F_u}$ of X'_u is isomorphic to $S_u^{[2]}$.

If we choose ϕ generally, we may assume that the X'_u are typical and that $\mathrm{Pic}(S_u)$ is generated by the polarization f'. Let Π denote $\mathrm{Pic}(\overline{F_u})$, which is equal to $\mathbb{Z}f' \oplus \mathbb{Z}\delta_2$ in our case. This is a lattice with respect to the canonical form of discriminant

$$\mathrm{disc}(\Pi) = -2\mathrm{deg}\ (S_u). \quad \dagger$$

On the other hand, Π is the saturation of $\mathbb{Z}g + \mathbb{Z}\delta_2$. Specializing to $S^{[2]}$ we can compute

$$\begin{aligned} \Pi &= \text{saturation } \big(\mathbb{Z}g + \mathbb{Z}\delta_2\big) \\ &= \text{saturation } \big(\mathbb{Z}(2f_6 - 3\delta) + \mathbb{Z}(2f_4 - 3\delta)\big) \\ &= \mathbb{Z}(2f_6 - 3\delta) + \mathbb{Z}(f_6 - f_4) \end{aligned}$$

with discriminant

$$\mathrm{disc}(\Pi) = -4(n^2 + n + 1).\ddagger$$

Combining † and ‡, we compute that the K3 surfaces S_u have degree

$$d(n) = 2(n^2 + n + 1)$$

and the X_u are special of discriminant $d(n)$. We conclude that for $n > 1$ the Fano variety of a generic special cubic of discriminant $d(n)$ is equal to $S^{[2]}$, where S is a degree $d(n)$ K3 surface. □

We should remark that we are actually proving a very weak version of surjectivity of the Torelli map for the symplectic varieties that occur as Fano varieties of cubic fourfolds. Specifically, the pure limiting Hodge structures parametrized by $\mathcal{D}_6$ actually arise from smooth symplectic varieties, i.e. the varieties $S^{[2]}$ where S is a degree six K3 surface. This also explains the computation of the limiting mixed Hodge structure H^4_{lim} in § 4.2.

There are a number of ways these examples may be generalized. We need not assume that the polarizations f_6 and f_4 actually generate the Picard lattice P of S. Then the saturated lattice Π would not be equal to $\mathbb{Z}(2f_6 - 3\delta) + \mathbb{Z}(f_6 - f_4)$, but would contain it as a subgroup of finite index a. In this situation, the discriminant would be equal to

$$d(n, a) = 2\frac{n^2 + n + 1}{a^2}.$$

Unfortunately, not every integer of the form $d(n, a)$ can be obtained in this way, because every pair (n, a) arising from this construction satisfies

$$24 \equiv n^2 (\text{mod a}).$$

A second way to generalize this argument is to replace $\tilde{\mathcal{C}}_6$ by some other divisor $\mathcal{C}_d$ parametrizing special cubic fourfolds whose Fano varieties are of the form $S^{[2]}$. We can try to repeat the argument given above, allowing the K3 surfaces to acquire quartic polarizations. Making precise statements can be quite difficult, because one needs explicit descriptions of two complicated closed sets: the complement $\mathcal{D}_d \backslash \mathcal{C}_d$ (i.e. the K3 surfaces corresponding to singular cubic fourfolds) and the locus in $\mathcal{C}_d$ where the isomorphism between the Fano varieties and the blown-up symmetric squares breaks down. We must exclude from our statements any 'bad' discriminants d', such that $\mathcal{D}_{d'} \cap \mathcal{D}_d$ lies entirely in these two closed subsets. Of course, if we are willing to exclude unspecified lists of 'bad' discriminants, then we can avoid describing these two sets. The computational complexity of the proof above reflects the difficulty in solving these problems, even along $\tilde{\mathcal{C}}_6$ where the relevant K3 surface is easy to describe.

Finally, Mukai's philosophy suggests that whenever we have an associated K3 surface S, the Fano variety F might be interpretted as a suitable moduli space of simple sheaves on S. It would be interesting to find such interpretations when F cannot be a blown-up symmetric square.

7 Notation

X a smooth cubic hypersurface in $\mathbb{P}^5$

L the cohomology lattice $H^4(X,\mathbb{Z})$

L^0 the primitive cohomology $H^4(X,\mathbb{Z})$

$\langle , \rangle$ the intersection forms on these lattices

h^2 the square of the hyperplane class in L

F the Fano variety of lines on X

g the hyperplane class in M

M the cohomology group $H^2(F,\mathbb{Z})$

M^0 the primitive cohomology $H^2(F,\mathbb{Z})^0$

$(,)$ Beauville's canonical form on M

$\alpha : L \longrightarrow M$ the Abel-Jacobi map

H the hyperbolic plane lattice $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

E_8 the positive definite even lattice associated to the Dynkin diagram E_8

B the lattice $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

S a K3 surface

$S^{[2]}$ the Hilbert scheme of length two subschemes of S

2δ the divisor on $S^{[2]}$ corresponding to the subschemes supported at a single point

Γ the automorphisms of L preserving the class h^2

Γ^+ an index-two subgroup of Γ

$\mathcal{D}'$ the local period domain for cubic fourfolds

$\mathcal{D} = \Gamma^+ \backslash \mathcal{D}'$ the global period domain for cubic fourfolds

$\mathcal{C}$ the moduli space of cubic fourfolds

$\tau : \mathcal{C} \hookrightarrow \mathcal{D}$ the period map

T an algebraic surface in X not homologous to a complete intersection

$A(X) = H^{2,2}(X) \cap H^4(X,\mathbb{Z})$ the lattice of algebraic classes

K a positive definite rank-two sublattice of L, containing the class h^2

Γ^+_K the subgroup of Γ^+ stabilizing K

$\mathcal{D}'_K$ the Hodge structures for which K is contained in the lattice of algebraic classes

$[K]$ the Γ^+ orbit of K

$\mathcal{D}_{[K]}$ the Hodge structures for which some lattice in the class $[K]$ is contained in the lattice of algebraic classes

$\mathcal{C}_{[K]}$ the corresponding divisor of $\mathcal{C}$

$\mathcal{D}_{[K]}^{\text{lab}} \cong \Gamma_K^+ \backslash \mathcal{D}'_K$ the labelled special Hodge structures of type $[K]$

$\mathcal{C}_{[K]}^{\text{lab}}$ the corresponding divisor of $\mathcal{C}$

$d(\mathcal{L})$ the discriminant group of a lattice

$q_{\mathcal{L}}$ the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form on the discriminant group of an even lattice

$\mathcal{C}_d$ the moduli space of special cubic fourfolds with discriminant d

$\mathcal{C}_d^{\text{lab}}$ the labelled special cubic fourfolds of discriminant d

K_d the lattice of algebraic classes on a special labelled cubic fourfold of discriminant d

$K_d^{\perp}$ the nonspecial cohomology lattice of discriminant d, i.e. the orthogonal complement to K_d in L

W_{X,K_d} the nonspecial cohomology of (X, K_d), the complexification of $K_d^{\perp}$ in $H^4(X,\mathbb{C})^0$

Λ_d^0 lattice isomorphic to the primitive cohomology of a degree d K3 surface

Λ lattice isomorphic to the middle cohomology of a K3 surface

$\mathcal{N}'_d$ the local period domain for degree d K3 surfaces

Σ_d the subgroup of$\text{Aut}(\Lambda)$ fixing a degree d polarization

Σ_d^+ an index-two subgroup of Σ_d

$\mathcal{N}_d = \Sigma_d^+ \backslash \mathcal{N}'_d$ the global period domain

G_d^+ the subgroup of Γ_d^+ acting trivially on K_d

$\mathcal{D}_d^{\text{mar}}$ the marked special Hodge structures of discriminant d

$\mathcal{C}_d^{\text{mar}}$ the marked special cubic fourfolds of discriminant d

$\text{Isom}(d(\mathcal{L}), d(\mathcal{K}))$ the set of isomorphisms of the discriminant groups

References

[AK] Altman, A. and Kleiman, S., 1977, Foundations of the theory of Fano schemes, *Compositio Math.* 34, 3-47.

[BB] Baily, W. L. and Borel, A., 1966, Compactifications of arithmetic quotients of bounded symmetric domains, *Ann. of Math.* 84, 442-528.

[B] Beauville, A., 1983, Variétés kählериennes dont la première classe de Chern est nulle, *J. Differential Geom.* 18, 755-782.

[B2] Beauville, A., 1985, Surjectivité de l'application des périodes, in *Géométrie des Surfaces K3: Modules et Périodes-Séminaire Palaiseau*, Astérisque 126, Société mathématique de France, Paris, pp. 122-128.

[BD] Beauville, A. and Donagi, R., 1985, La variété des droites d'une hypersurface cubique de dimension 4, *C.R. Acad. Sc. Paris, Série I* 301, 703-706.

[Bo] Borel, A., 1972, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, *J. Differential Geom.* 6, 543-560.

[Cl] Clemens, C.H., 1977, Degenerations of Kähler manifolds, *Duke Math J.* 44, 215-290.

[CS] Conway, J.H. and Sloane, N.J.A., 1988, On the Classification of Integral Quadratic Forms, in *Sphere Packings, Lattices and Groups*, Springer, New York, pp. 352-405.

[D] Debarre, O., 1984, Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles, *C.R. Acad. Sc. Paris, Série I* 299, 681-684.

[Do] Dolgachev, I., 1983, Integral quadratic forms, in *Séminaire Bourbaki Exposé 611*, Astérisque 105-106, Société mathématique de France, Paris, pp. 251-275.

[Fa] Fano, G., 1943, Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate rationali de 4° ordine, *Comment. Math. Helv.* 15, 71-80.

[G] Gauss, C.F., 1966, *Disquisitiones Arithmeticae*, Yale University Press, New Haven.

[GrS] Griffiths, P. and Schmid, W., 1973, Recent developments in Hodge theory, in *International Colloquium on Discrete Subgroups of Lie Groups and Applications to Moduli*, Tata Institute of Fundamental Research and Oxford University Press, Bombay, pp. 31-127.

[Ha1] Hassett, B., 1996, *Special Cubic Hypersurfaces of Dimension Four,* Harvard University Thesis, Cambridge, Mass.

[Ha2] Hassett, B., 1997, Some rational cubic fourfolds, to appear in the *Journal of Algebraic Geometry.*

[Iz] Izadi, E., 1997, A Prym construction for the cohomology of a cubic hypersurface, preprint in alg-geom/9701012.

[La] Laszlo, Y., 1989, Théorème de Torelli générique pour les intersections complètes de trois quadriques de dimension paire. *Invent. math.* 98, 247-264.

[LP] Loojenga, E. and Peters, C., 1981, Torelli theorem for Kähler K3 surfaces, *Compositio Math.* 42, 145-186.

[Mo] Morin, U., 1940, Sulla razionalità dell'ipersuperficie cubic generale dello spazio lineare S^5, *Rend. Sem. Mat. Univ. Padova* 11, 108-112.

[MFK] Mumford, D., Fogarty, J., and Kirwan, F., 1994, *Geometric Invariant Theory*, Springer, Berlin.

[Mu1] Mukai, S., 1984, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. math.* 77, 101-116.

[Mu2] Mukai, S., 1987, On the moduli space of bundles on K3 surfaces, I, *Vector Bundles on Algebraic Varieties, Proc. Bombay Conference, 1984,* Tata Institute of Fundamental Research Studies, no. 11, Oxford University Press, 341-413.

[Mu3] Mukai, S., 1988, Moduli of vector bundles on K3 surfaces, and symplectic manifolds, *Sugaku Expositions* 1, 139-174.

[Ni] Nikulin, V.V., 1980, Integral symmetric bilinear forms and some of their applications, *Math. USSR Izvestija* 14, no. 1, 103-167.

[SD] Saint-Donat, B., 1974, Projective models of K3 surfaces, *Amer. J. Math.* 96, 602-639.

[Sa] Satake, I., 1980, *Algebraic Structures of Symmetric Domains*, Princeton University Press, Princeton.

[Se] Serre, J.P., 1973, *A Course in Arithmetic*, Springer, New York.

[Si] Siu, Y.T., 1981, A simple proof of the surjectivity of the period map of K3 surfaces, *Manuscripta Math.* 35, 311-321.

[Tr1] Tregub, S.L., 1984, Three constructions of rationality of a cubic fourfold, *Moscow Univ. Math. Bull.* 39, no. 3, 8-16.

[Tr2] Tregub, S.L., 1993, Two remarks on four dimensional cubics, *Russian Math. Surveys* 48, no. 2, 206-208.

[V] Voisin, C., 1986, Théorème de Torelli pour les cubiques de $\mathbb{P}^5$, *Invent. math.* 86, 577-601.

[Za] Zarhin, Y. , 1990, Algebraic cycles over cubic fourfolds, *Boll. Un. Mat. Ital. B* (7), no. 4b, 833-847.

[Zu] Zucker, S., 1977, The Hodge conjecture for cubic fourfolds, *Compositio Math.* 34, 199-209.