RATIONALITY OF FANO THREEFOLDS OF DEGREE 18 OVER NONCLOSED FIELDS

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1. Introduction

Manin [Man93] proposed to study (uni)rationality of Fano threefolds over nonclosed fields, in situations where geometric (uni)rationality is known. In cases where the Picard group is generated by the canonical class, i.e., those of rank and index one, he assigned an 'Exercise' [Man93, p. 47] to explore the rationality of degree $12, 16, 18$, and 22. See [IP99, p. 215] for a list of geometrically rational Fano threefolds of rank one.

We have effective criteria for deciding the rationality of surfaces over nonclosed fields – the relevant invariant is encoded in the Galois action on the geometric Picard group. This invariant is trivial for Fano threefolds considered above. Our main result is:

Theorem 1. *Let X be a Fano threefold of degree 18 defined over a field k of characteristic zero and admitting a k-rational point. Then X is rational if and only if X admits a conic over k.*

Here a conic means a geometrically connected curve of degree two – possibly non-reduced or reducible.

Kuznetsov and Prokhorov [KP19] complete the study of rationality for geometrically rational Fano threefolds of Picard rank one over nonclosed fields. In particular, they address the degree 16 case where – assuming the existence of a rational point – rationality is equivalent to the existence of a twisted cubic curve. For Fano threefolds of degrees 12 and 22, rationality holds if and only if there is a rational point.

A key step in the proof of the *only if* direction in Theorem 1 is the analysis of torsors over intermediate Jacobians, as presented in [HT19a, BW19]. The other direction uses deformation and specialization techniques. While much recent work on rationality has focused on applications of specialization to show the *failure* of (stable) rationality [Voi15, CTP16, HKT16, Tot16, HPT18, Sch19, NS19, KT19], here we use it to *prove* rationality, avoiding complicated case-by-case arguments

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for special geometric configurations; see Theorem 8. This technique was also used to analyze rationality for cubic fourfolds [RS19].

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2. Projection constructions

We work over a field k of characteristic zero.

Let $X \subset \mathbb{P}^r$ be a smooth Fano threefold of Picard rank one, embedded by the minimal very ample multiple of the anticanonical divisor. Fix a center consisting of a point $x \in X$ or a smooth curve $\ell \subset X$ and consider the blowup $\sigma : \widetilde{X} \to X$ along that center. Assume that

- $-K_{\widetilde{X}}$ is nef and big;
- there are no effective divisors $D \subset X$ such that $(-K_{\widetilde{X}})^2 \cdot D = 0$.

Then by [IP99, Lem. 4.1.1] there exists an $n \geq 1$ such that $|-nK_{\widetilde{X}}|$ determines a birational morphism

$$\phi' : \widetilde{X} \to X'$$

to a normal variety with (at worst) terminal singularities. The morphism ϕ' is a small resolution and an isomorphism if and only if X' is nonsingular; this happens precisely when $\widetilde{X}$ is also Fano.

After flopping rational curves as necessary, we obtain a model $\widetilde{X}^+$ of $\widetilde{X}$ with semiample anticanonical class. Let $\chi : \widetilde{X} \dashrightarrow \widetilde{X}^+$ denote the induced flop, $\phi : \widetilde{X}^+ \to Y$ the contraction of the other extremal ray (which need not be birational), and $\psi : X \dashrightarrow Y$ the composed map (cf. [IP99, (4.1.1)]):

$$(1) \qquad \begin{array}{ccccc} \widetilde{X} & & \overset{\chi}{\dashrightarrow} & & \widetilde{X}^+ \\ & \searrow^{\phi'} & & {}^{\phi^+}\swarrow & \\ \sigma\downarrow & & X' & & \downarrow\phi \\ & & & & \\ X & & \overset{\psi}{\dashrightarrow} & & Y \end{array}$$

The contraction ϕ' may often be understood in terms of projections. We suppose that X is anticanonically embedded and $n = 1$.

- Assume the center is a line $\ell \subset X$. The induced rational map on X may be interpreted as projection from ℓ. Other lines incident to ℓ are contracted by ϕ'.

- Assume the center is a point $x \in X$. The anticanonical system on $\widetilde{X}$ corresponds to anticanonical divisors on X with multiplicity ≥ 2 at x. The induced map ϕ' may be interpreted as double projection from x, i.e., projection from the tangent space at x. Conics containing x are contracted by ϕ'.

When we refer to m-fold projection along a point or curve, this means imposing zeros of multiplicity m along the exceptional divisor of σ.

For the remainder of this section, X denotes a smooth Fano threefold of degree 18 over k.

2.1. **Projection from lines.** The variety of lines $R_1(X)$ is nonempty and connected of pure dimension one [IP99, Prop. 4.2.2] and sweeps out a divisor in X with class $-3K_X$ [IP99, Th. 4.2.7]. For generic X, $R_1(X)$ is a smooth curve of genus ten [IP99, Th. 4.2.7]. If $R_1(X)$ is smooth then X admits no nonreduced conics [KPS18, Rem. 2.1.7].

Suppose that $\ell \subset X$ is a line. Then double projection along ℓ induces a birational map as in Diagram (1)

$$\widetilde{X} \dashrightarrow \widetilde{X}^+ \to Y,$$

where $Y \subset \mathbb{P}^4$ is a smooth quadric hypersurface [IP99, Th. 4.3.3]. This flops the three lines incident to ℓ and contracts a divisor

$$D \in |-2K_{\widetilde{X}^+} - 3E^+|$$

to a smooth curve $C \subset Y$ of degree seven and genus two.

Since Y admits a k-rational point it is rational over k; the same holds true for X.

Proposition 2. *If X is a Fano threefold of degree* 18 *admitting a line over k then X is rational.*

2.2. **Projection from conics.** We discuss the structure of the variety $R_2(X)$ of conics on X:

- $R_2(X)$ is nonempty of pure dimension two [IP99, Th. 4.5.10].
- $R_2(X)$ is geometrically isomorphic to the Jacobian of a genus two curve C [IM07, Prop. 3] [KPS18, Th. 1.1.1].
- Through each point of X there pass finitely many conics [IP99, Lem. 4.2.6]; indeed, through a generic such point we have nine conics [Tak89, 2.8.1].
- Given a conic $D \subset X$, double projection along D induces a fibration [IP99, Cor. 4.4.3,Th. 4.4.11]

$$X \dashrightarrow \widetilde{X}^+ \stackrel{\phi}{\to} \mathbb{P}^2$$

in conics with quartic degeneracy curve.

2.3. **Projection from points.** We recall the results of Takeuchi [Tak89] presented in [IP99, Th. 4.5.8]. Let $\widetilde{X}$ denote the blowup of X at x, with exceptional divisor E.

Proposition 3. *Suppose we have a point $x \in X(k)$ and let $\widetilde{X}$ denote the blowup of X at x. We assume that*

- *x does not lie on a line in X;*
- *there are no effective divisors D on X such that $(K_{\widetilde{X}})^2 \cdot D = 0$.*

Then triple-projection from x gives a fibration

$$X \stackrel{\sim}{\dashrightarrow} \widetilde{X}^+ \stackrel{\phi}{\to} \mathbb{P}^1$$

in sextic del Pezzo surfaces.

We offer a more detailed analysis of double projection from a point $x \in X(k)$ not on a line. By [IP99, § 4.5] the projection morphism

$$\tilde{\phi} : \widetilde{X} \to \mathbb{P}^7$$

is generically finite onto its image $\overline{X}$ and the Stein factorization

$$\widetilde{X} \stackrel{\phi'}{\to} X' \stackrel{\bar{\phi}}{\to} \overline{X}$$

yields a Fano threefold of genus six with canonical Gorenstein singularities. The condition precluding effective divisors D with $(K_{\widetilde{X}})^2 \cdot D = 0$ means that ϕ' admits no exceptional divisors. The nontrivial fibers of ϕ' are all isomorphic to $\mathbb{P}^1$'s, with the following possible images in X:

(1) a conic in X through x;
(2) a quartic curve of arithmetic genus one in X, spanning a $\mathbb{P}^3$, with a singularity of multiplicity two at x;
(3) a sextic curve of arithmetic genus two in X, spanning a $\mathbb{P}^4$, with a singularity of multiplicity three at x.

Moreover, if ϕ' does not contract any surfaces then the exceptional divisor E over x is embedded in $\mathbb{P}^7$ as a Veronese surface.

The quartic curves on X with node at a fixed point x have expected dimension 0. The sextic curves on X with transverse triple point at a fixed point x have expected dimension -1. Indeed, we have:

Proposition 4. [IP99, Prop. 4.5.1] *Retain the notation above. For a generic $x \in X$*

- *the quartic and sextic curves described above do not occur;*
- *ϕ' is a small contraction;*
- *the rational map $X \stackrel{\sim}{\dashrightarrow} \widetilde{X}^+$ factors as follows*
  (1) *blow up the point x;*
  (2) *flop the nine conics through x;*

- *ϕ restricts to the proper transform E^+ of E as an elliptic fibration associated with cubics based at nine points.*

3. Unirationality constructions

In this section, we consider the following question inspired by [Man93, p. 46]:

Question 5. Let X be a Fano threefold of degree 18 over k. Suppose $X(k) \neq \emptyset$. Is X is unirational over k?

From our perspective, unirationality is more delicate than rationality as we lack a specialization theorem for smooth families in this context. We cannot apply the theorem of [KT19] – as we do in the proof of Theorem 8 – to reduce to configurations in general position.

The geometric constructions below highlight some of the issues that arise.

3.1. **Using a point.**

Proposition 6. *Let X be a Fano threefold of degree 18 over k admitting a point $x \in X(k)$ satisfying the condition in Proposition 3. Then X is unirational over k and rational points are Zariski dense.*

Proof. We retain the notation from Proposition 3. Note that the proper transforms of lines $L \subset E^+$ give trisections of our del Pezzo fibration

$$\phi : \widetilde{X}^+ \to \mathbb{P}^1.$$

Basechanging to L yields

$$\phi_L : \widetilde{X}^+ \times_{\mathbb{P}^1} L \to L,$$

a fibration of sextic del Pezzo surfaces with a section. Thus the generic fiber of ϕ_L is rational over $k(L)$ by [Man66, p. 77]. Since $L \simeq \mathbb{P}^1$, the total space of the fibration is rational over k. As it dominates $\widetilde{X}^+$, we conclude that X is unirational. □

If the rational points are Zariski dense then we can find one where Proposition 3 applies. However, if we are given only a single rational point on X we must make a complete analysis of degenerate cases as partly described in Section 2.3. In addition, we must consider cases where there exist lines over $\bar{k}$ passing through our given rational point.

For instance, consider the case where a single line $x \in \ell \subset X$. To resolve the double projection at x, we must take the following steps:

- blow up x to obtain an exceptional divisor $E_1 \simeq \mathbb{P}^2$;

- blow up the proper transform ℓ' of the line ℓ with

$$N_{\ell'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

to obtain an exceptional divisor $E_2 \simeq \mathbb{F}_1$.

Let E_1' and denote the proper transform of E_1 in the second blowups. The linear series resolving the double projection is

$$h - 2E_1' - E_2$$

which takes $E_1' \simeq \mathbb{F}_1$ to a cubic scroll, E_2 to a copy of $\mathbb{P}^2$, and the (-1)-curve on E_2 to an ordinary singularity on the image. The induced contraction

$$\phi' : \widetilde{X} \to X' \subset \mathbb{P}^7$$

has degree

$$(h - 2E_1' - E_2)^3 = 10.$$

Thus X' admits a 'degenerate Veronese surface' consisting of a cubic scroll and a plane meeting along a line coinciding with the (-1)-curve of the scroll; X' has an ordinary singularity along that line.

Of course, the most relevant degenerate cases for arithmetic purposes involve multiple lines through x conjugated over the ground field. It would be interesting to characterize the possibilities.

3.2. **Using a point and a conic.** Here is another approach: Let X admit a point $x \in X(k)$ and a conic $D \subset X$ defined over k. The results recalled in Section 2.2 imply that X is birational over k to

$$\phi : \widetilde{X}^+ \to \mathbb{P}^2,$$

a conic bundle degenerating over a plane quartic curve B.

Suppose there exists a rational point on $\widetilde{X}^+$ whose image $p \in \mathbb{P}^2$ is not contained in the degeneracy curve. Consider the pencil of lines through p. The corresponding pencil of surfaces on $\widetilde{X}^+$ are conic bundles over $\mathbb{P}^1$ with four degenerate fibers and the resulting fibration admits a section. Such a surface is either isomorphic to a quartic del Pezzo surface or birational to such a surface [KST89, p. 48]. It is a classical fact that a quartic del Pezzo surface with a rational point is unirational.

The argument works even when p is a smooth point or node of B. Here we necessarily have higher-order ramification over the nodes – this is because the associated generalized Prym variety is compact – which we can use to produce a section of the resulting pencil of degenerate quartic del Pezzo surfaces. However, there is trouble when p is a cusp of B.

4. Rationality results

Our first statement describes the rationality construction under favorable genericity assumptions:

Proposition 7. *Let X be a Fano threefold of degree 18 over k. Assume that:*

- *there exists an $x \in X(k)$ satisfying the conditions of Proposition 3 so X is birational to a fibration $\phi : \widetilde{X}^+ \to \mathbb{P}^1$ in sextic del Pezzo surfaces;*
- *there exists an irreducible curve $M \subset X$, disjoint from the indeterminacy of $X \overset{\sim}{\dashrightarrow} \widetilde{X}^+$, with degree prime to three.*

Then X is rational over k.

Proof. We saw in the proof of Proposition 6 that the generic fiber S of ϕ is a sextic del Pezzo surface admitting a rational point of a degree-three extension. Our assumptions imply that $S \cdot M = \deg(M)$ which is prime to three so applying [Man66, p. 77] we conclude that S is rational over $k(\mathbb{P}^1)$ and X is rational over the ground field. □

We now show these genericity assumptions are not necessary:

Theorem 8. *Let X be a Fano threefold of degree 18 over k. Assume that X admits a rational point x and a conic D, both defined over k. Then X is rational.*

Proof. Let B denote the Hilbert scheme of all triples (X, x, D) of objects described in the statement. This is smooth and connected over the moduli stack of degree 18 Fano threefolds; indeed, we saw in Section 2.2 that the parameter space of conics on X is an abelian surface. The moduli stack itself is a smooth Deligne-Mumford stack since Kodaira vanishing gives $H^i(T_X) = 0$ for $i = 2, 3$ and $H^0(T_X) = 0$ by [Pro90]. The classification of Fano threefolds shows that the moduli stack is connected. Thus B is smooth and connected.

Consider the universal family

$$(\mathcal{X} \xrightarrow{\pi} B, \mathbf{x} : B \to \mathcal{X}, \mathcal{D} \subset \mathcal{X}),$$

where π is smooth and projective. The generic fiber of π is rational over $k(B)$ as the genericity conditions of Proposition 7 are tautologically satisfied – see Proposition 4 for details. The specialization theorem [KT19, Th. 1] implies that every k-fiber of π is rational over k. This theorem assumes the base is a curve. However, our parameter space B is smooth so Bertini's Theorem implies that each $b \in B(k)$ may be connected to the generic point by a curve smooth at b. □

5. Analysis of principal homogeneous spaces

5.1. **Proof of Theorem 1.** One direction is Theorem 8; we focus on the converse. Suppose that X is rational over the ground field. Let C be the genus two curve whose Jacobian $J(C)$ is isomorphic to the intermediate Jacobian $IJ(X)$ over k. The mechanism of [HT19a, § 5] gives a principal homogeneous space P over $J(C)$ with the property that the Hilbert scheme $\mathcal{H}_d$ parametrizing irreducible curves of degree d admits a morphism

$$\mathcal{H}_d \to P_d$$

descending the Abel-Jacobi map to k, where $[P_d] = d[P]$ in the Weil-Châtelet group of $J(C)$. By Theorem 22 of [HT19a], if X is rational then $P \simeq \mathrm{Pic}^i(C)$ for $i = 0$ or 1. In particular, we have

$$R_1(X) \hookrightarrow P$$

and by the known results of Section 2.2

$$R_2(X) \simeq P_2 \simeq J(C).$$

Indeed, since C has genus two we have identifications

$$J(C) = \mathrm{Pic}^0(C) \simeq \mathrm{Pic}^2(C),$$

which gives the desired interpretation of P_2 whether $P = \mathrm{Pic}^0(C)$ or $\mathrm{Pic}^1(C)$. As a consequence, $R_2(X)$ admits a k-rational point.

5.2. **A corollary to Theorem 1.** Retain the notation of the previous section. Without assumptions on the existence of points or conics on X defined over k, we know that

$$18[P] = 0 \text{ and } 9[R_2(X)] = 0$$

in the Weil-Châtelet group. This allows us to deduce an extension of our main result:

Corollary 9. *Let X be Fano threefold of degree of degree 18 over k with $X(k) \neq \emptyset$. Suppose that X admits a curve of degree prime to three, defined over k. Then X is rational.*

Our assumption means that $2[P] = 0$, whence $[R_2(X)] = 0$ and X admits a conic defined over k. Hence Theorem 1 applies.

5.3. **Generic behavior.** There are examples over function fields where the principal homogeneous space is not annihilated by two:

Proposition 10. *Over $k = \mathbb{C}(\mathbb{P}^2)$, there exist examples of X such that the order of $[P]$ is divisible by three.*

Proof. Let S be a complex K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}h$ where $h^2 = 18$. Mukai [Muk88] has shown that S arises as a codimension-three linear section of a homogeneous space $W \subset \mathbb{P}^{13}$ arising as the closed orbit for the adjoint representation of G_2

$$S = \mathbb{P}^{10} \cap W.$$

Consider the associated net of Fano threefolds

$$\varpi : \mathcal{X} \to \mathbb{P}^2$$

obtained by intersecting W with codimension-two linear subspaces

$$\mathbb{P}^{10} \subset \mathbb{P}^{11} \subset \mathbb{P}^{13}.$$

Write X for the generic fiber over $\mathbb{C}(\mathbb{P}^2)$.

Let $R_2(\mathcal{X}/\mathbb{P}^2)$ denote the relative variety of conics. This was analyzed in [IM07, § 3.1]: The conics in fibers of ϖ cut out pairs of points on S, yielding a birational identification and natural abelian fibration

$$S^{[2]} \overset{\sim}{\dashrightarrow} R_2(\mathcal{X}/\mathbb{P}^2) \overset{\psi}{\to} \mathbb{P}^2.$$

The corresponding principal homogeneous space has order divisible by nine; its order is divisible by three if it is nontrivial.

These fibrations are analyzed in more depth in [MSTVA17, § 3.3] and [KR13]. Let T denote the moduli space of rank-three stable vector bundles V on S with $c_1(V) = h$ and $\chi(V) = 6$. Then we have

- T is a K3 surface of degree two;
- the primitive cohomology of S arises as an index-three sublattice of the primitive cohomology of T

  $$H^2(S,\mathbb{Z})_{prim} \subset H^2(T,\mathbb{Z})_{prim},$$

  compatibly with Hodge structures;
- the Hilbert scheme $T^{[2]}$ is birational to the relative Jacobian fibration of the degree-two linear series on T

  $$\mathcal{J} \to \mathbb{P}^2;$$

- the relative Jacobian fibration of ψ is birational to $\mathcal{J}$ over $\mathbb{P}^2$.

The last statement follows from [Saw07, p. 486] or [Mar06, § 4]: The abelian fibration ψ is realized as a twist of the fibration $\mathcal{J} \to \mathbb{P}^2$; the twisting data is encoded by an element $\alpha \in \mathrm{Br}(T)[3]$ annihilating $H^2(S,\mathbb{Z})_{prim}$ modulo three.

Now suppose that ψ had a section. Then $\mathcal{J}$ and $S^{[2]}$ would be birational holomorphic symplectic varieties. The Torelli Theorem implies that their transcendental degree-two cohomology – $H^2(T,\mathbb{Z})_{prim}$ and $H^2(S,\mathbb{Z})_{prim}$ respectively – are isomorphic. This contradicts our computation above. □

5.4. **Connections with complete intersections?** Assume k is algebraically closed and X a Fano threefold of degree 18 over k. Kuznetsov, Prokhorov, and Shramov [KPS18] have pointed out the existence of a smooth complete intersection of two quadrics $Y \subset \mathbb{P}^5$ with

$$R_1(Y) \simeq R_2(X), \tag{2}$$

Both have intermediate Jacobian isomorphic to the Jacobian of a genus two curve C.

Now suppose that X and Y are defined over a nonclosed field k with $IJ(X) \simeq IJ(Y)$. In general, we would not expect $R_2(X)$ and $R_1(Y)$ to be related as principal homogeneous spaces; for example, we generally have $9[R_2(X)] = 0$ and $4[R_1(Y)] = 0$ (see [HT19b]).

Verra [Ver18] has found a direct connection between complete intersections of quadrics and *singular* Fano threefolds of degree 18. Suppose we have a twisted cubic curve

$$R \subset Y \subset \mathbb{P}^5,$$

which forces Y to be rational. Consider the linear series of quadrics vanishing along R; the resulting morphism

$$\mathrm{Bl}_R(Y) \to \mathbb{P}^{11}$$

collapses the line residual to R in $\mathrm{span}(R) \cap Y$. Its image X_0 is a nodal Fano threefold of degree 18.

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