CUBIC FOURFOLDS OF DISCRIMINANT 24 AND RATIONALITY

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Let X be a smooth complex cubic fourfold; when is it rational? Classically, this was studied for specific examples obtained via uniform geometric constructions [Fan43, BD85, Tre93]. Today, we consider the moduli space $\mathcal{C}$ of smooth cubic fourfolds. The locus $\mathcal{C}_{\mathrm{rat}} \subset \mathcal{C}$ of rational members is a countable union of Zariski-closed subsets $\mathcal{C}_{\varrho} \subset \mathcal{C}$ [KT19], which we seek to enumerate.

All known examples of rational parametrizations $\mathbb{P}^4 \overset{\sim}{\dashrightarrow} X$ blow up a K3 surface. Thus we characterize cubic fourfolds whose middle cohomology – as a Hodge structure – contains the primitive cohomology of some K3 surface: This is an explicit countable union of divisors in $\mathcal{C}$ [Has00]. (We follows its notation.) Kuznetsov conjectured that this locus is precisely $\mathcal{C}_{\mathrm{rat}}$ [Kuz10, AT14, Add16, Has16]; in particular, the very general cubic fourfold is irrational. Finding irrational cubic fourfolds remains a challenging problem, despite promising ideas of Katzarkov, Kontsevich, Pantev, and Yu.

We focus on proving rationality for cubic fourfolds associated with K3 surfaces. Russo and Staglianò [RS19, RS23] found new divisors in $\mathcal{C}_{26}, \mathcal{C}_{38}, \mathcal{C}_{42} \subset \mathcal{C}$ parametrizing rational X. Numerous examples have also been found in codimension two [Has99, AHTVA19, FL23]. A common theme is assigning *twisted* K3 surfaces to cubic fourfolds, i.e., pairs (S, α) where S is a K3 surface and $\alpha \in \mathrm{Br}(S)$ a Brauer class, such that we have saturated embeddings of Hodge structures

$$T(S, \alpha) \hookrightarrow H^4(X, \mathbb{Z})(1).$$

Here $T(S) \subset H^2(S, \mathbb{Z})$ is transcendental cohomology and $T(S, \alpha) \subset T(S)$ is a finite index sublattice. The classification of cubic fourfolds with associated twisted K3 surfaces is in [Huy17]; X should be rational whenever $\alpha = 0$. In [Has99] and [AHTVA19], S has degree two and α has order two and three, respectively. These form divisors $\mathcal{C}_8, \mathcal{C}_{18} \subset \mathcal{C}$, each containing countably many codimension-two loci in $\mathcal{C}_{\mathrm{rat}}$.

We offer a new example in this vein: Consider cubic fourfolds X with twisted K3 surfaces (S, α), where S has degree six and α has order two, forming a divisor in $\mathcal{C}_{24} \subset \mathcal{C}$. These are rational whenever $\alpha = 0$, a countable union of codimension-two loci; see Theorems 1.11 and 2.1.

Section 1 presents our rationality construction via *nodal* sextic del Pezzo surfaces in X. This differs substantially from [AHTVA19], where we considered smooth sextic del Pezzo surfaces. Section 2 puts this geometry in Hodge-theoretic terms, allowing an explicit lattice theoretic description of where the construction applies. The condition is expressed in terms of the hyperkähler geometry of the variety of lines $F_1(X)$. Section 3 is the bridge between the hyperkähler geometry – which can be understood in uniform terms – and the del Pezzo fibrations. Section 4 gives context for a key construction needed in Section 3, formulated in Theorem 4.2.

Ultimately, to automate rationality constructions we want to sidestep clever classical constructions, replacing them with moduli-theoretic analysis. We return to our example from the perspective of twisted sheaves on K3 surfaces in Section 5; the main result is Theorem 5.3. The eight-dimensional hyperkähler manifolds arising also have intriguing properties; see Proposition 5.5.

Here is the high-level strategy for the rationality proof in the complex case: Let X be a smooth complex cubic fourfold with variety of lines $F_1(X)$. Suppose that X is special of discriminant 24, i.e. it has Hodge class $W \in H^4(X, \mathbb{Z})$ with degree six and self-intersection 20 (see (5)). The variety of lines contains divisors $E, E' \subset F_1(X)$, both ruled over a degree-six K3 surface S; each ruling yields a scroll $T \subset X$ (§2.2). The scroll is degree six and generically has two nodes (§3.1). A cubic fourfold containing such a scroll contains a nodal sextic del Pezzo surface W and *vice versa* (§3.2); the constructions use three-dimensional scrolls. The linear system of quadrics cutting out W gives a fibration over $\mathbb{P}^2$ (§1.3) whose fibers are also nodal sextic del Pezzo surface (§1.4). Rationality follows provided the fibration has a multisection of odd degree (§1.5), because sextic del Pezzo surfaces with zero-cycles of degree one are rational.

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1. Rationality construction

We work an algebraically-closed base field k.

1.1. **Sextic del Pezzo surfaces:** Let $\widetilde{W} \subset \mathbb{P}^6$ denote a sextic del Pezzo surface, isomorphic to $\mathbb{P}^2$ blown up in three non-collinear points

p_1, p_2, p_3, and embedded anti-canonically. It contains a distinguished hexagon of lines

$$E_1, \ell_{12}, E_2, \ell_{23}, E_3, \ell_{13},$$

where E_i is the exceptional divisor over p_i and ℓ_{ij} is the proper transform of the line joining p_i and p_j. We have intersection numbers

$$E_i^2 = \ell_{ij}^2 = -1, \quad i, j \in \{1, 2, 3\}$$
$$E_1\ell_{12} = \ell_{12}E_2 = E_2\ell_{23} = \ell_{23}E_3 = E_3\ell_{13} = \ell_{13}E_1 = 1$$

with the remaining intersections zero. The complement to this hexagon in $\widetilde{W}$ is a two-dimensional torus $U \simeq k^* \times k^*$.

The three *conic fibrations* arise from the linear series

$$|C_i| : \widetilde{W} \xrightarrow{\gamma_i} \mathbb{P}^1, \quad C_i = \ell_{ij} + E_j = \ell_{ik} + E_k. \tag{1}$$

The two *blowup realizations* come from the series

$$|R| : \widetilde{W} \xrightarrow{b} \mathbb{P}^2, \quad R = \ell_{12} + E_1 + E_2 = \ell_{13} + E_1 + E_3 = \ell_{23} + E_2 + E_3,$$

$$|R'| : \widetilde{W} \xrightarrow{b'} \mathbb{P}^2, \quad R' = E_1 + \ell_{12} + \ell_{13} = E_2 + \ell_{12} + \ell_{23} + E_3 + \ell_{13} + \ell_{23}.$$

Remark 1.1. Let $\widetilde{W}$ be a sextic del Pezzo surface over an arbitrary field K. Each conic fibration on $\widetilde{W}$ yields a étale algebra of dimension three over K. Given a rational point

$$s \in \widetilde{W}(K) \subset \mathbb{P}^6$$

away from the hexagon of lines, double projection from s gives a birational map to a quadric surface

$$\Pi(s) : \mathrm{Bl}_s(\widetilde{W}) \xrightarrow{\sim} QS \subset \mathbb{P}^3, \tag{2}$$

blowing down the three conics

$$s \in C_1(s), C_2(s), C_3(s) \subset \widetilde{W}, \quad C_i(s) = \gamma_i^{-1}(\gamma_i(s)).$$

1.2. **Nodal sextic del Pezzo surfaces.** Choose distinct points $w_+, w_- \in U$ that are generic in the sense that $\gamma_i(w_+) \neq \gamma_i(w_-)$ for $i = 1, 2, 3$. Project from a point on the secant line joining these points

$$x \in \mathrm{Sec}(w_+, w_-) \setminus \{w_0, w_-\}$$

to obtain:

$$\pi_x : \mathbb{P}^6 \dashrightarrow \mathbb{P}^5$$
$$\widetilde{W} \longrightarrow W := \pi_x(\widetilde{W})$$
$$\{w_+, w_-\} \mapsto w_0 := \pi_x(\mathrm{Sec}(w_+, w_-)).$$

The resulting surface $W \subset \mathbb{P}^5$ will be called a *nodal sextic del Pezzo surface*.

Proposition 1.2. *A nodal sextic del Pezzo surface W has the following properties:*

- *$\widetilde{W}$ is the normalization of W, whose only singularities is a node (transverse self-intersection) at w_0.*
- *W contains nodal plane curve curves N, N', singular at w_0, whose normalizations $\widetilde{N}, \widetilde{N}'$ are the unique members of $|B|$ and $|B'|$ containing $\{w_+, w_-\}$.*
- *The spanning planes $P \supset N$ and $P' \supset N'$ are contained in all the quadrics vanishing along W.*
- *W depends on two moduli as an abstract variety and three moduli as a polarized variety.*

Proof. The ideal of $\widetilde{W}$ is generated in degree two, so secant lines not contained in $\widetilde{W}$ meet it in exactly two points. We assumed that w_+ and w_- are not on the six lines, nor are both on the same conic. It follows that they cannot be extended to a set of four coplanar points on $\widetilde{W}$ [EGH96, Prop. 1], so x lies on a unique secant to $\widetilde{W}$. This implies that W has the indicated singularities.

Our genericity assumption for $\{w_+, w_-\}$ also ensures that $\widetilde{N}$ and $\widetilde{N}'$ are smooth twisted cubics on $\widetilde{W}$, thus they map to nodal cubic plane curves under projection. Since $P \cap W$ and $P' \cap W$ are plane cubic curves, these planes must be contained in the linear system of quadrics vanishing on W.

For the moduli statement, we observe that the choice of $\{w_+, w_-\}$, modulo the torus action, depends on two parameters. The choice of $x \in \mathrm{Sec}(w_+, w_-)$ – equivalent to stipulating the embedding line bundle on W – involves one additional parameter encoding an identification

$$\mathcal{O}_{\widetilde{W}}(1)|_{w_+} \simeq \mathcal{O}_{\widetilde{W}}(1)|_{w_-}.$$

$\square$

1.3. **Equations and linear series.** We continue to assume $W \subset \mathbb{P}^5$ is a nodal sextic del Pezzo surface.

Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r$ is *d-regular*, in the sense of Castelnuovo and Mumford, if $H^i(\mathcal{F}(d-i)) = 0$ for all $i \geq 1$. See [Eis05, ch. 4] for the properties of regularity, for instance:

- If $\mathcal{F}$ is d-regular then $\mathcal{F}(d)$ is globally generated [Eis05, Cor. 4.18].
- Let $H \subset \mathbb{P}^r$ be a hyperplane that is not a zero divisor of X; then $\mathcal{I}_X$ is d-regular iff $\mathcal{I}_{X \cap H}$ is d-regular and $H^1(\mathcal{I}_X(d-1)) = 0$ [Eis05, Ex. 4.14]. The last condition is equivalent to the surjectivity of $\Gamma(\mathcal{O}_{\mathbb{P}^r}(d-1)) \to \Gamma(\mathcal{O}_X(d-1))$.

Proposition 1.3. *The ideal sheaf $\mathcal{I}_W$ is 3-regular thus generated in degree ≤ 3*

$$\mathcal{I}_W = \langle Q_0, Q_1, Q_2, F, F' \rangle \,, \tag{3}$$

where the Q_i are quadratic forms and F and F' are cubic forms.

Remark 1.4. The surface

$$S = \{Q_0 = Q_1 = Q_2 = 0\} = P \cup_N W \cup_{N'} P'$$

is a degenerate K3 surface of degree eight. The forms F and F' may be chosen to cut out $N \subset P$ and $N' \subset P$ respectively.

Proof. A generic hyperplane section $C \subset X$ is a smooth curve of genus one and degree six, thus $\mathcal{I}_C \subset \mathcal{O}_{\mathbb{P}^4}$ is 3-regular [Eis05, §8A]. It remains then to analyze $H^1(\mathcal{I}_W(2))$; suppose $H^1(\mathcal{I}_W(2)) \neq 0$. We deduce that $h^0(\mathcal{I}_W(2)) > 3$ and also $h^0(\mathcal{I}_C(2)) > 3$, as $W \subset \mathbb{P}^5$ is nondegenerate; but then $h^1(\mathcal{I}_C(2)) \neq 0$, violating the 3-regularity of $\mathcal{I}_C$.

We therefore have

$$\Gamma(\mathcal{O}_{\mathbb{P}^5}(3)) \twoheadrightarrow \Gamma(\mathcal{O}_W(3)) \simeq k^{36},$$

hence $h^0(\mathcal{I}_W(3)) = 20$, giving (3). $\square$

Propositions 1.2 and 1.3 imply:

Corollary 1.5. *The moduli space of incidences*

$$\mathcal{V} = \{(W, X) : \textit{nodal sextic del Pezzo surface } W \subset X \textit{ cubic fourfold}\}$$

is unirational of dimension $3 + 19 = 22$.

Proposition 1.6. *Assume k has characteristic zero. Then a generic cubic fourfold $X \supset W$ is smooth.*

Proof. Consider the blow up $\widetilde{P} := \mathrm{Bl}_W(\mathbb{P}^5)$. This resolves the base locus of the linear series $\Gamma(\mathcal{I}_W(3))$, as the ideal of W is generated in degree three. Consider the resulting basepoint-free linear series on $\widetilde{P}$.

We analyze the singularities of $\widetilde{P}$ in étale-local coordinates centered at $w_0 \in W \subset \mathbb{P}^5$:

$$(0,0,0,0,0) \in \{u = vy = vz = xy = xz = 0\}.$$

The relevant affine patch on $\widetilde{P}$, where we see smooth hypersurfaces with tangent space $T_w W$, has coordinates and equations

$$(u, v, x, y, z, b, c, d) \quad u = vye, z = yb, x = vc, d = bc,$$

and thus is nonsingular. Overall, $\widetilde{P}$ has one isolated singularity, locally equivalent to a cone over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. The Bertini Theorem implies the generic member of our linear series is smooth. $\square$

Proposition 1.3 also yields:

Corollary 1.7. *Let $W \subset X$ be a nodal sextic del Pezzo surface in a smooth cubic fourfold. If X does not contain a plane then $\mathcal{I}_W \subset \mathcal{O}_X$ is generated in degree two. The quadratic equations for W induce*

$$\phi_W : \mathrm{Bl}_W(X) \to \mathbb{P}(\Gamma(\mathcal{I}_W(2))^\vee) \simeq \mathbb{P}^2, \tag{4}$$

a smooth fourfold fibered over $\mathbb{P}^2$.

Proof. Actually, we need only assume that X does not contain the planes P and P' described in Remark 1.4. This ensures that $\Gamma(\mathcal{I}_W(2))$ – regarded as a linear series on X – cuts out W. Blowing up resolves the indeterminacy and yields the morphism ϕ_W. Now $\mathrm{Bl}_W(X)$ is the blow-up of a smooth fourfold along a surface whose only singularities are two smooth branches meeting transversally; such blowups are smooth. (We will return to this local geometry in Lemma 1.9.) □

1.4. **Residuation constructions.** We continue to assume that W is a nodal sextic del Pezzo surface singular at w_0.

We start with an observation on smooth hyperplane sections $C \subset W$. These have genus one and degree six so $C \subset \mathbb{P}^4$ is neither linearly normal nor cut out by quadrics; $\mathcal{I}_C$ is not 2-regular! It is 3-regular so we have, as in (3),

$$\mathcal{I}_C = \langle q_0, q_1, q_2, f, f' \rangle .$$

Choose a pencil in $\mathrm{span}(q_0, q_1, q_2)$ and an element in $\mathrm{span}(f, f')$, both generic; after relabelling this could be written

$$D = \{q_0 = q_1 = f = 0\} = C \cup_B C'.$$

A liaison computation – D is a nodal complete intersection with $\omega_D \simeq \mathcal{O}_D(2)$ – yields

$$\mathcal{O}_C(B) = \mathcal{O}_C(2), \quad \mathcal{O}_{C'}(B) = \mathcal{O}_{C'}(2),$$

and C' also has genus one and degree six. This constructive is involutive in the sense that C is one of the curves C'' arising when this process is applied to C'.

We return to analyzing sextic del Pezzo surfaces:

Proposition 1.8. *Assume k has characteristic zero. Let $W \subset X \subset \mathbb{P}^5$ be a nodal sextic del Pezzo surface embedded in a smooth cubic fourfold that contains no planes. Given a generic pencil*

$$\Lambda \in \Gamma(\mathcal{I}_W(2)) \simeq k^3$$

with base locus $Y_\Lambda \subset \mathbb{P}^5$, the residual surface

$$Y_\Lambda \cap X = W \cup W'$$

is also a nodal sextic del Pezzo surface.

Proof. The most direct approach is to compute the invariants of W'. Since W is a nodal sextic del Pezzo surface we have

$$\chi(\mathcal{O}_W(n)) = \chi(\mathcal{O}_{\widetilde{W}}(n)) - 1 = 3n^2 + 3n.$$

Since $Y_\Lambda \cap X$ is a complete intersection

$$\chi(\mathcal{O}_{Y_\Lambda \cap X}(n)) = 6n^2 - 6n + 7.$$

The conductor curve $B = W \cap W'$ is a bit complicated. Its partial normalization $\widehat{B} \subset \widetilde{W}$ has nodes at w_+, w_-; write B^ν for the full normalization. Adjunction

$$\omega_{Y_\Lambda \cap X} = \mathcal{O}_{Y_\Lambda \cap X}(1), \quad \omega_{Y_\Lambda \cap X}|_{\widetilde{W}} = \omega_{\widetilde{W}}(\widehat{B})$$

implies $[\widehat{B}] = -2K_{\widetilde{W}}$ and has arithmetic genus seven. Thus B^ν has genus five and B is obtained from it by gluing four points – the preimages of $w_\pm$ – to a quadruple point. In conclusion, B has arithmetic genus eight, degree 12, and Hilbert polynomial

$$\chi(\mathcal{O}_B(n)) = 12n - 7.$$

Putting everything together

$$\begin{aligned}\chi(\mathcal{O}_{W'}(n)) &= \chi(\mathcal{O}_{Y_\Lambda \cap X}(n)) - \chi(\mathcal{O}_W(n)) + \chi(\mathcal{O}_B(n)) \\ &= (6n^2 - 6n + 7) - (3n^2 + 3n) + (12n - 7) = 3n^2 + 3n.\end{aligned}$$

Furthermore, since W' is the residual to W in a complete intersection of two quadrics, it also has a node (transverse double point) at w_0. The partial normalization $\widehat{B}' \subset W'$ of B also has two nodes and arithmetic genus seven. The gluing $B^\nu \to \widehat{B}'$ differs from the gluing $B^\nu \to \widehat{B}$.

Lemma 1.9. *Consider the blow up $\beta : \mathrm{Bl}_W(X) \to X$ and the morphism associated with the quadratic equations of W (see (4) above):*

$$\phi_W : \mathrm{Bl}_W(X) \to \mathbb{P}(\Gamma(\mathcal{I}_W(2))^\vee) \simeq \mathbb{P}^2.$$

- *$\beta^{-1}(w_0) \simeq \mathbb{P}^1 \times \mathbb{P}^1$;*
- *given a smooth local surface $w \in \Sigma \subset X$ meeting each branch of W along a smooth curve, Σ lifts to $\mathrm{Bl}_W(X)$ and meets $\mathbb{P}^1 \times \mathbb{P}^1$ transversally at a point;*
- *the proper transform of W' in $\mathrm{Bl}_W(X)$ equals the normalization of $w_0 \in W'$;*
- *the canonical sheaf*

  $$\omega_{\mathrm{Bl}_W(X)} = \beta^*\omega_X(E) = (\beta^*\mathcal{O}_X(-3))(E)$$

  where E is the exceptional divisor of β.

Proof. This is easy to understand through local analytic coordinates on X

$$\{u = v = y = z = 0\} = w_0 \in W = \{uy = uz = vy = vz = 0\}.$$

The blowup of W has equations

$$zA - yB = vA - uC = vB - uD = zC - yD = AD - BC,$$

which is nonsingular. The fiber over w_0 is

$$\{AD - BC = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Since Σ meets W as a Cartier divisor it lifts to $\mathrm{Bl}_W(X)$. Writing

$$T_w\Sigma = \{a_0u + a_1v = b_0y + b_1z = 0\}$$

we see that its intersection with the fiber over w_0 is

$$[A, B, C, D] = [a_1b_1, -a_1b_0, -a_0b_1, a_0b_0].$$

In particular, a pair of such surfaces meeting transversally at the origin is separated in the blowup. The last statement follows from adjunction and the formula for the canonical class of a blowup. $\square$

The last step of the proof of Proposition 1.8 requires that k have characteristic zero: The generic fiber $\widetilde{W}'$ of ϕ_W is smooth by the Bertini Theorem. Let W' denote its image in X and recall that

$$\chi(\mathcal{O}_{\widetilde{W}'}(n)) = \chi(\mathcal{O}_W(n)) + 1 = 3n^2 + 3n + 1,$$

as expected for a sextic del Pezzo surface.

Consider the linear series $\Gamma(X, \mathcal{I}_W(3))$ and the induced linear series

$$\Gamma(\mathrm{Bl}_W(X), \beta^*\mathcal{O}_X(3)(-E)) \simeq \Gamma(\omega^{-1}_{\mathrm{Bl}_W(X)}),$$

which is basepoint-free by Proposition 1.3. Furthermore, Corollary 1.7 implies

$$\Gamma(\mathrm{Bl}_W(X), \beta^*\mathcal{O}_X(2)(-E))$$

is also basepoint free; for each nonzero section $s \in \Gamma(X, \mathcal{I}_W(2))$ we have

$$\Gamma(\mathrm{Bl}_w(X), \beta^*\mathcal{O}_X(1)) \overset{s}{\hookrightarrow} \Gamma(\mathrm{Bl}_W(X), \beta^*\mathcal{O}_X(3)(-E)).$$

For smooth fibers $\widetilde{W}'$ of ϕ_W, adjunction yields

$$\Gamma(\mathrm{Bl}_W(X), \omega^{-1}_{\mathrm{Bl}_W(X)}) \twoheadrightarrow \Gamma(\widetilde{W}', \omega^{-1}_{\widetilde{W}'})$$

and for s nonvanishing along $\widetilde{W}'$ we obtain

$$\Gamma(W', \mathcal{O}_{W'}(1)) \hookrightarrow \Gamma(\widetilde{W}', \omega_{\widetilde{W}'}).$$

Thus $\omega^{-1}_{\widetilde{W}'}$ is ample hence $\widetilde{W}'$ is a sextic del Pezzo surface. $\square$

Remark 1.10. We offer a more conceptual way to understand Proposition 1.8, which applies provided k does not have characteristic two.

Suppose we are given two intersecting planes in $\mathbb{P}^5_{t,u,v,x,y,z}$

$$P = \{t = u = v = 0\}, P' = \{t = y = z = 0\}.$$

Consider

$$\Gamma(\mathcal{I}_{P\cup P'}(2)) = t\Gamma(\mathcal{O}_{\mathbb{P}^5}(1)) + \mathrm{span}(uy, uz, vy, vz)$$

which has dimension ten; all members have tangent space $t = 0$ at $p = [0, 0, 0, 1, 0, 0]$. Any complete intersection Y of two such quadrics contains two additional planes R and R' of the form

$$\{t = a_0u + a_1v = b_0y + b_1z = 0\};$$

$P \cup R \cup P' \cup R$ is a cone over a cycle of four rational curves. After coordinate change, we write

$$R = \{t = v = y = 0\}, \quad R' = \{t = z = u = 0\},$$

and the defining quadrics as

$$tL_1 + uy = tM_1 + vz = 0, \quad L_1, M_1 \in k[t, u, v, x, y, z] \text{ linear}.$$

Furthermore, we may assume

$$L_1 = x + at + L_2,\ M_1 = x + bt + M_2, \quad a, b \in k, \quad L_2, M_2 \in k[u, v, y, z].$$

The point p is generically an ordinary double point of Y. Projecting from p – which entails eliminating x – yields a quadric hypersurface

$$Q = \{t((a - b)t + (L_2 - M_2)) = vz - uy\} \subset \mathbb{P}^4_{t,u,v,y,z},$$

which is smooth provided Y is chosen generically. The inverse of projection

$$\pi_p : Y \overset{\sim}{\dashrightarrow} Q$$

blows up the locus

$$Z = \{t = vz = uy = 0\}.$$

We see that Y has four additional ordinary double points along the lines $P \cap R, R \cap P', P' \cap R', R' \cap P$, i.e., $|\operatorname{Sing}(Y)| = 5$. Now $\operatorname{Aut}(Q)$ acts transitively on the configurations Z. Indeed, it acts transitively on smooth hyperplane sections $Q_0 = \{t = 0\} \subset Q$, and the stabilizer of Q_0 surjects onto $\operatorname{Aut}(Q_0)$; the last statement is clear for $\{t^2 + vz - uy\}$, for example.

In particular, Y has automorphisms leaving $P \cup R \cup P' \cup R'$ invariant but permuting the individual components via the dihedral group of order eight. The same action applies to nodal sextic del Pezzo surfaces, i.e.,

$$Y \cap \{q = 0\} = P \cup P' \cup W, \quad Y \cap \{r = 0\} = R \cup R' \cup W',$$

where q and r are quadratic forms vanishing on $P \cup P'$ and $R \cup R'$ respectively. Thus the resulting surfaces W and W' have the same geometric properties.

1.5. **Rationality proof.** With these preliminaries in hand, proving rationality is routine:

Theorem 1.11. *Let k have characteristic zero, X a smooth cubic fourfold that contains no planes, and $W \subset X$ a nodal sextic del Pezzo surface. Assume X contains an algebraic two-cycle M such that $M \cdot W$ is odd. Then X is rational.*

Proof. We only need to assume that X does not contain the two planes described in Remark 1.4.

Proposition 1.8 gives a fibration in sextic del Pezzo surfaces

$$\phi_W : \mathrm{Bl}_W(X) \to \mathbb{P}^2.$$

A generic fiber maps to a surface with class

$$[W'] = 4h^2 - [W], \quad h \text{ hyperplane class},$$

as they are residual in a pencil of quadrics. Note that

$$M \cdot W \equiv M \cdot W' \pmod 2.$$

If $\phi_W^{-1}(p)$ is rational for $p \in \mathbb{P}^2$ generic then X is rational as well. We focus on this generic fiber $\widetilde{W}_p$. A sextic del Pezzo surface is rational iff it admits a zero-cycle of degree relatively prime to six [AHTVA19, Prop. 8]. In our situation $\widetilde{W}_p$ comes with a length-two subscheme – the pre-image of the singularity w_0 – so we only need a zero-cycle of odd degree. Restricting M to $\widetilde{W}_p$ gives what we require. □

We describe the exceptional locus of the birational map produced Theorem 1.11. Globalizing (2) over $\mathbb{P}^2$, we obtain

$$\begin{tikzcd} \Pi_W : \mathrm{Bl}_W(X) \arrow[rr, dashed, "\sim"] \arrow[dr, "\phi_W"'] & & \mathcal{Q} \arrow[dl] \\ & \mathbb{P}^2 & \end{tikzcd}$$

where $\mathcal{Q} \to \mathbb{P}^2$ is a quadric surface bundle. Consider the Hilbert scheme of conics in fibers of ϕ_W

$$\mathrm{Conics}(\mathrm{Bl}_W(X)/\mathbb{P}^2) \to \mathbb{P}^2$$

inducing a generically-finite morphism $S_W \to \mathbb{P}^2$ of degree three (see Remark 1.1). Thus we conclude:

Remark 1.12. The birational parametrization $\mathbb{P}^4 \overset{\sim}{\dashrightarrow} X$ of Theorem 1.11 blows up a surface birational to the triple cover $S_W \to \mathbb{P}^2$ parametrizing conic fibrations in fibers of ϕ_W.

2. Hodge theory and hyperkähler geometry

In this section we assume $k = \mathbb{C}$.

2.1. **Statement of results.** Suppose that X is a smooth cubic fourfold containing a nodal sextic del Pezzo surface W. The lattice of cycles generated by W and the square of the hyperplane class h has intersection form

$$\begin{array}{c|cc} & h^2 & W \\ \hline h^2 & 3 & 6 \\ W & 6 & 20 \end{array} \tag{5}$$

and thus is special of discriminant 24 in the sense of [Has00, §3]. Note the lattice involution

$$h^2 \mapsto h^2, \quad W \mapsto 4h^2 - W$$

reflecting the residuation construction of Proposition 1.8.

We clarify precisely where Theorem 1.11 applies:

Theorem 2.1. *Let X be a smooth cubic fourfold with hyperplane class h admitting a saturated lattice of algebraic two-cycles*

$$\langle h^2, W, M \rangle$$

with intersection form

$$\begin{array}{c|ccc} & h^2 & W & M \\ \hline h^2 & 3 & 6 & m \\ W & 6 & 20 & a \\ M & m & a & s \end{array} \tag{6}$$

with a odd. Then X is rational.

The Integral Hodge Conjecture is true for cubic fourfolds [Voi13, Th. 1.4] so we obtain:

Corollary 2.2. *Let X be a smooth complex cubic fourfold with a saturated collection of Hodge classes (6). Then X is rational.*

The smooth cubic fourfolds admitting a collection of algebraic cycles (or Hodge classes) of type (5) is an *irreducible* divisor $\mathcal{C}_{24}$ in the moduli space $\mathcal{C}$ [Has00, §3.2]. Since rationality is preserved under specialization of smooth projective varieties [KT19], Theorem 2.1 follows immediately from

Proposition 2.3. *A generic element of $\mathcal{C}_{24}$ contains nodal sextic del Pezzo surfaces.*

In light of Corollary 1.5 and the fact that $\dim(\mathcal{C}_{24}) = 19$, we also deduce

Corollary 2.4. *For generic $[X] \in \mathcal{C}_{24}$, the Hilbert scheme of nodal sextic del Pezzo surfaces in X contains two connected components $\mathcal{H}_{dP}(X)$ and $\mathcal{H}'_{dP}(X)$, each of dimension three and interchanged via residuation.*

The remainder of this section – with support from Sections 3 and 4 – is devoted to the proof of Proposition 2.3.

2.2. **Hyperkähler geometry of the variety of lines.** We recall constructions of [BD85]: Let X be a smooth cubic fourfolds with polarization h and $F_1(X)$ its variety of lines with polarization g. Then $F_1(X)$ is a smooth hyperkähler fourfold, deformation equivalent to the Hilbert scheme of pairs of points on a K3 surface. The incidence correspondence induces a homomorphism of Hodge structures

$$\iota : H^4(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})(-1)$$

and an isomorphism on primitive cohomology

$$\iota : (h^2)^\perp \simeq g^\perp.$$

On primitive cohomology, the intersection form on X coincides (up to sign) with the Beauville-Bogomolov form $(,)$ form on $F_1(X)$. Note that $(g,g) = 6$ and $(g, H^2(F_1(X),\mathbb{Z})) = 2\mathbb{Z}$.

For instance, (5) gives intersection form

$$\begin{array}{c|cc} & h^2 & p \\ \hline h^2 & 3 & 0 \\ p & 0 & 8 \end{array}, \quad p = W - 2h^2,$$

hence we have Beauville-Bogomolov form

$$\begin{array}{c|cc} & g & \varpi \\ \hline g & 6 & 0 \\ \varpi & 0 & -8 \end{array}, \quad \varpi = \iota(p). \tag{7}$$

The orthogonal complement complement to these divisor classes

$$\{g, \varpi\}^\perp \subset H^2(F_1(X),\mathbb{Z})$$

is an even lattice of discriiminant 24 and signature $(2,19)$, isomorphic to (see [Has00, §2,3])

$$\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus (8) \oplus U \oplus (-E_8)^2,$$

where E_8 is the positive definite lattice associated with the corresponding root system and

$$U \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is hyperbolic. Now this lattice admits a canonical index-two extension

$$\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \oplus (2) \oplus U \oplus (-E_8)^2 \tag{8}$$

which is isomorphic to

$$(-6) \oplus U \oplus U \oplus (-E_8)^2. \tag{9}$$

Indeed, fixing basis elements for the first two summands of (8)

$$(a,a) = (b,b) = -2, (a,b) = -1, (c,c) = 2$$

then

$$w = 3c + 2(a+b), u_2 = c + a, v_2 = c + b$$

is the basis for the first two summands of (9). Now (9) is the primitive cohomology of a K3 surface of degree six. Surjectivity of the Torelli map for K3 surfaces yields (cf. [Huy17, Th. 1.4]):

Proposition 2.5. *Let X be a discriminant 24 special cubic fourfold with distinguished divisors $g, \varpi \in \mathrm{Pic}(F_1(X))$ as above. Then there exists a K3 surface S and a divisor $f \in \mathrm{Pic}(S)$ satisfying $f \cdot f = 6$ such that*

$$\{g, \varpi\}^{\perp} \hookrightarrow f^{\perp} \subset H^2(S, \mathbb{Z})$$

as an index-two sublattice. The data (S, α), where

$$\alpha \in H^2(S, \mathbb{Z}/(1/2)\mathbb{Z})/(H^2(S,\mathbb{Z}) + \mathbb{Z}(f/2))$$

determines the index-two sublattice, is a twisted K3 surface with Brauer class $[\alpha] \in \mathrm{Br}(S)[2]$.

We are not asserting that f is a polarization on S.

We return to divisors on $F_1(X)$, specifically

$$E = g + \varpi, E' = g - \varpi, \quad E^2 = (E')^2 = -2$$

with Beauville-Bogomolov pairing

$$\begin{array}{c|cc} & g & E \\ \hline g & 6 & 6 \\ E & 6 & -2 \end{array}. \tag{10}$$

Assume that X is generic in the divisor of cubic fourfolds of discriminant 24. Moreover the main result of [BHT15] – see also [BM14, Th. 12.4] – implies that E and E' are effective divisors with $\mathbb{P}^1$-fibrations

$$E, E' \to S$$

classified by α. See [HT16] for explications of the Bayer-Macrì classification and descriptions of the exceptional loci in small dimensions. As we shall see in Theorem 5.3, $F_1(X)$ is a moduli space of α-twisted sheaves on S.

Let R and R' denote the generic fibers of $E \to S$ and $E' \to S$ respectively. The homology classes are readily computed via the Beauville-Bogomolov form; see [HT09, §4] for context. Given $v \in H^2(F_1(X), \mathbb{Z})$ we have

$$v \cdot R = (v, E), \quad v \cdot R' = (v, E'),$$

and in particular $g \cdot R = g \cdot R' = 6$. Thus tracing R and R' through the incidence correspondence linking $F_1(X)$ and X, we obtain sextic ruled surfaces

$$T, T' \subset X,$$

each deforming in family parametrized by S. Note that

$$[T'] = 4h^2 - [T]$$

in the cohomology of X.

3. Synthetic geometry

3.1. **Geometry of the scrolls.** While hyperkähler theory tells us the sextic scrolls exist, a direct construction clarifies what they look like. The double point formula guarantees that T and T' are singular; *assuming* they have only nodes (transverse double points) then each has two singularities (see [HT01, §7]). This is in fact the case:

Proposition 3.1. *Let X be a discriminant 24 special cubic fourfold that is generic with this property. Then X admits two families $\mathcal{H}_{ss}(X)$ and $\mathcal{H}'_{ss}(X)$ of sextic scrolls, generically with two nodes; each family is parametrized by the K3 surface S described in Proposition 2.5.*

These surfaces are called *two-nodal sextic scrolls.*

Most of this is already proven except for the explicit construction of the families:

Lemma 3.2. *Let $\widetilde{T} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ be a smooth sextic scroll associated with the linear series $\Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,3))$. Choose four generic points*

$$t_{1+}, t_{1-}, t_{2+}, t_{2-} \in \mathbb{P}^1 \times \mathbb{P}^1$$

and a generic line $\ell \subset \mathbb{P}^7$ *such that*

$$\ell \cap \mathrm{Sec}(t_{1+}, t_{1-}) \neq \emptyset, \ \ell \cap \mathrm{Sec}(t_{2+}, t_{2-}) \neq \emptyset.$$

Projecting from ℓ

$$\pi_\ell : \widetilde{T} \to T \subset \mathbb{P}^5$$

yields a sextic scroll T *with nodes* t_1 *and* t_2. *This surface is contained in cubic fourfolds and the generic such variety is smooth.*

The argument is very similar to the corresponding statement for nodal sextic del Pezzo surfaces. It can also be obtained by direct computation for an example over a convenient finite field. For our purposes, it is most useful to outline the key properties:

- $\dim \Gamma(\mathcal{I}_T(2)) = 2$ and $\dim \Gamma(\mathcal{I}_T(3)) = 18$;
- $T \subset Y$ a complete intersection of two quadrics;
- the ideal of T is generated in degrees ≤ 3

  $$\mathcal{I}_T = \langle Q_0, Q_1, F_0, F_1, F_2, F_3, F_4, F_5 \rangle$$

  hence a generic cubic fourfold $X \supset T$ is smooth;
- fixing $X \supset T$, the intersection

  $$X \cap Y = T \cup T'$$

  where T' is also a two-nodal sextic scroll.

As a dividend, we obtain

Corollary 3.3. *The moduli space of incidences*

$$\mathcal{U} = \{(T, X) : \ \textit{two-nodal sextic scroll } T \subset X \ \textit{cubic fourfold}\}$$

is unirational of dimension $2+2+17 = 21$ *and has a natural involution*

$$\begin{array}{ccc} \mathcal{U} & \dashrightarrow & \mathcal{U} \\ T & \mapsto & T'. \end{array}$$

3.2. **Relating the scrolls and del Pezzos.** Proposition 2.3 involves a synthetic construction for the nodal sextic del Pezzo surfaces starting from the two-nodal sextic scrolls. This suffices because discriminant 24 cubic fourfolds generically contain the latter surfaces by Proposition 3.1. We carry out this construction in several stpes.

First construction. Each sextic del Pezzo surfaces $\widetilde{W} \subset \mathbb{P}^6$ admits three conic bundle structures (see (1)):

$$\gamma_i : \widetilde{W} \to \mathbb{P}^1, \quad C_i = \gamma_i^* \mathcal{O}_{\mathbb{P}^1}(1).$$

Each has a factorization through a $\mathbb{P}^2$ bundle

$$\begin{tikzcd} \widetilde{W} \arrow[r, hook] \arrow[dr, "\gamma_i"'] & \widetilde{V}_i \arrow[d, "\varpi_i"] \\ & \mathbb{P}^1 \end{tikzcd}$$

where $\widetilde{V}_i = \mathbb{P}((\gamma_{i*}(\omega_{\gamma_i}^{\vee}))^{\vee})$. The Leray spectral sequence gives

$$\Gamma(\widetilde{W}, \omega_{\gamma_i}^{\vee}(nC_i)) = \Gamma(\mathbb{P}^1, \gamma_{i*}(\omega_{\gamma_i}^{\vee}) \otimes \mathcal{O}_{\mathbb{P}^1}(n)).$$

Since

$$\Gamma(\widetilde{W}, \omega_{\gamma_i}^{\vee}(nC_i)) = \Gamma(\widetilde{W}, \mathcal{O}_{\widetilde{W}}((n-2)C_i) \otimes \mathcal{O}_{\widetilde{W}}(1)) \simeq k^{3n+1}$$

we find that

$$\gamma_{i*}(\omega_{\gamma_i}^{\vee}) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^2.$$

Thus

$$\widetilde{V}_i \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$$

and we have a factorization

$$\widetilde{W} \subset \widetilde{V}_i \subset \mathbb{P}^6$$

through a quartic threefold ruled in planes.

Writing $\xi = [\mathcal{O}_{\widetilde{V}_i}(1)]$ and f for the class of a fiber of ϖ_i, we compute

$$\xi^3 = 4, \ \xi^2 f = 1, \ [\widetilde{W}] = 2\xi - 2f.$$

A computation. Let $\pi_x : \widetilde{W} \to \mathbb{P}^5$ denote the projection from a point $x \in \mathrm{Sec}(w_+, w_-)$ as in Section 1.2.

Proposition 3.4. *There exists a unique section σ of ϖ_i with the following properties:*

- *$\sigma(\mathbb{P}^1) \subset \widetilde{V}_i \subset \mathbb{P}^6$ is a conic;*
- *$\sigma(\gamma_i(w_\pm)) = w_\pm$;*
- *the projection*
$$\pi_x : \widetilde{V}_i \to V_i$$
pinches $\widetilde{V}_i$ along $\sigma(\mathbb{P}^1)$ via an involution of the conic gluing w_+ and w_-.

Proof. For concreteness we set $i = 1$. In Section 1.2 we assumed that $\gamma_1(w_+) \neq \gamma_1(w_-)$, so it is possible for a section to contain both of them. The curves E_1 and ℓ_{23} are sections of γ_1 spanning the distinguished line-subbundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^2) \subset \widetilde{V}_1$. Our assumptions also guarantee that w_+ and w_- are outside this line. The sections with image of degree two are given by

$$\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2),$$

indexed by elements of $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^1})) \simeq \mathbb{P}^4$. We may rescale so that the coordinates of w_+ and w_- are 1, i.e.

$$w_+ = [a_{1+}, a_{2+}, 1], \quad w_- = [a_{1-}, a_{2-}, 1].$$

Linear algebra gives a unique section the desired property.

The double-point formula [Ful98, Th. 9.3] applied to the projection

$$\pi_x : \widetilde{V}_1 \to \mathbb{P}^5$$

gives the class of the locus in $\widetilde{V}_1$ sitting over the singularities of the image: It is the residual intersection, to two lines in fibers of ϖ_1, in a codimension-two linear section of $\widetilde{V}_1 \subset \mathbb{P}^6$; its class equals to the class of $\sigma(\mathbb{P}^1)$. We compute that

$$\sigma(\mathbb{P}^1) \cap \widetilde{W} = \{w_+, w_-\}.$$

Let Π denote the span of $\sigma(\mathbb{P}^1)$ in $\mathbb{P}^6$. Since $x \in \mathrm{Sec}(w_+, w_-)$, it also sits on Π. Projection from x induces a degree-two morphism

$$\sigma(\mathbb{P}^1) \to \mathrm{Sing}(V_1),$$

to the line in V_1 along which it is singular. Moreover, we have

$$w_0 = \pi_x(w_\pm) = \pi_x(\sigma(\mathbb{P}^1) \cap \widetilde{W}) \in \mathrm{Sing}(V_1).$$

$\square$

Remark 3.5. While $V_1 \subset \mathbb{P}^5$ is a threefold of degree four containing W, it is not a complete intersection of two quadrics! Indeed,

$$\dim \Gamma(\mathcal{I}_{V_1}(2)) = 1$$

as codimension-two linear sections of V_1 are rational normal quartic curves in $\mathbb{P}^3$.

Suppose $W \subset X$, a cubic fourfold. The conics in W arising as fibers of γ_i are residual to lines in X. Computing in $\widetilde{V}_i$ we find

$$X \cap \widetilde{V}_i = \widetilde{W} \cup \widetilde{T}_i, \quad [\widetilde{T}_i] = \xi + 2f,$$

where $\widetilde{T}_i \subset \mathbb{P}^6$ is a sextic scroll with

$$\widetilde{T}_i \cap \sigma(\mathbb{P}^1) = \{w_+, w_-, t_{2+}, t_{2-}\}. \tag{11}$$

Projection π_x takes

$$\widetilde{T}_i \to T_i$$

with nodes at the images of (11).

Proposition 3.6. *Let X be a generic cubic fourfold containing a nodal sextic del Pezzo surface W and*

$$W \subset V_i \subset \mathbb{P}^5$$

the threefold spanned by the conics from $|C_i|$. We have a residual intersection

$$X \cap V_i = T_i \cup W$$

where T_i is a sextic scroll with two nodes $t_1 := w_0$ and t_2.

Corollary 3.7. *Each $W \subset X$ admits* three *residual two-nodal sextic scrolls T_1, T_2, and T_3, indexed by the conic fibrations on $\widetilde{W}$.*

Reversing the construction. We return to the situation of Lemma 3.2: $\widetilde{T} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ is a smooth sextic scroll, with four generic points

$$t_{1+}, t_{1-}, t_{2+}, t_{2-} \in \mathbb{P}^1 \times \mathbb{P}^1$$

and a line $\ell \subset \mathbb{P}^7$ such that

$$\ell \cap \mathrm{Sec}(t_{1+}, t_{1-}) \neq \emptyset, \ \ \ell \cap \mathrm{Sec}(t_{2+}, t_{2-}) \neq \emptyset.$$

Now choose $p \in \ell$, not on the secant lines, so we have a factorization

$$\pi_\ell : \mathbb{P}^7 \overset{\pi_p}{\dashrightarrow} \mathbb{P}^6 \overset{\pi_{p'}}{\dashrightarrow} \mathbb{P}^5$$

where $p' = \pi_p(\ell)$. From the perspective $\widetilde{T}$, the point p is not special: A generic point of $\mathbb{P}^7$ is contained in the span of some pair of secant lines for $\widetilde{T}$. Now the projected scroll is contained in a distinguished three-dimensional degree-four scroll

$$\pi_p(\widetilde{T}) \subset \Sigma_p \to \mathbb{P}^6$$

where the fibrations $\widetilde{T} \to \mathbb{P}^1$ and $\Sigma_p \to \mathbb{P}^1$ are compatible; see Theorem 4.2 for an explanation why. Projecting from p' yields

$$T \subset \pi_{p'}(\Sigma_p).$$

If X is a cubic fourfold containing T then residual intersection yields a nodal sextic del Pezzo surface

$$X \cap \Sigma_p = T \cup W_p.$$

We summarize this as follows:

Proposition 3.8. *Let X be a generic cubic fourfold containing a two-nodal sextic scroll T. If ℓ is the line of projection for T then for generic $p \in \ell$ we obtain a nodal sextic del Pezzo surface $W_p \subset X$, residual to T in the projection of a degree-four three-dimensional scroll projected to $\mathbb{P}^5$.*

This completes the proof of Proposition 2.3, assuming the constructions of Section 4.

3.3. **Speculation on the correspondence.** Let X be a generic cubic fourfold of discriminant 24, $\mathcal{H}_{dP}(X)$ and $\mathcal{H}'_{dP}(X)$ the families of nodal sextic del Pezzo surfaces in X, and $\mathcal{H}_{ss}(X)$ and $\mathcal{H}'_{ss}(X)$ the families of two-nodal sextic scrolls in X. Assume that

$$[W] = 4h^2 - T$$

for $W \in \mathcal{H}_{dP}(X)$ and $T \in \mathcal{H}_{ss}(X)$ via the residuation constructions of Propositions 3.6 and 3.8.

Recall that

(1) given $W \in \mathcal{H}_{dP}(X)$, the $W' \in \mathcal{H}'_{dP}(X)$ directly residual to W are parametrized by a $\mathbb{P}^2$ and *vice versa* (Prop. 1.8);
(2) $\mathcal{H}_{dP}(X)$ and $\mathcal{H}'_{dP}(X)$ are three-dimensional (Cor. 2.4);
(3) $\mathcal{H}_{ss}(X)$ and $\mathcal{H}'_{ss}(X)$ are isomorphic to the degree-six K3 surface S (Props. 2.5 and 3.1);
(4) given $W \in \mathcal{H}_{dP}(X)$, we have three $T_1, T_2, T_3 \in \mathcal{H}_{ss}(X)$ directly residual to W (Cor. 3.7);
(5) given $T \in \mathcal{H}_{ss}(X)$, the $W \in \mathcal{H}_{dP}(X)$ directly residual to T are parametrized by a $\mathbb{P}^1$ (Prop. 3.8).

Consider items (1) and (2) and the residuation correspondence

$$\mathcal{R} = \{(W, W') : W \cup W' = \{Q_\lambda = 0\}_{\lambda \in \mathbb{P}^1} \cap X\} \subset \mathcal{H}_{dP}(X) \times \mathcal{H}'_{dP}(X).$$

Fixing $W \in \mathcal{H}_{dP}(X)$, let $\mathcal{R}_W \simeq \mathbb{P}^2$ denote the elements directly residual to W. For generic

$$W'_t, t \in \mathcal{R}_W$$

we have

$$W \in \mathcal{R}_{W'_t} \simeq \mathbb{P}^2.$$

In other words, each W is a contained a family of divisors parametrized by $\mathbb{P}^2$, with the generic member isomorphic to $\mathbb{P}^2$. This resembles the duality correspondence

$$\{(p, L) : L(p) = 0\} \subset \mathbb{P}^3 \times (\mathbb{P}^3)^\vee.$$

The K3 surface appearing in item (3) naturally embeds

$$S \subset Q \subset \mathbb{P}^4$$

where Q is a smooth quadric threefold. Lines $\ell \subset Q$ are three-secant to S, inducing

$$\begin{aligned} \mathbb{P}^3 \simeq F_1(Q) &\hookrightarrow S^{[3]} \\ \ell &\mapsto \ell \cap S \end{aligned}$$

which resembles the correspondence in item (4). Furthermore, through each point $s \in S$, there is a conic parametrizing the lines

$$s \in \ell \subset Q,$$

i.e. the projective tangent cone of $T_s Q \cap Q$. This is similar to what we observe in item (5).

Further, for each line $\ell \subset Q$, projection induces

$$\mathrm{Bl}_{\ell \cap S}(S) \to \mathbb{P}^2$$

a degree-three cover. On the other hand, conics in fibers of

$$\phi_W : \mathrm{Bl}_W(X) \to \mathbb{P}^2$$

give a degree-three morphism $S_W \to \mathbb{P}^2$. When ϕ_W has a section, the resulting birational parametrization of X blows up a surface birational to S_W by Remark 1.12.

Question 3.9. Let X be a generic cubic fourfold of discriminant 24 and $S \subset \mathbb{P}^4$ the degree-six K3 surface from Prop. 2.5. Is $\mathcal{H}_{dP}(X)$ isomorphic to $\mathbb{P}^3$, realized as the lines ℓ three-secant to S? Can we identify the surfaces $\mathrm{Bl}_{\ell \cap S}(S)$ and S_W?

4. Background on scrolls

We work over a field k of characteristic zero.

Fix positive integers a and r and consider the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-a)^s \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)^{r-s}$$

for some $s = 1, \ldots, r$. The dual bundle $\mathcal{E}^\vee$ is generated by

$$s(a+1) + (r-s)(a+2) = r(a+2) - s$$

global sections, yielding

$$\mathcal{E} \hookrightarrow \Gamma(\mathbb{P}^1, \mathcal{E}^\vee)^\vee \otimes \mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}^{r(a+2)-s}.$$

Its projectivization

$$\Sigma := \mathbb{P}(\mathcal{E}) \subset \mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{E}^\vee)^\vee) \simeq \mathbb{P}^{r(a+2)-s-1} = \mathbb{P}^n, \quad n := r(a+2) - s - 1,$$

is a *generic scroll* of dimension r and degree

$$d := as + (r-s)(a+1) = r(a+1) - s.$$

Here "generic" refers to the splitting type of $\mathcal{E}$. Since $r + d = n + 1$ the variety Σ is of minimal degree.

Automorphisms of $\mathcal{E}$ take the form

$$\begin{pmatrix} C_1 & L \\ 0 & C_2 \end{pmatrix}$$

where C_1 is an $s \times s$ invertible matrix of constant forms, C_2 an $(r-s) \times (r-s)$ invertible matrix of constant forms, and L is an $s \times (r-s)$ matrix of linear forms. Thus

$$\dim(\mathrm{Aut}(\mathcal{E})) = s^2 + (r-s)^2 + 2s(r-s) = r^2$$

and

$$\dim(\mathrm{Aut}(\Sigma)) = r^2 + 2.$$

The Hilbert scheme of generic scrolls

$$\mathrm{Hilb}(\Sigma \subset \mathbb{P}^n) \tag{12}$$

thus has dimension $n^2 + 2n - r^2 - 2$.

Now assume $r \geq 2$ so that

$$\mathrm{Pic}(\Sigma) = \mathbb{Z}f + \mathbb{Z}h$$

where h is the hyperplane class and f is the fiber of $\Sigma = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. The space

$$\Gamma(\Sigma, \mathcal{O}_\Sigma(h+2f)) = \Gamma(\mathbb{P}^1, \mathcal{E}^\vee(2)) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a+2)^s \oplus \mathcal{O}_{\mathbb{P}^1}(a+3)^{r-s})$$

has dimension $r(a+4) - s = n + 2r + 1$. The Hilbert scheme of flags

$$\mathrm{Hilb}(\Xi \subset \Sigma \subset \mathbb{P}^n), \quad [\Xi] = h + 2f, \tag{13}$$

a $\mathbb{P}^{n+2r}$ bundle over (12), has dimension

$$n^2 + 3n - r^2 + 2r - 2.$$

The number of moduli – modding out by the action of PGL_{n+1} – is

$$\#\,\mathrm{Moduli}(\Xi \subset \Sigma \subset \mathbb{P}^n) = n - r^2 + 2r - 2. \tag{14}$$

Assume that Ξ is smooth. We represent $\Xi = \mathbb{P}(\mathcal{F})$ where $\mathcal{F}$ is a rank-$(r-1)$ vector bundle. The equation for Ξ, a section

$$s_\Xi \in \Gamma(\Sigma, \mathcal{O}_\Sigma(h+2f)) \simeq \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{E}^\vee),$$

yields exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{E}^\vee \to \mathcal{F}^\vee \to 0,$$
$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

In particular,

$$\deg(\mathcal{F}^\vee) = \deg(\mathcal{E}^\vee) + 2 = r(a+1) - s + 2, \quad \chi(\mathcal{F}^\vee) = \chi(\mathcal{E}^\vee) + 1.$$

Example 4.1. Suppose $r = s = 2$ so that

$$\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}(a)^2.$$

For generic Σ in the linear series $|h+2f|$ we have

$$\mathcal{F}^\vee = \mathcal{O}_{\mathbb{P}^1}(2a+2) = \mathcal{O}_{\mathbb{P}^1}(n+1).$$

When $s = 1$ so that

$$\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)$$

then generically

$$\mathcal{F}^\vee = \mathcal{O}_{\mathbb{P}^1}(2a+3) = \mathcal{O}_{\mathbb{P}^1}(n+1).$$

Now take $r = 3$. For $s = 3$ we have $\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}(a)^3$ and generically

$$\mathcal{F}^\vee = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(1+3a/2)^2 & \text{if } a \text{ even} \\ \mathcal{O}_{\mathbb{P}^1}((3a+1)/2) \oplus \mathcal{O}_{\mathbb{P}^1}((3a+3)/2) & \text{if } a \text{ odd.} \end{cases}$$

When $s = 2$, $\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}(a)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)$ and generically

$$\mathcal{F}^\vee = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(1+3a/2) \oplus \mathcal{O}_{\mathbb{P}^1}(2+3a/2) & \text{if } a \text{ even} \\ \mathcal{O}_{\mathbb{P}^1}((3a+3)/2)^2 & \text{if } a \text{ odd.} \end{cases}$$

When $s = 1$, $\mathcal{E}^\vee = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)^2$ and generically

$$\mathcal{F}^\vee = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(2+3a/2)^2 & \text{if } a \text{ even} \\ \mathcal{O}_{\mathbb{P}^1}((3a+1)/2) \oplus \mathcal{O}_{\mathbb{P}^1}((3a+3)/2) & \text{if } a \text{ odd.} \end{cases}$$

We have seen that $\Gamma(\mathbb{P}^1, \mathcal{E}^\vee) \subset \Gamma(\mathbb{P}^1, \mathcal{F}^\vee)$ has codimension one. Thus $\Xi \subset \mathbb{P}^n$ is not linearly normal but arises as the projection of a scroll

$$\widetilde{\Xi} = \mathbb{P}(\mathcal{F}) \subset \mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{F}^\vee)^\vee) \simeq \mathbb{P}^{n+1}$$

from a point p; recall that

$$2\dim(\widetilde{\Xi}) = 2(r-1) < 2r + ar - s - 1 = \dim(\mathbb{P}^n)$$

so the projection to $\mathbb{P}^n$ is smooth for generic p. The Hilbert scheme

$$\mathrm{Hilb}(\widetilde{\Xi} \subset \mathbb{P}^{n+1} \ni p), \tag{15}$$

has dimension

$$(n+1)^2 + 2(n+1) - (r-1)^2 - 2 + (n+1) = n^3 + 5n - r^2 + 2r + 1.$$

Modding out by the action of PGL_{n+2} we get

$$\#\,\mathrm{Moduli}(\widetilde{\Xi} \subset \mathbb{P}^{n+1} \ni p) = n - r^2 + 2r - 2. \tag{16}$$

The equality between (14) and (16) suggests analyzing fibers of the morphism

$$\Phi : \mathrm{Hilb}(\Xi \subset \Sigma \subset \mathbb{P}^n)/\,\mathrm{PGL}_{n+1} \to \mathrm{Hilb}(\widetilde{\Xi} \subset \mathbb{P}^{n+1} \ni p)/\,\mathrm{PGL}_{n+2},$$

which we expect to be generically finite.

Theorem 4.2. *Let* $\widetilde{\Xi} \subset \mathbb{P}^{n+1}$ *be a generic scroll of dimension* $r-1 < n/2$ *and* $p \in \mathbb{P}^{n+1}$ *a generic point. Then the projection*

$$\pi_p : \widetilde{\Xi} \to \Xi \subset \mathbb{P}^n$$

is contained in a unique scroll of dimension r*. In particular,* Φ *is birational.*

Proof. Given $\widetilde{\Xi} = \mathbb{P}(\mathcal{F}) \subset \mathbb{P}^{n+1}$, projections from $p \in \mathbb{P}^{n+1}$ correspond to elements $\lambda \in \Gamma(\mathbb{P}^1, \mathcal{F}^\vee)^\vee$. However, Serre duality gives

$$\Gamma(\mathbb{P}^1, \mathcal{F}^\vee)^\vee = \mathrm{Ext}^1(\mathcal{F}^\vee, \omega_{\mathbb{P}^1}) = \mathrm{Ext}^1(\mathcal{F}^\vee, \mathcal{O}_{\mathbb{P}^1}(-2))$$

hence an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{E}^\vee \to \mathcal{F}^\vee \to 0.$$

Since $\mathcal{F}^\vee$ has generic splitting type, the same holds for $\mathcal{E}^\vee$ for generic p. Repeating the analysis above, we deduce that

$$\Xi \subset \mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{E}^\vee)^\vee)$$

and is associated with a section of $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{E}^\vee) = \Gamma(\mathbb{P}^1, \mathcal{E}^\vee(2))$. □

Example 4.3. cf. [Par07] Suppose that $\widetilde{\Xi} \subset \mathbb{P}^{n+1}$ is a rational normal curve. For odd $n = 2a+1$, a generic $p \in \mathbb{P}^{2a+2}$ lies on a curve of $(a+2)$-secant $(a+1)$ planes, parametrized by $\mathbb{P}^1$ [RS00, §1.5]. For even $n = 2a+2$, a generic $p \in \mathbb{P}^{2a+3}$ lies on a $(a+2)$-secant $(a+1)$ plane, which is unique by a classical result of Sylvester [RS00, §0.4].

In the even case, let $\xi_1 + \cdots + \xi_{a+2} \in \mathrm{Sym}^{a+2}(\widetilde{\Xi})$ denote the divisor such that $p \in \mathrm{span}(\xi_1, \ldots, \xi_{a+2})$. The corresponding projected points

$$\xi_1, \ldots, \xi_{a+2} \in \Xi \subset \mathbb{P}^{2a+2}$$

span a $\mathbb{P}^a$. The Kapranov construction [Kap93] gives a canonical rational normal curve

$$\xi_1, \ldots, \xi_{a+2} \in R \subset \mathbb{P}^a$$

and an isomorphism $\iota : R \xrightarrow{\sim} \Xi$ respecting $\xi_1, \ldots, \xi_{a+2}$. The join of R and Ξ via ι is isomorphic to

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)),$$

with R the distinguished section and $[\Xi] = R + (a+3)f = h + 2f$, where f is the fiber over $\mathbb{P}^1$ and h is the hyperplane class. Hence Σ is obtained as this join.

In the odd case, let $(\xi_1 + \cdots + \xi_{a+2})_t, t \in \mathbb{P}^1$, denote the divisors with secants containing p. The corresponding linear series has graph

$$R \subset \widetilde{\Xi} \times \mathbb{P}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

with bidegree $(a+2,1)$. After projection to $\mathbb{P}^{2a+1}$, each divisor $(\xi_1 + \cdots + \xi_{a+2})_t$ spans a $\mathbb{P}^a_t$. Putting this all together, we obtain

$$\Sigma := \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^a \times \mathbb{P}^1 \subset \mathbb{P}^{2a+1}$$

with Σ embedded via forms of bidegree $(a,1)$. We again have $[\Xi] = h + 2f$ on Σ.

We refer the reader to [Nag99, HKP23] for further analysis of these constructions.

5. Twisted sheaf interpretations

In this section, the base field $k = \mathbb{C}$. Our goal is to explain why $F_1(X)$ is isomorphic to a moduli space of α-twisted sheaves on (S, f).

5.1. **Twisted cohomology.** We follow [HS05, HS06].

Let S be a projective K3 surface with Mukai lattice

$$\widetilde{H}(S,\mathbb{Z}) = \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$$

with bilinear form

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2)\rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

Let

$$\alpha \in \mathrm{Br}(S) = \mathrm{H}^2(S,\mathbb{Q}) / \left(\mathrm{Pic}(S)_{\mathbb{Q}} + \mathrm{H}^2(S,\mathbb{Z})\right)$$

and $B \in \mathrm{H}^2(S,\mathbb{Q})$ mapping to α. Write

$$\widetilde{H}(S,B,\mathbb{Z}) = \widetilde{H}(S,\mathbb{Z})$$

as a lattice with weight-two Hodge structure on $\widetilde{H}(S,\mathbb{C})$ induced by

$$\exp(B)\omega = \omega + (B \wedge \omega), \quad \omega \in \mathrm{H}(\Omega^2_S).$$

The same lattice may be regarded as a mixed Hodge structure $\mathrm{H}^*(S,B,\mathbb{Z})$ with weight filtration

$$W_i := \exp(B) \oplus_{j \le i} \mathrm{H}^j(S,\mathbb{Q}).$$

Applying $\exp(-B)$, we may understand

$$\exp(-B) : \widetilde{H}(S,B,\mathbb{Z}) \hookrightarrow \widetilde{H}(S,\mathbb{Q})$$

as a sub Hodge structure. Primitive Mukai vectors refer to vectors primitive with respect to this twisted cohomology.

The bilinear form is invariant under the action via $\exp(B)$:

$$\exp(B)(r,D,s) = (r, D + Br, s + B \wedge D + \frac{B^2 r}{2})$$

thus

$$\begin{aligned}\langle \exp(B)(r,D,s), \exp(B)(r,D,s)\rangle &= D^2 + 2B \wedge Dr + B^2 r^2 \\ &\quad - 2r(s + B \wedge D + B^2 r/2) \\ &= D^2 - 2rs = \langle (r,D,s),(r,D,s)\rangle .\end{aligned}$$

Example 5.1. Take $B = f$ a polarization. Then the weight-zero part is generated by $(I + f + \frac{f^2}{2}) \cdot 1$ and the weight-two part is obtained by adding

$$\{\gamma + f \wedge \gamma : \gamma \in \mathrm{H}^2(S,\mathbb{Z})\}.$$

For Mukai vectors associated with moduli spaces of simple sheaves $\mathcal{E} \to S$, i.e.,

$$(r,D,s) = v(\mathcal{E}),$$

we have

$$\exp(f)v(\mathcal{E}) = v(\mathcal{E} \otimes \mathcal{O}_S(f)).$$

When our focus is on projective bundles $\mathbb{P}(\mathcal{E}) \to S$, it is natural to consider orbits of Mukai vectors under exponentiation by line bundles.

Remark 5.2. For general B, twisted Hodge classes in $\widetilde{H}(S,B,\mathbb{Z})$ include

$$[pt] \in \mathrm{H}^4(S,\mathbb{Z}), \mathrm{Pic}(S) \subset \mathrm{H}^2(S,\mathbb{Z}), r\left(1 + B + \frac{B^2}{2}\right),$$

where r is a positive integer such that rB and $rB^2/2$ are integral.

We refer the reader to [HS05, §2] for the general analysis of the twisted Picard group $\mathrm{Pic}(S,B)$.

5.2. **Moduli results.** Existence theorems for moduli of twisted sheaves are due to Yoshioka [Yos06]. He takes, as input, a smooth Brauer-Severi fibration

$$p : Y \to S$$

representing $\alpha \in \mathrm{Br}(S)$. Let $\mathrm{Coh}(S,Y)$ denote coherent sheaves on Y that are locally, in the analytic or étale topology on S, tensor products of coherent sheaves pulled-back from S and line bundles. There is a distinguished vector bundle $\mathcal{G}$ on Y corresponding to the non-trivial extension

$$0 \to \mathcal{O}_Y \to \mathcal{G} \to T_{Y/S} \to 0.$$

A coherent sheaf $\mathcal{F}$ on Y belongs to $\mathrm{Coh}(S,Y)$ iff the "global generation map"

$$p^*p_*(\mathcal{G}^\vee \otimes \mathcal{F}) \to \mathcal{G}^\vee \otimes \mathcal{F}$$

is an isomorphism [Yos06, Lem. 1.5]. The minimum of

$$\{\mathrm{rk}(\mathcal{F}) > 0 : \mathcal{F} \in \mathrm{Coh}(S, Y)\}$$

is the order of α. We recall the basic existence results:

- Assume that v is a primitive Mukai vector and H is a general polarization with respect to v. Then all $\mathcal{G}$-twisted semistable sheaves $\mathcal{F}$ with $v_G(\mathcal{F}) = v$ are G-twisted stable. Thus $M_H^{Y,\mathcal{G}}(v)$ is a projective manifold when it is non-empty [Yos06, Thm. 3.11].
- Assume in addition that $r > 0$ and $\langle v, v\rangle \geq -2$. Then $M_H^{Y,\mathcal{G}}(v)$ is non-empty and deformation equivalent to a punctual Hilbert scheme of a K3 surface. When $\langle v, v\rangle = 0$ then it is a K3 surface [Yos06, Thm. 3.16].
- The second cohomology of the associated moduli space is isomorphic to $v^\perp$ or $v^\perp/\mathbb{Z}v$ depending on whether $\langle v, v\rangle = 0$ or $\neq 0$ [Yos06, Thm. 3.19].

Below we will write $M_v(S, \alpha)$ for the moduli space $M_H^{Y,\mathcal{G}}(v)$, suppressing the polarization and choice of representative for α.

5.3. **Application to the variety of lines.** Fix an isomorphism

$$\mathrm{H}^2(S, \mathbb{Z}) \simeq U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$$

where

$$U \simeq \langle u_i, v_i\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i = 1, 2, 3.$$

Let (S, f) have degree six; we choose the lattice isomorphism so that

$$f = u_1 + 3v_1,$$

in the first U factor. The discriminant group of $f_6^\perp$ is generated by $(u_1 - 3v_1)/6$.

Set $B = \frac{v_1+u_2-v_2}{2}$ so that $B^2 = -[pt]/2$ and $B \wedge f = [pt]/2$; as before, α denotes the corresponding element of $\mathrm{Br}(S)[2]$. We obtain (see Remark 5.2) integral Hodge classes

$$\{2 - (v_1 + u_2 - v_2), f, [pt]\}$$

and intersection matrix

$$\begin{pmatrix} -2 & -1 & -2 \\ -1 & 6 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

Consider the Mukai vector

$$v = 2 - (v_1 + u_2 - v_2) + f = 2 + u_1 + 2v_1 - u_2 + v_2$$

whose orthogonal complement has elements

$$g = -6 + 3(v_1 + u_2 - v_2) - f + 2[pt] = -6 - u_1 + 3u_2 - 3v_2 + 2[pt]$$

and

$$E = 2 - (v_1 + u_2 - v_2) + f + [pt] = 2 + u_1 + 2v_1 - u_2 + v_2 + [pt].$$

Since we have

$$\begin{array}{c|cc} & g & E \\ \hline g & 6 & 6 \\ E & 6 & -2 \end{array},$$

which coincides with (10), the moduli space of twisted sheaves $M_v(S, \alpha)$ has Picard group isomorphic to the Picard group $F_1(X)$. We refer to [BM14, Thm. 5.7(a),§8] for details on the birational geometry of such moduli spaces. The divisor E may be interpreted as the projectivization of a rigid rank-two α-twisted sheaf on S.

We summarize this discussion:

Theorem 5.3. *Let $F_1(X)$ denote the variety of lines on a cubic fourfold of discriminant 24. Then $F_1(X)$ is birational to a moduli space $M_v(S, \alpha)$ where S is a K3 surface, $f \in \mathrm{Pic}(S)$ with $f^2 = 6$, and $\alpha \in \mathrm{Br}(S)[2]$ and v are as specified above.*

Proof. This follows from the Torelli Theorems for hyperkähler fourfolds of $K3^{[2]}$ type [Mar11, Thm. 9.8] and for polarized K3 surfaces. □

The classification of Brauer classes $\alpha \in \mathrm{Br}(S)[2]$ – up to monodromy of the polarized surface (S, f) – is given in [vG05, Prop. 9.2]. There are three orbits, characterized via discriminant quadratic forms.

Question 5.4. Starting from the étale $\mathbb{P}^1$ bundle $E \to S$, restriction of scalars yields a $(\mathbb{P}^1)^3$ bundle

$$\mathbf{R}_{Z/S^{[3]}} E \to S^{[3]};$$

here $Z \to S^{[3]}$ is the universal subscheme. Restricting to the distinguished $\mathbb{P}^3 \subset S^{[3]}$ (see Question 3.9) gives

$$\mathbf{R}_{Z/S^{[3]}} E \times_{S^{[3]}} \mathbb{P}^3 \to \mathbb{P}^3.$$

A relative hyperplane section is a fibration in sextic del Pezzo surfaces. Is this related to the universal family $\widetilde{\mathcal{W}} \to \mathcal{H}_{dP}$ of normalized sextic del Pezzo surfaces in X?

5.4. **Higher-dimensional hyperkähler manifolds.** Given a cubic fourfold X, the variety of lines $F_1(X)$ is the first of an infinitely sequence of hyperkähler manifolds associated with X [BLM$^+$21, §29]. The next example, discovered in [LLSvS17], is obtained as follows: Consider the (quasi-projective) Hilbert scheme $\mathcal{H}_{tc}(X)$ of twisted cubic curves $\mathbb{P}^1 \simeq R \subset X$. When the cubic surface $\mathrm{span}(R) \cap X$ is smooth, $|R|$ is a two-dimensional linear series on this cubic surface, one of 72 such series. The resulting dominant map to the Grassmannian $\mathcal{H}_{tc}(X) \to \mathrm{Gr}(4,6)$ factors

$$\mathcal{H}_{tc}(X) \to G_\circ(X) \to \mathrm{Gr}(4,6),$$

generically a $\mathbb{P}^2$-bundle followed by a degree-72 cover. Combining the results of [LLSvS17] and [AL17] with [BLM$^+$21, Thm. 29.2(2)], we find that $G_\circ(X)$ admits a compactification by a hyperkähler manifold $G(X)$ with the following properties:

- $G(X)$ has dimension eight and is deformation equivalent to the length-four Hilbert scheme of a K3 surface;
- for generic X, $G(X)$ admits a polarization $g \in H^2(G(X),\mathbb{Z})$ with $(g,g) = 2$ and $(g, H^2(G(X),\mathbb{Z})) = 2\mathbb{Z}$.

Here $(,)$ denotes the Beauville-Bogomolov form.

Suppose that X is special of discriminant 24 with no additional algebraic cycles. Using the same results of [BLM$^+$21], we find (cf. (7)):

$$\mathrm{Pic}(G(X)) = \langle g, \varpi \rangle, \quad \begin{array}{c|cc} & g & \varpi \\ \hline g & 2 & 0 \\ \varpi & 0 & -8 \end{array}.$$

The divisor class $3g - 2\varpi$, satisfying $(3g - 2\varpi, 3g - 2\varpi) = -14$, has implications for the birational geometry of $G(X)$.

Proposition 5.5. *Suppose that X is a special cubic fourfold of discriminant 24 and $G(X)$ the associated hyperkähler eightfold. If $[X] \in \mathcal{C}_{24}$ is generic then there exists an embedding*

$$\mathbb{P}^4 \hookrightarrow G(X).$$

Proof. This is a special case of [HT16, §5.3] but we sketch the key ideas. Recall the notation of [BM14, §12]. The presence of a $\mathbb{P}^4 \hookrightarrow M_v$, where v is the Mukai vector, corresponds to algebraic classes

$$\begin{array}{c|cc} & v & a \\ \hline v & 6 & 3 \\ a & 3 & -2 \end{array}$$

in the Mukai lattice. Note that $v - 2a \in v^\perp = H^2(M_v,\mathbb{Z})$ satisfies $(v-2a., v-2a) = -14$ and $(v - 2a, H^2(M_v,\mathbb{Z})) = 2\mathbb{Z}$. Thus $3g - 2\varpi$

and $v - 2a$ have the same numerical characteristics, and consequently the same birational interpretation by the main result of [BHT15]. $\square$

In Section 1.1, we saw that each sextic del Pezzo surface has a pair $|B|, |B'|$ of two-dimensional linear series of twisted cubic curves.

Question 5.6. How is the $\mathbb{P}^4 \subset G(X)$ related to the locus in $\mathcal{H}_{tc}(X)$ arising from twisted cubic curves in nodal sextic del Pezzo surfaces?

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