A VERY GENERAL QUARTIC DOUBLE FOURFOLD IS NOT STABLY RATIONAL

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ABSTRACT. We prove that a very general double cover of the projective four-space, ramified in a quartic threefold, is not stably rational.

1. INTRODUCTION

In this note, we consider quartic double fourfolds, i.e., hypersurfaces X_f in the weighted projective space $\mathbb{P}(2, 1, 1, 1, 1, 1)$, with homogeneous coordinates (s, x, y, z, t, u), given by a degree four equation of the form

$$s^2 + f(x, y, z, t, u) = 0. \tag{1.1}$$

The failure of stable rationality for cyclic covers of projective spaces has been considered by Voisin [Voi15b], Beauville [Bea16b], Colliot-Thélène–Pirutka [CTP16a], and Okada [Oka16]. We work over an uncountable ground field k of characteristic zero. Our main result is

Theorem 1. *Let $f \in k[x, y, z, t, u]$ be a very general degree four form. Then X_f is not stably rational.*

Rationality properties of quartic double fourfolds were recently investigated by Beauville [Bea15, Bea16a] and C. Voisin [Voi15a] (for quartic double fourfolds singular along a line), who used the new technique of specializing an integral decomposition of the diagonal [Voi15b, CTP16b, Tot16]. The main difficulty is to construct a special X in the family (1.1) with following properties:

- (**O**) Obstruction: the second unramified cohomology group $H^2_{nr}(X)$ (or another birational invariant obstructing the universal CH_0-triviality) does not vanish,
- (**R**) Resolution: there exists a resolution of singularities $\beta : \tilde{X} \to X$, such that the morphism β is universally CH_0-trivial,

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(see, e.g., [CTP16b] or Sections 2 and 4 of [HPT16] for definitions).

The verification of both properties for potential examples is notoriously difficult. The paper [Bea15] proposed an example satisfying the second property, but the analysis of the first property contained a gap [Bea16a]. The preprint [Voi15a] relied on this analysis to show that certain quadric surface bundles over $\mathbb{P}^2$ were not generally stably rational.

Our main goal here is to produce an X satisfying both properties (O) and (R). We have a candidate example:

$$V : s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0. \tag{1.2}$$

The singular locus of V is a connected curve, consisting of 4 components: two nodal cubics, a conic, and a line. How do we find this example? We may transform equation (1.2) to

$$yzs^2 + xzt_1^2 + xyu_1^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))v_1^2 = 0. \tag{1.3}$$

Precisely, we homogenize with respect to the variables s, t, u, via an additional variable v, multiply through by yz, and absorb the squares into the variables t_1, u_1, and v_1. The resulting equation gives a bidegree $(2, 2)$ hypersurface

$$V' \subset \mathbb{P}^2 \times \mathbb{P}^3,$$

birational to V via the coordinate changes. In [HPT16] we proved that this V' satisfies both properties (O) and (R). In particular, V also satisfies (O), since unramified cohomology is a birational invariant.

Instead of the direct verification of property (R) for this V (so that we could take $X = V$), we found it more transparent to take an alternative approach, applying the specialization argument twice: First we can specialize a very general X_f to a quartic double fourfold X which is singular along a line ℓ (contained in the ramification locus); we choose X to be very general subject to this condition. The main part of our argument is then to show that X is not stably rational. We show that the blowup morphism

$$\beta : \tilde{X} := \mathrm{Bl}_\ell(X) \to X$$

is universally CH_0-trivial and that $\tilde{X}$ is smooth, i.e., X satisfies (R). Furthermore, there exists a quadric bundle structure $\pi : \tilde{X} \to \mathbb{P}^2$, with degeneracy divisor a smooth octic curve. In Section 2 we analyze this geometry. We consider a degeneration of these quadric bundles to a fourfold X' which is birational to V', and thus satisfies (O). The singularities of X' are similar to those considered in [HPT16]; the verification of the required property (R) for X' is easier in this presentation. This is the content of Section 3. In Section 4 we give the argument for failure of stable rationality of very general (1.1).

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2. Geometry of quartic double fourfolds

Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. From the equation (1.1) we see that the quartic double fourfold X is singular precisely along the singular locus of the quartic threefold $Y \subset \mathbb{P}^4$ given by $f = 0$.

We will consider quartic threefolds Y double along ℓ. These form a linear series of dimension

$$\binom{8}{4} - 5 - 12 = 53$$

and taking into account changes of coordinates—automorphisms of $\mathbb{P}^4$ stabilizing ℓ—we have 34 free parameters.

Let $\beta : \tilde{X} \to X$ be the blowup of X along ℓ. We will analyze its properties by embedding it into natural bundles over $\mathbb{P}^2$.

We start by blowing up ℓ in $\mathbb{P}^4$. Projection from ℓ gives a projective bundle structure

$$\varpi : \mathrm{Bl}_\ell(\mathbb{P}^4) \to \mathbb{P}^2$$

where we may identify

$$\mathrm{Bl}_\ell(\mathbb{P}^4) \simeq \mathbb{P}(\mathcal{E}), \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

Write h for the hyperplane class on $\mathbb{P}^2$ and its pullbacks and ξ for the first Chern class of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Taking global sections

$$\mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \twoheadrightarrow \mathcal{E}^\vee$$

induces morphisms

$$\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 5}) \simeq \mathbb{P}^4 \times \mathbb{P}^2;$$

projecting onto the first factor gives the blow up. Its exceptional divisor

$$E \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}) \simeq \mathbb{P}^1 \times \mathbb{P}^2$$

has class $\xi - h$.

Let $\tilde{Y} \subset \mathbb{P}(\mathcal{E})$ denote the proper transform of Y, which has class

$$4\xi - 2E = 2\xi + 2h.$$

Conversely, divisors in this linear series map to quartic hypersurfaces in $\mathbb{P}^4$ singular along ℓ. Since $2\xi + 2h$ is very ample in $\mathbb{P}(\mathcal{E})$ the generic such divisor is smooth. The morphism ϖ realizes $\tilde{Y}$ as a conic bundle over $\mathbb{P}^2$; its defining equation q may also be interpreted as a section of the vector bundle $\mathrm{Sym}^2(\mathcal{E}^\vee)(2h)$. Let $\gamma : \tilde{Y} \to Y$ denote the resulting resolution; its exceptional divisor $F = \tilde{Y} \cap E$ is a divisor of bidegree $(2, 2)$ in $E \simeq \mathbb{P}^1 \times \mathbb{P}^2$. The conic bundle $F \to \ell$ has a section since $k(\ell)$ is a C_1-field, hence γ is universally CH_0-trivial.

Let $\tilde{X} \to \mathbb{P}(\mathcal{E})$ denote the double cover branched over $\tilde{Y}$, i.e., $s^2 = q$. This naturally sits in the projectization of an extension

$$0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0,$$

where $\mathcal{L}$ is a line bundle. Note the natural maps

$$\mathrm{Sym}^2(\mathcal{E}^\vee) \hookrightarrow \mathrm{Sym}^2(\mathcal{F}^\vee) \twoheadrightarrow \mathcal{L}^{-2},$$

and their twists

$$\mathrm{Sym}^2(\mathcal{E}^\vee)(2h) \hookrightarrow \mathrm{Sym}^2(\mathcal{F}^\vee)(2h) \twoheadrightarrow \mathcal{L}^{-2}(2h);$$

the last sheaf corresponds to the coordinate s. Since we are over $\mathbb{P}^2$ the extension above must split; furthermore, the coordinate s induces a trivialization

$$\mathcal{L}^{-2}(2h) \simeq \mathcal{O}_{\mathbb{P}^2}.$$

Thus we conclude

$$\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

The divisor $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ is generically smooth; let $\beta : \tilde{X} \to X$ denote the induced resolution of X. Its exceptional divisor is a double cover of E branched over F (of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a divisor of bidegree $(2, 2)$) thus a quadric surface bundle over $\mathbb{P}^1$. As any such bundle admits a section, it follows that β is universally CH_0-trivial.

We summarize the key elements we will need:

Proposition 2. *Let $X \to \mathbb{P}^4$ be a double fourfold, ramified along a quartic threefold $Y \subset \mathbb{P}^4$. Assume that Y is singular along a line ℓ and generic subject to this condition. Let $\beta : \tilde{X} \to X$ be the blowup of X along ℓ. Then $\tilde{X}$ is smooth and β universally CH_0-trivial.*

Regarding $\tilde{X} \subset \mathbb{P}(\mathcal{F})$, there is an induced quadric surface fibration

$$\pi : \tilde{X} \to \mathbb{P}^2.$$

Let D denote the degeneracy curve, naturally a divisor in

$$\det(\mathcal{F}^\vee(2h)) \simeq \mathcal{O}_{\mathbb{P}^2}(8).$$

The analysis above gives an explicit determinantal description of the defining equation of D. Choose homogeneous forms

$$c \in \Gamma(\mathcal{O}_{\mathbb{P}^2}),\ F_1, F_2, F_3 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(2)),\ G_1, G_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(3)), H \in \Gamma(\mathcal{O}_{\mathbb{P}^2}(4))$$

so that the symmetric matrix associated with $\tilde{X}$ takes the form:

$$\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & F_1 & F_2 & G_1 \\ 0 & F_2 & F_3 & G_2 \\ 0 & G_1 & G_2 & H \end{pmatrix}$$

We fix coordinates to obtain a concrete equation for $\tilde{X}$. Let (x, y, z) denote coordinates of $\mathbb{P}^2$, or equivalently, linear forms on $\mathbb{P}^4$ vanishing along ℓ. Let s denote a local coordinate trivializing $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$, t and u coordinates corresponding to $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \subset \mathcal{F}$, and v to $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{F}$. Then we have

$$\tilde{X} = \{cs^2 + F_1t^2 + 2F_2tu + F_3u^2 + 2G_1tv + 2G_2uv + Hv^2 = 0\}, \tag{2.1}$$

where F_1, F_2, F_3, G_1, G_2, and H are homogeneous in x, y, z.

Finally, we interpret the degeneration curve in geometric terms. Ignoring the constant, we may write

$$D = (F_1F_3 - F_2^2)H - F_3G_1^2 + 2F_2G_1G_2 - F_1G_2^2 = 0.$$

Modulo $F_1F_3 - F_2^2$ we have

$$-F_3G_1^2 + 2F_2G_1G_2 - F_1G_2^2 = 0$$

which is equal to

$$\frac{-1}{F_1}(F_2G_1 - F_1G_2)^2 = \frac{-1}{F_3}(F_3G_1 - F_2G_2)^2.$$

Thus we conclude that D is tangent to a quartic plane curve

$$C = \{F_1F_3 - F_2^2\} = 0$$

at 16 points. *Every* smooth quartic plane curve admits multiple such representations: Surfaces

$$\{a^2F_1 + 2abF_2 + b^2F_3 = 0\} \subset \mathbb{P}^1_{a,b} \times \mathbb{P}^2$$

are precisely degree two del Pezzo surfaces equipped with a conic bundle structure, the conic structures indexed by non-trivial two-torsion points of the branch curve C. One last parameter check: The moduli space of pairs (C, D) consisting of a plane quartic and a plane octic tangent at 16 points depends on

$$14 + 44 - 16 - 8 = 34$$

parameters. This is compatible with our first parameter count.

Remark 3. *Smooth* divisors $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ as above necessarily have trivial Brauer group. This follows from Pirutka's analysis [Pir16]: if the degeneracy curve is smooth and irreducible then there cannot be unramified second cohomology. It also follows from a singular version of the Lefschetz hyperplane theorem. Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$ so that $[\tilde{X}] = 2\zeta + 2h$. This is almost ample: the line bundle $\zeta + h$ contracts the distinguished section $s : \mathbb{P}^2 \to \mathbb{P}(\mathcal{F})$ associated with the summand $\mathcal{O}_{\mathbb{P}^1}(1) \subset \mathcal{F}$ to a point but otherwise induces an isomorphism onto its image. In particular, $\zeta + h$ induces a small contraction in the sense of intersection homology. The homology version of the Lefschetz Theorem of Goresky-MacPherson [GM88, p. 150] implies that $0 \simeq H^3(\mathbb{P}(\mathcal{F}), \mathbb{Z}) \xrightarrow{\sim} H^3(\tilde{X}, \mathbb{Z})$.

3. Singularities of the special fiber

We specialize (2.1) to:

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0. \tag{3.1}$$

Proposition 4. *The fourfold $X' \subset \mathbb{P}(\mathcal{F})$ defined by (3.1) admits a resolution of singularities $\beta' : \tilde{X}' \to X'$ such that β' is universally* CH_0*-trivial.*

The remainder of this section is a proof of this result.

3.1. **The singular locus.** A direct computation in `Magma` (or an analysis as in [HPT16, Section 5]) yields that the singular locus of (3.1) is a connected curve consisting of the following components:

- Singular cubics:

$$E_z := \{v^2 y(y-x)^2 + u^2 x = z = s = t = 0\}$$
$$E_y := \{v^2 z(z-x)^2 + t^2 x = y = s = u = 0\}$$

- Conics:

$$R_x := \{u^2 - 4v^2 + t^2 = x = z - y = s = 0\}$$
$$C_x := \{zu^2 + yt^2 = s = v = x = 0\}$$

The nodes of E_z and E_y are

$$\mathfrak{n}_z := \{z = s = t = y - x = u = 0\}$$
$$\mathfrak{n}_y := \{y = s = u = z - x = t = 0\},$$

respectively. Here R_x and C_x intersect transversally at two points,

$$\mathfrak{r}_\pm := \{u \pm it = v = s = z - y = x = 0\};$$

R_x is disjoint from E_z and E_y, and the other curves intersect transversally in a single point (in coordinates $(x, y, z) \times (s, t, u, v)$):

$$\begin{aligned} E_z \cap E_y =& \mathfrak{q}_x := (1, 0, 0) \times (0, 0, 0, 1), \\ E_z \cap C_x =& \mathfrak{q}_y := (0, 1, 0) \times (0, 0, 1, 0), \\ E_y \cap C_x =& \mathfrak{q}_z := (0, 0, 1) \times (0, 1, 0, 0). \end{aligned}$$

This configuration of curves is similar to the one considered in [HPT16], but the singularities are different.

3.2. **Local étale description of the singularities and resolutions.** The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

The main normal form is

$$a^2 + b^2 + c^2 = p^2q^2 \tag{3.2}$$

which is singular along the locus

$$\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}.$$

This is resolved by successively blowing up along these components in either order. Indeed, after blowing up the first component, using $\{A, B, C, P\}$ for homogeneous coordinates associated with the corresponding generators of the ideal, we obtain

$$A^2 + B^2 + C^2 = P^2q^2.$$

The exceptional fibers are isomorphic to a non-singular quadric hypersurface (when $q \neq 0$) or a quadric cone (over $q = 0$). Dehomogenizing by setting $P = 1$, we obtain

$$A^2 + B^2 + C^2 = q^2$$

which is resolved by blowing up $\{A = B = C = q = 0\}$. This has ordinary threefold double points at each point, so the exceptional fibers are all isomorphic to non-singular quadric hypersurfaces.

There are cases where

$$\{a = b = c = p = 0\} \cup \{a = b = c = q = 0\}$$

are two branches of the same curve. For example, this could arise from

$$a^2 + b^2 + c^2 = (m^2 - n^2 - n^3)^2 \tag{3.3}$$

by setting $p = m-n\sqrt{1+n}$ and $q = m+n\sqrt{1+n}$. Of course, we cannot pick one branch to blow up first. We therefore blow up the origin first, using homogeneous coordinates A, B, C, D, P, Q corresponding to the generators to obtain

$$A^2 + B^2 + C^2 = P^2q^2 = Q^2p^2.$$

The resulting fourfold is singular along the stratum

$$A = B = C = q = p = 0$$

as well as the proper transforms of the original branches. Indeed, on dehomogenizing $P = 1$ we obtain local affine equation

$$A^2 + B^2 + C^2 = Q^2p^2;$$

this is singular along $\{A = B = C = p = 0\}$, the locus where the exceptional divisor is singular, and $\{A = B = C = Q = 0\}$, and proper transform of $\{a = b = c = q = 0\}$. The local affine equation is the same as (3.2); we resolve by blowing up the singular locus of the exceptional divisor followed by blowing up the proper transforms of the branches. This descends to a resolution of (3.3).

3.3. **Summary of the resolution.**

Blowup steps. Below we construct the resolution β' as a sequence of blowups:

(1) Blow up the nodes $\mathfrak{n}_z$ and $\mathfrak{n}_y$; the resulting fourfold is singular along rational curves R_z and R_y in the exceptional locus, meeting the proper transforms of E_z and E_y transversally in two points sitting over $\mathfrak{n}_z$ and $\mathfrak{n}_y$, respectively.
(2) The exceptional divisors are quadric threefolds singular along R_z and R_y.
(3) At this stage, the singular locus consists of six smooth rational curves, the proper transforms of E_z, E_y, R_x, C_x and the new curves R_z and R_y, with a total of nine nodes. (This is the configuration appearing in [HPT16, Section 5].)
(4) The local analytic structure is precisely as indicated in Section 3.2. Thus we can blow up the six curves in any order to obtain a resolution of singularities. The fibers are either the Hirzebruch surface $\mathbb{F}_0$ or a union of Hirzebruch surfaces $\mathbb{F}_0 \cup_\Sigma \mathbb{F}_2$ where $\Sigma \simeq \mathbb{P}^1$ with self intersections $\Sigma^2_{\mathbb{F}_0} = 2$ and $\Sigma^2_{\mathbb{F}_2} = -2$.

For concreteness, we blowup in the order

$$R_z, R_y, E_z, E_y, C_x, R_x.$$

3.4. **Computation in local charts.** We exploit the symmetry under the involution exchanging $y \leftrightarrow z$, $t \leftrightarrow u$ and $t \leftrightarrow -t$. It suffices then to analyze E_z, C_x, and R_x and the distinguished point $\mathfrak{n}_z$ and intersection points of the components.

Analysis along the curve C_x. Recall the equation of X':

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0$$

and the equation of C_x: $zu^2 + yt^2 = s = v = x = 0$. We order coordinates $(x, y, z), (s, t, u, v)$ and write intersections

- $C_x \cap R_x = (0, 1, 1) \times (0, 1, \pm i, 0)$;
- $C_x \cap E_z = (0, 1, 0) \times (0, 0, 1, 0)$;
- $C_x \cap E_y = (0, 0, 1) \times (0, 1, 0, 0)$.

We use the symmetry between t and u to reduce the number of cases. *Chart* $u = 1$, $z = 1$. We extract equations for the exceptional divisor $\mathbf{E}$ obtained by blowing up C_x. In this chart, C_x takes the form

$$1 + yt^2 = s = v = x = 0$$

and X' is

$$s^2 + x(yt^2 + 1) + v^2y(y - 1)^2 + v^2xG = 0,$$

where v^2xG are the 'higher order terms', i.e., terms that vanish of order at least three along C_x, these terms always vanish at the exceptional divisor and do not affect the smoothness of the blow up, so that in the analysis below we often omit these terms.

Now we analyse the local charts of the blow up:

(1) $\mathbf{E} : yt^2 + 1 = 0$, $s = s_1(yt^2 + 1), x = x_1(yt^2 + 1), v = v_1(yt^2 + 1)$, the equation for X', up to removing higher order terms, in new coordinates is:

$$s_1^2 + x_1 + v_1^2y(y - 1)^2 = 0,$$

this is smooth and rational. The exceptional divisor

$$s_1^2 + x_1 + v_1^2y(y - 1)^2 = 0, yt^2 + 1 = 0$$

is rational, and its fibers over C_x are rational as well.

(2) $\mathbf{E} : x = 0$, $s = s_1x, v = v_1x, yt^2 + 1 = wx$, equation of X':

$$s_1^2 + w + v_1^2y(y - 1)^2 = 0, yt^2 + 1 = wx,$$

smooth;

(3) $\mathbf{E} : s = 0$, $x = x_1s, v = v_1s, yt^2 + 1 = ws$:

$$1 + wx_1 + v_1^2y(y - 1)^2 = 0, yt^2 + 1 = sw,$$

smooth.

(4) $\mathbf{E}: v=0, s=s_1v, x=x_1v, yt^2+1=wv$, equation of X' is
$$s_1^2+wx_1+y(y-1)^2=0, yt^2+1=wv,$$
which has at most ordinary double singularity (corresponding to $C_x\cap R_x=\mathfrak{r}_\pm$) of type
$$a^2+b^2+cd=0,\ a=b=c=d=0.$$
This is resolved by one blowup.

Chart $u=1$, $y=1$. In this chart C_x is $z+t^2=s=v=x=0$ and X' is
$$s^2+x(t^2+z)+v^2z(z-1)^2+v^2xG=0,$$
where v^2xG are the 'higher order terms'. We analyze local charts of the blow up:

(1) $\mathbf{E}: t^2+z=0,\ s=s_1(t^2+z), x=x_1(t^2+z), v=v_1(t^2+z)$, the equation for X', up to removing higher order terms, in new coordinates is:
$$s_1^2+x_1+v_1^2z(z-1)^2=0,$$
this is smooth and rational. The exceptional divisor
$$s_1^2+x_1+v_1^2z(z-1)^2=0, t^2+z=0$$
is rational, and its fibers over C_x are rational as well.

(2) $\mathbf{E}: x=0,\ s=s_1x, v=v_1x, t^2+z=wx$, equation of X':
$$s_1^2+w+v_1^2z(z-1)^2=0, t^2+z=wx,$$
smooth.

(3) $\mathbf{E}: s=0,\ x=x_1s, v=v_1s, t^2+z=ws$:
$$1+wx_1+v_1^2z(z-1)^2=0, t^2+z=sw,$$
smooth.

(4) $\mathbf{E}: v=0, s=s_1v, x=x_1v, t^2+z=vw$, equation of X' is
$$s_1^2+wx_1+z(z-1)^2=0, t^2+z=wv,$$
or, up to removing the higher order terms
$$s_1^2+wx_1+z(t^2+1)^2=0, z=-t^2+wv,$$
this has at most ordinary double singularities
$$s_1=w=x_1=0, t=\pm i$$
(where we meet the proper transform of R_x) of type
$$a^2+b^2+cd=0, a=b=c=d$$

resolved as above by one blowup.

Analysis near $\mathfrak{n}_z$. Center the coordinates by setting $\xi = y - 1$

$$s^2 + (\xi+1)t^2 + zu^2 + (\xi+1)z(\xi^2 + z^2 - 2z(\xi+2)) = 0.$$

Note that E_z is given by

$$(\xi+1)\xi^2 + u^2 = z = s = t = 0.$$

We regroup terms

$$s^2 + (\xi+1)t^2 + z(u^2 + (\xi+1)\xi^2) + (\xi+1)z^2(z - 2\xi - 4) = 0.$$

Provided $\xi \neq -1, -2$ this is étale-locally equal to

$$s_1^2 + t_1^2 + z_1(u^2 + (\xi+1)\xi^2) + z_1^2 = 0$$

which is equivalent to normal form (3.3). When $\xi = -1$ we are at the point $\mathfrak{q}_x$, which we analyze below. A local computation at $\xi = -2$ shows that the singularity is resolved there by blowing up E_z and the exceptional fiber there is isomorphic to $\mathbb{F}_0$. In other words, we have ordinary threefold double points there as well.

Blowing up the singular point $\mathfrak{n}_z$ *of* E_z. The point $\mathfrak{n}_z$ lies in the chart $x = 1, v = 1$, where we now make computations. The equation of the point (and the locus we blow up) is

$$s = t = u = z = y - 1 = 0.$$

The equation of X can be written as:

$$s^2 + yt^2 - 2z^2y(y+1) + zu^2 + z^3y + yz(y-1)^2 = 0.$$

The curve E_z has equations

$$y(y-1)^2 + u^2 = z = s = t = 0.$$

Now we compute the charts for the blow up and we extract equations for the exceptional divisor $\mathbf{E}$:

(1) $\mathbf{E} : s = 0$. The change of variables is $u = su_1, t = st_1, z = sz_1, y = 1 + y_1 s$. Then the equation of X' (resp. the exceptional divisor $\mathbf{E}$), up to removing the higher order terms, is:

$$1 + t_1^2(1 + y_1 s) - 2z_1^2(1 + sy_1)(2 + sy_1) = 0$$

(resp. $1 + t_1^2 - 2z_1^2 = 0$), so that the blow up and the exceptional divisor are smooth, and $\mathbf{E}$ is rational.

(2) $\mathbf{E}: t = 0$. The change of variables is $s = s_1 t, u = u_1 t, z = z_1 t, y = 1 + y_1 t$; the equations are

$$s_1^2 + (1 + y_1 t) - 2z_1^2(1 + y_1 t)(2 + y_1 t) = 0,$$

and E is given by

$$s_1^2 + 1 - 4z_1^2 = 0,$$

so that the blowup is smooth at any point of the exceptional divisor.

(3) $\mathbf{E}: z = 0$, the change of variables is $s = s_1 z, u = u_1 z, y = 1 + y_1 z$; we obtain

$$s_1^2 + (1 + y_1 z)t_1^2 - 2(1 + y_1 z)(2 + y_1 z) = 0$$

and the equation of $\mathbf{E}$ is $s_1^2 + t_1^2 - 4 = 0$, so that the blow up is smooth at any point of the exceptional divisor.

(4) $\mathbf{E}: y_1 := y - 1 = 0$, the change of variables is $z = z_1 y_1, s = s_1 y_1, u = u_1 y_1, t = t_1 y_1$; the equations are

$$s_1^2 + t_1^2(1+y_1) - 2z_1^2(1+y_1)(2+y_1) + u_1^2 y_1 z + z_1^3 y_1(1+y_1) + z_1(1+y_1) = 0,$$

this is smooth, as well as the exceptional divisor ($y_1 = 0$).

(5) $\mathbf{E}: u = 0$, the change of variables is $s = s_1 u, t = t_1 u, z = z_1 u, y = 1 + y_1 u$; the equations for the proper transform of X' are

$$s_1^2 + (1+y_1 u)t_1^2 - 2z_1^2(1+y_1 u)(2+y_1 u) + u z_1 + u z_1^3(1+y_1 u) + z_1 u y_1^2(1+y_1 u) = 0,$$

and the proper transform of E_z is given by

$$(1 + y_1 u)y_1^2 + 1 = z_1 = s_1 = t_1 = 0.$$

The exceptional divisor

$$\mathbf{E}: s_1^2 + t_1^2 - 4z_1^2 = 0$$

is singular along $s_1 = t_1 = z_1 = u = 0$ (and y_1 is free). The resulting curve is denoted $R_z \simeq \mathbb{P}^1$; note that R_z meets the proper transform of E_z at two points $y_1 = \pm i$.

Blowing up R_z. For the analysis of singularities we can remove higher order terms, so that the equation of the variety (resp. R_z) is given by:

$$s_1^2 + t_1^2 - 4z_1^2 + u z_1 + u z_1 y_1^2 = 0$$

and $s_1 = t_1 = z_1 = u = 0$.

The charts for the new blow up with exceptional divisor $\mathbf{E}'$ are:

(1) $\mathbf{E}' : s_1 = 0$, then after the usual change of variables for a blow up, we obtain the equation:

$$1 + t_2^2 - 4z_2^2 + u_2 z_2 + u_2 z_2 y_1^2 = 0,$$

which is smooth.

(2) $\mathbf{E}' : t_1 = 0$ is similar to the previous case.

(3) $\mathbf{E}' : z_1 = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4 + u_2 + u_2 y_1^2 = 0,$$

that is smooth;

(4) $\mathbf{E}' : u = 0$, we obtain the equation

$$s_2^2 + t_2^2 - 4z_2^2 + z_2(1 + y_1^2) = 0,$$

which has ordinary double points at $s_2 = t_2 = z_2 = y_1^2 + 1 = 0$. These are resolved by blowing up the proper transform of E_z.

Analysis near $\mathfrak{q}_x$. Dehomogenize

$$s^2 + xyt^2 + xzu^2 + yz(x^2 + y^2 + z^2 - 2(xy + xz + yz))v^2 = 0$$

by setting $v = 1$ and $x = 1$ to obtain

$$s^2 + yt^2 + zu^2 + yz(1 + y^2 + z^2 - 2(y + z + yz)) = 0.$$

We first analyze at $\mathfrak{q}_x$, the origin in this coordinate system. Note that $1 + y^2 + z^2 - 2(y + z + yz) \neq 0$ here and thus its square root can be absorbed (étale locally) nto s, t, and u to obtain

$$s_1^2 + yt_1^2 + zu_1^2 + yz = 0.$$

Setting $y_1 = y + u_1^2$ and $z_1 = z + t_1^2$ gives

$$s_1^2 + y_1 z_1 = t_1^2 u_1^2,$$

which is equivalent to the normal form (3.2). (The blow up over the generic point of E_z was analyzed previously.)

Blowing up R_x. Similar to the analysis of singularities near R_z, see also [HPT16, Section 5.2 (4)]

3.5. **Exceptional fibers.** The local computations above provide the following description of the exceptional fibers:

- Over the nodes $\mathfrak{n}_z$ and $\mathfrak{n}_y$: The exceptional fiber has two three-dimensional components. One is the standard resolution of a quadric threefold singular along a line, that is,
$$\mathbf{F}' = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)).$$
The other is a quadric surface fibration $\mathbf{F}'' \to \mathbb{P}^1$, over R_z and R_y respectively, smooth except for two fibers corresponding to the intersections with E_z and E_y; the singular fibers are unions $\mathbb{F}_0 \cup \mathbb{F}_2$ as indicated above. The intersection $\mathbf{F}' \cap \mathbf{F}''$ is along the distinguished subbundle
$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \subset \mathbf{F}'$$
which meets the smooth fibers of $\mathbf{F}'' \to \mathbb{P}^1$ in hyperplanes and the singular fibers in smooth rational curves in $\mathbb{F}_2$ with self-intersection 2.
- Over E_z: the exceptional divisor is a quadric surface fibration over $\mathbb{P}^1$, with two degenerate fibers of the form $\mathbb{F}_0 \cup \mathbb{F}_2$ corresponding to the intersections with C_x and E_y.
- Over E_y: the exceptional divisor is a quadric surface fibration with one degenerate fiber, corresponding to the intersection with C_x.
- Over C_x: a quadric surface fibration with two degenerate fibers corresponding to the intersections with R_x.
- Over R_x: a smooth quadric surface fibration.

In each case, the fibers of β' are universally CH_0-trivial.

4. Proof of the Theorem 1

We recall implications of the "integral decomposition of the diagonal and specialization" method, following [Voi15b], [CTP16b].

Theorem 5. [Voi15b, Theorem 2.1], [CTP16b, Theorem 1.14 and Theorem 2.3] *Let*
$$\phi : \mathcal{X} \to B$$
be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ so that the fiber
$$X := \phi^{-1}(b)$$
satisfies the following conditions:

- *X admits a desingularization*

$$\beta : \tilde{X} \to X,$$

where the morphism β is universally CH_0*-trivial,*
- *$\tilde{X}$ is not universally* CH_0*-trivial.*

Then a very general fiber of ϕ is not universally CH_0*-trivial and, in particular, not stably rational.*

We apply this twice: Consider a family of double fourfolds X_f ramified along a quartic threefold $f = 0$, as in (1.1). Let X' be the fourfold given by (3.1) and let V' be the bidegree $(2, 2)$ hypersurface defined in (1.3).

(1) As mentioned in the introduction, V' satisfies property (O); this is an application of Pirutka's computation of unramified second cohomology of quadric surface bundles over $\mathbb{P}^2$ [Pir16]. By construction, X' is birational to V'. Proposition 4 and Section 3.3 yield property (R) for X'. We conclude that very general hypersurfaces $\tilde{X} \subset \mathbb{P}(\mathcal{F})$ given by Equation 2.1, in Section 2, following Proposition 2, fail to be universally CH_0-trivial.

(2) By Proposition 2, the resolution morphism $\beta : \tilde{X} \to X$ is universally CH_0-trivial; here X is a double fourfold, ramified along a quartic which is singular along a line. A second application of Theorem 5 to the family of double fourfolds ramified along a quartic threefold completes the proof of Theorem 1.

References

[Bea15] Arnaud Beauville. A very general quartic double fourfold or fivefold is not stably rational. *Algebr. Geom.*, 2(4):508–513, 2015.

[Bea16a] Arnaud Beauville. Erratum: A very general quartic double fourfold or fivefold is not stably rational (algebraic geometry 2, no. 4 (2015), 508-513). *Algebr. Geom.*, 3(1):137–137, 2016.

[Bea16b] Arnaud Beauville. A very general sextic double solid is not stably rational. *Bull. Lond. Math. Soc.*, 48(2):321–324, 2016. `arXiv:1411.7484`.

[CTP16a] Jean-Louis Colliot-Thélène and Alena Pirutka. Cyclic covers that are not stably rational. *Izvestiya RAN, Ser. Math.*, 80(4), 2016. `arXiv:1506.0042v2`.

[CTP16b] Jean-Louis Colliot-Thélène and Alena Pirutka. Hypersurfaces quartiques de dimension 3 : non rationalité stable. *Ann. Sci. ENS*, 49(2):735–801, 2016.

[GM88] Mark Goresky and Robert MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[HPT16] Brendan Hassett, Alena Pirutka, and Yuri Tschinkel. Stable rationality quadric surface bundles over surfaces, 2016. arXiv:1603.09262.

[Oka16] Takuzo Okada. Stable rationality of cyclic covers of projective spaces, 2016. arXiv:1604.08417.

[Pir16] Alena Pirutka. Varieties that are not stably rational, zero-cycles and unramified cohomology, 2016. arXiv:1603.09261.

[Tot16] Burt Totaro. Hypersurfaces that are not stably rational. *J. Amer. Math. Soc.*, 29(3):883–891, 2016.

[Voi15a] Claire Voisin. (Stable) rationality is not deformation invariant, 2015. arXiv:1511.03591.

[Voi15b] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

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