ON STABLE RATIONALITY OF FANO THREEFOLDS AND DEL PEZZO FIBRATIONS

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1. Introduction

Recent breakthroughs of Voisin [Voi15], developed and amplified by Colliot-Thélène and Pirutka [CTP16b, CTP16a], Beauville [Bea16b], and Totaro [Tot16], have reshaped the classical study of rationality questions for higher-dimensional varieties. Failure of stable rationality is now known for large classes of rationally-connected threefolds. The key tool is (Chow-theoretic) integral decompositions of the diagonal, which necessarily exist for stably rational varieties. Integral decompositions of the diagonal specialize well, even to mildly singular varieties, connecting logically the stable rationality of various classes of varieties. This puts a premium on discovering appropriate degenerations linking different classes of rationally connected varieties. In this paper we exhibit novel degenerations of smooth Fano threefolds and use these to prove:

Theorem 1. *Let X be a very general smooth non-rational Fano threefold over $\mathbb{C}$. Assume that X is not birational to a cubic threefold. Then X is not stably rational.*

Here, 'very general' refers to the complement to a countable union of Zariski-closed proper subsets of the families enumerated in Sections 5 and 7.

While smooth cubic threefolds are all known to be non-rational, determining whether or not they are stably rational remains an open problem. No smooth cubic threefolds are known to be stably rational. However, Voisin [Voi14] has shown that the cubic threefolds where her techniques fail to apply, i.e., those admitting an integral decomposition of the diagonal, are dense in moduli.

Several common geometric threads, developed in collaboration with Kresch, unify our approach to Theorem 1. In [HKT16b], we showed

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that very general conic bundles over rational surfaces with sufficiently large discriminant fail to be stably rational. The conic bundle structures on cubic threefolds arising from projection from a line have quintic plane curves as their discriminants—too small for our techniques to apply. Nevertheless, conic bundles are a useful tool for analyzing stable rationality of Fano threefolds. Second, in [HKT16a] we classified quartic del Pezzo surfaces with mild singular fibers and maximal monodromy; previously [HT14] we showed that a number of these arise as specializations of Fano threefolds of index one. Together, these facilitate a streamlined approach to most families of non-rational Fano threefolds.

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2. Conic bundles over rational surfaces

We recall the set-up for the results of [HKT16b]: Let S be a smooth projective rational surface over $\mathbb{C}$. Fix a linear system $\mathcal{L}$ of effective divisors on S such that the general member is smooth and irreducible. Consider the space of pairs

$$\left\{ \; D \in \mathcal{L} \text{ nodal and reduced, } D' \to D \text{ étale of degree two } \right\} \to \mathcal{L}$$

and let $\mathcal{M}$ be one of its irreducible components. The curves with at worst nodes as singularities form an open dense subset in $\mathcal{L}$; the étale coverings of a projective variety are formally smooth over its deformation space [Gro71, Exp. IX,Prop. 1.7]. Hence $\mathcal{M}$ is unramified over its image in $\mathcal{L}$ and thus smooth.

Assume $\mathcal{M}$ contains a point $\{D' \to D\}$ with the following properties:

- the nodes of D are disjoint from the base locus of $\mathcal{L}$;
- D is reducible and for each irreducible component $D_1 \subset D$ the induced cover $D' \times_D D_1 \to D_1$ is non-trivial.

Results of Artin and Mumford [AM72] and Sarkisov [Sar82] allow us to assign to each point of $\mathcal{M}$ a conic bundle $X \to S$, unique up to birational equivalences over S. Essentially, $\mathcal{M}$ parametrizes ramification data for the associated Brauer elements in the function field of S, which determine them as S is rational. The condition on the distinguished point implies that the corresponding conic bundle has non-trivial Brauer group.

Using Voisin's decomposition of the diagonal technique, we proved in [HKT16b] that a very general point $[X] \in \mathcal{M}$ parametrizes a threefold that fails to be stably rational.

We first observe an obvious strengthening of the main theorem of [HKT16b]: $\mathcal{M}$ need not dominate the linear series $\mathcal{L}$ but can be any smooth irreducible parameter space of reduced nodal curves $D \in \mathcal{L}$ with étale double covers $D' \to D$. Let $\mathcal{K}$ denote the image of $\mathcal{M}$ in $\mathcal{L}$, so we have

$$\mathcal{M} \xrightarrow{\varphi} \mathcal{K} \subset \mathcal{L}$$

where φ is étale and a covering space over the open subset parametrizing smooth curves. We still insist that there is a reducible member whose nodes are disjoint from the base locus of $\mathcal{K}$, such that the cover over each component is non-trivial.

Second, our result is easiest to apply in cases where the monodromy action is large, e.g., when $\mathcal{M}$ parametrizes *all* non-trivial double covers of the general point $[D] \in \mathcal{K}$, or equivalently, when the monodromy representation on $H^1(D, \mathbb{Z}/2\mathbb{Z}) \setminus \{0\}$ is transitive. This is the case when $S = \mathbb{P}^2$ and $\mathcal{L}$ parametrizes plane curves of even degree; in odd degree there are two such orbits [Bea86]. Large monodromy actions make it easier to decide which component contains a given distinguished point $\{D' \to D\}$.

3. Classification of quartic del Pezzo fibrations and stable rationality

A quartic del Pezzo surface fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ is a flat projective morphism whose general fiber is a degree four del Pezzo surface, i.e., a smooth complete intersection of two quadrics in $\mathbb{P}^4$. We suppose π satisfies two non-degeneracy conditions:

- the discriminant is square-free, i.e., $\mathcal{X}$ is regular and the singular fibers are complete intersections of two quadrics with at most one ordinary singularity;
- the monodromy action on the Picard groups of the fibers is the full Weyl group $W(\mathsf{D}_5)$.

The fundamental invariant of such fibrations is the *height*

$$h(\mathcal{X}) = \deg(c_1(\omega_\pi)^3) = -2\deg(\pi_*\omega_\pi^{-1}),$$

an even integer (see [HKT16a, HT14] for more background). The principal results we require are [HKT16a, Th. 10.2]:

- under the non-degeneracy conditions we have $h(\mathcal{X}) \geq 8$;

- when $h(\mathcal{X}) = 8$ or 10, the moduli space of these fibrations has two irreducible components;
- when $h(\mathcal{X}) \geq 12$ the moduli space is irreducible.

When $h(\mathcal{X}) = 8, 10, 12$ the total space $\mathcal{X}$ is either rational or birational to a cubic threefold; see [HKT16a, §11] and [HT14, §8-10] for details. Thus we will focus on fibrations with heights at least fourteen. Note that Alexeev [Ale87] established non-rationality in these cases by relating the del Pezzo fibrations to conic bundles. (We will review this below.)

Theorem 2. *Let $\mathcal{X} \to \mathbb{P}^1$ be a fibration in quartic del Pezzo surfaces satisfying our non-degeneracy conditions, with $h(\mathcal{X}) \geq 14$, and very general in moduli. Then $\mathcal{X}$ fails to admit an integral decomposition of the diagonal and thus is not stably rational.*

Proof. We first reduce to the conic bundle case, following Alexeev. Choose a section $\sigma : \mathbb{P}^1 \to \mathcal{X}$, which we may assume is not contained in any line of the general fiber. Blowing up this section gives a cubic surface fibration with a distinguished line and projecting from this line gives a conic fibration:

$$\begin{array}{ccccc} \mathcal{L} & \hookrightarrow & \widetilde{\mathcal{X}} & \stackrel{\pi}{\to} & S \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{P}^1 & & \end{array}$$

Here $S \to \mathbb{P}^1$ is a rational ruled surface.

The conic bundle structure over S yields a discriminant curve $D \subset S$ and an étale double cover $D' \to D$. Note that $D' \to D$ coincides with the spectral data introduced in [HKT16a, §§2,8] and S is the natural ruled surface containing D described in [HKT16a, §10].

Using [HKT16a, §6] we pin down the numerical invariants: Suppose first that $h(\mathcal{X}) = 4n + 2$ for $n \geq 3$. Here the surface $S \simeq \mathbb{F}_1$, the Hirzebruch surface. Let ξ denote the (-1)-curve and f the class of a fiber. Then $[D] = 5\xi + (n+3)f$ which has genus $h(\mathcal{X}) - 4$. If $h(\mathcal{X}) = 4n$ for $n \geq 4$ then $S \simeq \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Here D has bidegree $(n, 5)$, also of genus $h(\mathcal{X}) - 4$.

The fundamental dictionary between del Pezzo fibrations and spectral data [HKT16a, Th. 10.1] implies that the $D \subset S$ arising from del Pezzo fibrations are general in the linear series $\mathcal{L} = |D|$. The analysis of [HKT16a, §3] shows that the monodromy acts on $H^1(D, \mathbb{Z}/2\mathbb{Z})$ via the full symplectic group, hence transitively on the non-trivial elements.

As indicated in Section 2, we let $\mathcal{M}$ parametrize étale connected double covers of nodal curves in $\mathcal{L}$, which is étale over $\mathcal{L}$. To apply the main result of Section 2, it suffices to exhibit a distinguished point in $\mathcal{L}$, i.e., a reducible curve $D = D_1 \cup D_2$ with D_1 and D_2 smooth of positive genus, intersecting transversally. For $S = \mathbb{F}_1$ take $D_1 \in |2\xi + 3f|$, the proper transform of a cubic plane curve, and $D_2 \in |3\xi + nf|$, a smooth curve of genus $2n-5 \geq 1$. For $S = \mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ take D_1 of bidegree $(2,2)$, an elliptic curve, and D_2 of bidegree $(n-2,3)$, of genus $2n-6 \geq 2$. □

4. Quartic del Pezzo fibrations and sextic double solids

Our goal in this section is:

Proposition 3. *The general sextic double solid arises as a deformation of a nodal birational model of a general height* 22 *quartic del Pezzo fibration* $\mathcal{X} \to \mathbb{P}^1$.

Let $V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4}$ and consider

$$V^* \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 9}$$

associated with global sections of V. Then we have morphisms

$$\mathbb{P}(V^*) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^8 \xrightarrow{\pi_2} \mathbb{P}^8,$$

where the composition collapses the distinguished section

$$\sigma : \mathbb{P}^1 \to \mathbb{P}(V^*)$$

arising from the $\mathcal{O}_{\mathbb{P}^1}$ summand. We use $\pi = \pi_1$ for the fibration over $\mathbb{P}^1$. Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(V^*)}(1))$ and $h = \pi^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)))$ so that $\xi^5 = 4\xi^4 h$.

A general height 22 quartic del Pezzo $\mathcal{X} \to \mathbb{P}^1$ admits an embedding

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathbb{P}(V^*) \\ & \searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

as a complete intersection of divisors of degrees $2\xi - h$ and 2ξ [HT14, §4, Case 5]. Let $\mathcal{Q} \to \mathbb{P}^1$ denote the former divisor, which necessarily contains σ. The second divisor $\mathcal{Q}'$ is a pull-back of a quadric hypersurface via π_2.

Consider projection from σ:

$$\varpi : \mathbb{P}(V^*) \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 4}) \simeq \mathbb{P}^1 \times \mathbb{P}^3$$

inducing a birational map

$$\mathcal{Q} \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^3.$$

Restricting to $\mathcal{X}$ yields a generically finite morphism

$$\phi : \mathcal{X} \to \mathbb{P}^3.$$

We compute its invariants via intersections in $\mathbb{P}(V^*)$. The pullback of the hyperplane class on $\mathbb{P}^3$ via ϕ is $\xi - h$. First, we have

$$\deg(\phi) = (\xi - h)^3 (2\xi)(2\xi - h) = 2$$

which means ϕ is a double cover. Its ramification divisor

$$R = K_{\mathcal{X}} - \phi^* K_{\mathbb{P}^4} = -\xi + h + 4(\xi - h) = 3(\xi - h)$$

maps to the branch surface $B \subset \mathbb{P}^3$ of degree six.

We interpret when ϕ fails to be finite. Points $p \in \mathbb{P}^3$ correspond to line subbundles

$$\sigma(\mathbb{P}^1) \hookrightarrow \mathcal{L}(p) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{F}_1 \hookrightarrow \mathbb{P}(V^*)$$

where $\mathbb{F}_1$ is the blowup of the projective plane at a point. Thus $\mathcal{Q} \cap \mathcal{L}(p)$ is the union of the (-1)-curve and the proper transform of a line ℓ and $\mathcal{Q}' \cap \mathcal{L}(p)$ is the proper transform of a conic *disjoint* from the (-1)-curve. These typically meet at two points but the conic might contain the line ℓ, i.e., $\phi^{-1}(p) = \ell$; this is a codimension-three condition on p and corresponds to singular points of B.

Proposition 4. *Let y_0, y_1, y_2, y_3 denote coordinates on $\mathbb{P}^3$.*

- *The equation for B takes the form* $\det(M) = 0$ *where*

$$M = \begin{pmatrix} L^2 & Q_0 & Q_1 \\ Q_0 & Q'_{00} & Q'_{01} \\ Q_0 & Q'_{01} & Q'_{11} \end{pmatrix},$$

  with L linear in the y_i and the remaining entries quadratic.
- *Conversely, the general such matrix arises from a height* 22 *fibration in quartic del Pezzo surfaces.*
- *When M is general, the singularities of B are of two types. The first type corresponds to the vanishing of the 2×2 minors of M; there are* 32 *such singularities. The second type corresponds to the locus*

$$L = Q_0 = Q_1 = 0;$$

  there are four such singularities. All 36 *singularities are nodes.*

This suffices to establish Proposition 3.

Proof. Let x_0 and x_1 denote homogeneous coordinates on $\mathbb{P}^1$ and their pullbacks to $\mathbb{P}(V^*)$. Designate generating global sections

$$y_0, y_1, y_2, y_3 \in \Gamma(\mathcal{O}_{\mathbb{P}(V^*)}(\xi - h)) \simeq \Gamma(\mathcal{O}_{\mathbb{P}^3}(1))$$

and

$$z, x_0y_0, x_1y_0, \dots, x_0y_3, x_1y_3 \in \Gamma(\mathcal{O}_{\mathbb{P}(V^*)}(\xi)).$$

After completing the square to eliminate the term linear in z, the defining equation $\mathcal{Q}'$ may be written in the form

$$z^2 = Q'_{00}x_0^2 + 2Q'_{01}x_0x_1 + Q'_{11}x_1^2,$$

where the Q'_{ij} are quadratic in the y_i. The defining equation for $\mathcal{Q}$ takes the form

$$-zL(y_0, y_1, y_2, y_3) + Q_0x_0 + Q_1x_1 = 0,$$

where L is linear and Q_0 and Q_1 are quadratic in the y_i. Eliminating z we obtain

$$x_0^2(Q'_{00}L^2 - Q_0^2) + 2x_0x_1(Q'_{01}L^2 - Q_0Q_1) + x_1^2(Q'_{11}L^2 - Q_1^2) = 0,$$

which is the defining equation for the image of $\mathcal{X}$ in $\mathbb{P}^1 \times \mathbb{P}^3$. The discriminant of this polynomial—regarded as a binary quadratic form in x_0 and x_1—can be written as

$$L^4((Q'_{01})^2 - Q'_{00}Q'_{11}) + L^2(-2Q'_{01}Q_0Q_1 + Q'_{00}Q_1^2 + Q'_{11}Q_0^2).$$

After dividing out by $-L^2$ we obtain $\det(M)$. This proves the first assertion. Reversing the algebra gives the second assertion.

We analyze the singularities of the hypersurface $\det(M) = 0$. In general, the singularities of the determinant of a symmetric 3×3 matrix of forms is given by the vanishing of the 2×2 minors. In geometric terms, this is the Veronese surface $\mathrm{Ver} \hookrightarrow \mathbb{P}^5$ which has degree four. If the entries are quadratic forms in $y_0, \dots, y_3$ then the image of the associated morphism $\mathbb{P}^3 \to \mathbb{P}^5$ has degree eight. Bézout's Theorem gives 32 transversal points of intersection, which are nodes of B.

However, we also have to take into account singularities of the entries. Given the form of the upper-left entry of M, these occur precisely when $L = 0$. (The other entries are general.) The determinantal hypersurface thus has additional singularities along the locus $L = Q_0 = Q_1 = 0$. Our generality assumption implies this is a complete intersection, thus we obtain four additional ordinary double points. □

5. Index one Fano threefolds

The tables in [IP99] enumerate non-rational Fano threefolds; see also the summary in [Bea16a, Section 2.3], which includes references to methods used to establish non-rationality. Our proofs of failure of stable rationality for very general members of these families are based on the degeneration method of Voisin and its strengthening by Colliot-Thélene and Pirutka: if a smooth threefold admits an integral decomposition of the diagonal the same holds true for a specialization with mild singularities [Voi15, Th. 1.1] and [CTP16b]. We will exhibit specializations not admitting such decompositions.

Let V be a smooth Fano threefold with $\mathrm{Pic}(V) = \mathbb{Z}K_V$, i.e., with rank one and index one. Its degree $d(V) = -K_V^3$ takes the following values [IP99]:

$$d(V) = 2, 4, 6, 8, 10, 12, 14, 16, 18, 22.$$

For each $d(V)$ there is an irreducible parameter space for the corresponding Fano threefolds. The cases $d(V) = 12, 16, 18$, and 22 are rational.

When $d(V) = 14$ the general $X \subset \mathbb{P}^9$ arises as a linear section of the Grassmannian $\mathrm{Gr}(2, 6)$. Projective duality gives a codimension ten section of the Pfaffian cubic hypersurface in $\mathbb{P}^{14}$, a cubic threefold V'. There is a birational map $V \dashrightarrow V'$; see [IM00, §1], for example, for additional details. This example is related to quartic del Pezzo fibrations: One of the two species of quartic del Pezzo fibrations of height ten $\mathcal{X} \to \mathbb{P}^1$ admits a natural morphism [HKT16a, §11]

$$\mathcal{X} \to V \subset \mathbb{P}^9;$$

the image is a nodal Fano threefold of degree 14. However, stable rationality of cubic threefolds (and birationally equivalent varieties) remains an open problem.

5.1. $d(V) = 2$**: Sextic double solids.** Failure of stable rationality in this case has been established by Beauville [Bea16b] and by Colliot-Thélène and Pirutka [CTP16a]. It also follows from Proposition 3: a general height 22 fibration in quartic del Pezzo surfaces fails to have an integral decomposition of the diagonal, by Theorem 2, but also arises as a nodal sextic double solid.

5.2. $d(V) = 4$**: Quartic threefolds.** Failure of stable rationality in this case has been established by Colliot-Thélène and Pirutka [CTP16b]. As above, we obtain an alternative proof: a general quartic del Pezzo fibration of height 20 fails an integral decomposition of the diagonal, by Theorem 2, but also admits a birational model as a determinantal quartic threefold with sixteen nodes [HT14, §11] (cf. [Che06, Th. 11]).

5.3. $d(V) = 6$**: Complete intersections of a quadric and a cubic in** $\mathbb{P}^5$**.** We proceed as before, using the fact that a general quartic del Pezzo fibration of height 18 admits a birational model as a complete intersection $\mathcal{Y} \subset \mathbb{P}^5$ with eight nodes. Indeed, realize

$$\mathcal{X} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subset \mathbb{P}^1 \times \mathbb{P}^5$$

as a complete intersection of forms of bidegree $(1,1)$, $(0,2)$, and $(1,2)$, as in Case 2 of [HT14, §4]. (Here we are using the irreducibility of the moduli space of quartic del Pezzo fibrations of height 18.) Let $\mathcal{Y} \subset \mathbb{P}^5$ denote the image of projection onto the second factor. Consider first the image in $\mathbb{P}^5_{[x_0,\dots,x_5]}$ of the locus cut out by the forms of bidegree $(1,1)$ and $(1,2)$:

$$sL_0 + tL_1 = sQ_0 + tQ_1 = 0,$$

with

$$L_0, L_1 \in \mathbb{C}[x_0, \dots, x_5]_1, \quad Q_0, Q_1 \in \mathbb{C}[x_0, \dots, x_5]_2.$$

Its equation is obtained by eliminating s and t, which yields

$$L_0Q_1 - L_1Q_0 = \det \begin{pmatrix} L_0 & L_1 \\ Q_0 & Q_1 \end{pmatrix} = 0.$$

This is a cubic fourfold $\mathcal{W}$ singular along the elliptic quartic curve

$$C = \{L_0 = L_1 = Q_0 = Q_1 = 0\}.$$

Let $Q \in \mathbb{C}[x_0, \dots, x_5]_2$ be the form of bidegree $(0,2)$, so that

$$\mathcal{Y} = \mathcal{W} \cap \{Q = 0\}.$$

This is a complete intersection of Q with $\mathcal{W}$, having eight nodes at the intersection $C \cap \{Q = 0\} = \{p_1, \dots, p_8\}$. The preimages of these nodes in $\mathcal{X}$ are distinguished sections of $\mathcal{X} \to \mathbb{P}^1$.

We establish failure of stable rationality for very general complete intersections as before: Theorem 2 gives failure of integral decomposition of the diagonal for very general quartic del Pezzo fibrations of height eighteen. Thus very general complete intersections $V \subset \mathbb{P}^5$ of a quadric and a cubic also lack such decompositions, so stable rationality fails.

5.4. $d(V) = 8$**: Complete intersections of three quadrics in** $\mathbb{P}^6$**.** Let $V \subset \mathbb{P}^6$ denote a complete intersection of three quadrics. Beauville [Bea77, §6.4,§6.23] has shown that V is birational to a conic fibration

$$X \to \mathbb{P}^2,$$

with discriminant $D \subset \mathbb{P}^2$ of degree seven, and a general plane curve of degree seven arises in this way. Thus the results in §2 apply: a very general such V fails to be stably rational.

We sketch an alternative proof using the fact that a general quartic del Pezzo fibration of height 16 admits a birational model as a complete intersection $\mathcal{Y} \subset \mathbb{P}^6$ with four nodes. Express the fibration

$$\mathcal{X} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) \subset \mathbb{P}^1_{[s,t]} \times \mathbb{P}^6_{[x_0,\ldots,x_6]}$$

as a complete intersection of two forms of bidegree $(1,1)$, and two quadratic forms from $\mathbb{P}^6$ (see Case 3 of [HT14, §4]). Write the bilinear forms as

$$sL_0 + tL_1 = sM_0 + tM_1 = 0, \quad L_0, L_1, M_0, M_1 \in \mathbb{C}[x_0, \ldots, x_6]_1$$

so that eliminating s and t gives

$$L_0M_1 - L_1M_0 = \det \begin{pmatrix} L_0 & L_1 \\ M_0 & M_1 \end{pmatrix} = 0,$$

defining a quadric hypersurface $\mathcal{W} \subset \mathbb{P}^6$ singular along $P = \{L_0 = L_1 = M_0 = M_1 = 0\}$. Let $\mathcal{Y}$ be the projection of $\mathcal{X}$ onto the second factor; it is the intersection of $\mathcal{W}$ with two arbitrary quadric hypersurfaces, and is singular where these both meet P. The failure of stable rationality for a very general complete intersection of three quadrics follows as before from Theorem 2.

5.5. $d(V) = 10$**: Complete intersections in** $\mathrm{Gr}(2,5)$**.** Fano threefolds V of this type are obtained by intersecting the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with two linear forms and one quadratic form.

In [HT14, §11] we showed that general quartic del Pezzo fibrations of height fourteen are birational to $\mathcal{Y} \subset \mathbb{P}^7$, where $\mathcal{Y}$ is a specialization of V with two nodes. Repeating the arguments above, we conclude that the very general such V fails to be stably rational.

6. Fano threefolds of index two

In this section we consider Fano threefolds V with $\mathrm{Pic}(V) = \mathbb{Z}\frac{K_V}{2}$, i.e., those of rank one and index two. Here the degree $d(V) = -K_V^3 = 8 \cdot \delta(V)$ where $\delta(V) \in \mathbb{N}$. The possible values are $\delta(V) = 1, 2, 3, 4, 5$; if

$\delta(V) = 4$ or 5 then V is rational, and when $\delta(V) = 3$ then V is a cubic threefold.

6.1. $\delta(V) = 1$**: Double cover of Veronese cone.** Let

$$\mathbb{P} := \mathbb{P}(1, 1, 1, 2) \subset \mathbb{P}^6$$

denote the cone over the Veronese surface Ver $\subset \mathbb{P}^5$; the vertex $p = [0, 0, 0, 1]$ is a terminal singularity of $\mathbb{P}$ of index 2. Let $B \subset \mathbb{P}$ denote the restriction of a general cubic hypersurface in $\mathbb{P}^6$, which has degree six in the natural grading on $\mathbb{P}$. Consider the double cover

$$\phi : V \to \mathbb{P}$$

branched over B. It is also ramified over p; its preimage $v_0 \in V$ is smooth.

We elaborate the geometry: Blowing up p gives a resolution

$$\beta : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \simeq \mathrm{Bl}_p(\mathbb{P}) \to \mathbb{P}.$$

Let ξ and h generate the Picard group of the projective bundle, where h is the pullback of the hyperplane class on $\mathbb{P}^2$ and $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-2))})$. Let

$$E \simeq \mathbb{P}^2 \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2))$$

denote the exceptional divisor; note that $[E] = \xi - 2h$. The divisor $B + E = 4\xi - 2h$ is divisible by two, so we obtain a double cover $V' \to \mathrm{Bl}_p(\mathbb{P})$ branched along E and B. The normal bundle $\mathcal{N}_{E/V'} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ so we can blown down V' along E; the resulting variety is V.

We may also regard V as a hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree six, which is clearly Fano of index two. Note that $h^1(\Omega^2_V) = 21$ [IP99, §12] and that V depends on 34 parameters.

We specialize B so it contains v_0, analyzing the resulting double cover $\phi : V \to \mathbb{P}$. (This imposes one condition so the construction depends on 33 parameters.) Let $\tilde{B}$ denote the proper transform of B with $[\tilde{B}] = 2\xi + 2h$, $\tilde{V} \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ the double cover branched along $\tilde{B}$, and $\tilde{E} \subset \tilde{V}$ the preimage of E. Note that $\tilde{E} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ as $\tilde{B}$ meets $E \simeq \mathbb{P}^2$ in a plane conic C. Moreover, applying the Hurwitz formula and adjunction yields

$$\mathcal{N}_{\tilde{E}/\tilde{V}} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2).$$

The induced birational morphism $\tilde{V} \to V$ resolves v_0 with exceptional divisor $\tilde{E} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In particular, $\tilde{V} \to V$ is universally CH_0-trivial (see [CTP16b, Prop. 1.8]). For the equivalence between universal CH_0-triviality and the existence of integral decompositions of the diagonal, see [ACTP16, Lemma 1.3] and [Voi15, §1].

We compute the invariants of $\tilde{V}$: The bundle structure

$$\varpi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \to \mathbb{P}^2$$

induces a morphism

$$\psi : \tilde{V} \to \mathbb{P}^2.$$

Since $\tilde{B}$ is a bisection of ϖ, ψ endows V with the structure of a conic bundle. Let $D \subset \mathbb{P}^2$ denote its discriminant curve, which coincides with the branch locus of $\varpi : \tilde{B} \to \mathbb{P}^2$. An adjunction computation implies $K_{\tilde{B}} = h|_{\tilde{B}}$ so D is a plane octic curve, generically smooth.

Now D and C are tangent at each point of their intersection, i.e.,

$$D \cap C = 2(z_1 + \ldots + z_8) = 2Z$$

with $\mathcal{I}_Z \subset \mathcal{O}_D$ the ideal sheaf. Thus D depends on $44 - 8 - 3 = 33$ parameters; moreover, the parameter space of smooth octic plane curves eight-tangent to C is birational to a projective bundle over $C^{[8]}$, thus irreducible. Furthermore, $\eta := \mathcal{O}_D(1) \otimes \mathcal{I}_Z$ is a two-torsion element of the Jacobian of D. The double cover $D' \to D$ associated with $\tilde{V} \to \mathbb{P}^2$ is classified by η. From it, we read off the cohomology of $\tilde{V}$:

$$\mathrm{IJ}(\tilde{V}) = \mathrm{Prym}(D' \to D).$$

The curve D has genus 21 so $h^1(\Omega^2_{\tilde{V}}) = 20$. Thus the singularity v_0 reduces this Hodge number by one.

Lemma 5. *There exists a specialization*

$$D \rightsquigarrow D_1 \cup D_2$$

of octic curves eight-tangent to C, such that D_1 and D_2 are transverse plane quartics each four-tangent to C. This satisfies the requirements of §2.

Proof. Consider the space of pairs (D_1, D_2) where D_1 and D_2 are plane quartics four-tangent to C, with D_1 and D_2 meeting transversally. Repeating the argument above, the plane quartics four-tangent to C are birational to a $\mathbb{P}^6$ bundle over $C^{[4]}$, an irreducible rational variety of dimension ten. It is easy to check that a general pair of such curves meets transversally, yielding a rational parameter space of dimension twenty. Write

$$D_1 \cap C = 2(z_1 + z_2 + z_3 + z_4) \quad D_2 \cap C = 2(z_5 + z_6 + z_7 + z_8)$$

and $\mathcal{I}_Z \subset \mathcal{O}_{D_1 \cup D_2}$ for the ideal sheaf of $Z = \{z_1, \dots, z_8\}$. Thus $\eta_0 := \mathcal{I}_Z(1)$ is two-torsion in the Picard group of $D_1 \cup D_2$ and restricts to non-trivial two-torsion elements on D_1 and D_2 because

$$z_1 + z_2 + z_3 + z_4 \not\equiv [\mathcal{O}_{D_1}(1)], \quad z_5 + z_6 + z_7 + z_8 \not\equiv [\mathcal{O}_{D_2}(1)].$$

Otherwise, these four-tuples of points would be collinear.

Linear algebra shows that we can smooth $D_1 \cup D_2$ to a smooth plane octic D tangent to C at $z_1, \dots, z_8$. As we saw in the proof of Theorem 2, this gives rise to a cover $D' \to D$ classified by the divisor η. As $D \rightsquigarrow D_1 \cup D_2$ we have $\eta \rightsquigarrow \eta_0$. □

Thus the results of §2 imply that $\tilde{V}$ fails to admit an integral decomposition of the diagonal. An application of the results of [CTP16b, §1] implies that a very general $V \subset \mathbb{P}(1,1,1,2,3)$ also fails to admit an integral decomposition of the diagonal, and thus is not stably rational.

6.2. $\delta(V) = 2$: **Quartic double solids.** Let V be a quartic double solid

$$\phi : V \to \mathbb{P}^3$$

with branch locus a degree four K3 surface B. When V is smooth we have $h^1(\Omega^2_V) = 10$. Voisin [Voi15] and Colliot-Thélène and Pirutka [CTP16a] established the failure of stable rationality for very general varieties in this class. Here we discuss how to approach this through conic bundle fibrations.

Now suppose V (or equivalently B) has a node p and write $\tilde{V} = \mathrm{Bl}_p(V)$. Projection from p gives a conic bundle structure

$$\pi : \tilde{V} \to \mathbb{P}^2$$

branched along a sextic plane curve D. The plane curve is typically smooth but admits a six-tangent conic curve C corresponding to the exceptional divisor of the induced resolution of B. Write

$$D \cap C = 2Z, \quad Z = z_1 + \dots + z_6$$

so that $\eta := \mathcal{O}_D(1) \otimes \mathcal{I}_Z$ is two-torsion on D. Here $\mathcal{I}_Z$ is the ideal sheaf of Z.

As we saw in the previous case, the parameter space of sextic plane curves six-tangent to a prescribed conic is irreducible, being a projective bundle over the Hilbert scheme $C^{[6]}$. We can specialize

$$D \rightsquigarrow D_1 \cup D_2,$$

where D_1 and D_2 are smooth plane cubics meeting transversally, each three-tangent to C. Thus η specializes to a two-torsion divisor on $D_1 \cup D_2$ that is non-trivial on each component.

An application of the results of §2 implies that the very general quartic double solid fails to have an integral decomposition of the diagonal, thus is not stably rational.

7. Fano threefolds of higher Picard rank

As before we write $d(V) = -K_V^3$.

7.1. $d(V) = 6, h^{1,2}(V) = 20$. The first case is double covers

$$V \to \mathbb{P}^1 \times \mathbb{P}^2$$

branched over a divisor of bidegree $(2, 4)$. These depend on

$$3 \times 15 - (1 + 3 + 8) = 33$$

parameters. Projection onto the second factor gives a conic bundle structure $V \to \mathbb{P}^2$ with octic discriminant curve $D \subset \mathbb{P}^2$. The equation of D is given by the vanishing of the determinant of a 2×2 *symmetric* matrix of quartic forms in three variables. In particular, D is not general in its linear series and each symmetric determinantal octic comes with a distinguished non-trivial two-torsion class, i.e., the one associated with the ramification data of $V \to \mathbb{P}^2$. This makes it hard to apply the methods of §2 directly.

However, there is a natural degeneration of such Fano threefolds to another class of rationally connected varieties: Fix distinct points $p, q \in \mathbb{P}^2$ and consider divisors $B_0 \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 4)$ whose fibers over $\mathbb{P}^1$ admit nodes at p and q. (Equivalently, these are singular along $\mathbb{P}^1 \times \{p, q\}$.) Consider the birational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

blowing up p and q and blowing down the line joining them. This takes quartic plane curves singular at p and q to bidegree $(2, 2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Using the induced birational map

$$\mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

we see that B_0 is mapped to $(2, 2, 2)$ divisor in the image. Conversely, $(2, 2, 2)$ divisors in the image all arise from this construction.

Lemma 6. *Let V_0 denote the double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a very general such B_0. Let $\widetilde{V}_0 \to V_0$ denote the blowup along $\mathbb{P}^1 \times \{p, q\}$. If $\widetilde{V}_0$ admits no integral decomposition of the diagonal then the same* *holds for the very general Fano variety V arising as a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a divisor of bidegree $(2,4)$.*

Proof. The singularities of V_0 are along the lines $\ell_p := \mathbb{P}^1 \times \{p\}$ and $\ell_q := \mathbb{P}^1 \times \{q\}$. The singularities of B_0 are ordinary double points along the general points of these lines and cusps (analytically isomorphic to $x^2 + y^3 = 0$) at a finite number of special points. For special $r \in \mathbb{P}^1$ the local singularity type of V_0 at (r,p) is of the form

$$w^2 = x^2 + ty^2 + y^3,$$

where $\{x,y\}$ are local coordinates of $\mathbb{P}^2$ centered at p, t is a local coordinate of $\mathbb{P}^1$ centered at r, and w is used to realize the double cover over $\mathbb{P}^1 \times \mathbb{P}^2$. Thus the singularities of V_0 are resolved by blowing up the lines

$$\widetilde{V_0} = \mathrm{Bl}_{\ell_p \cup \ell_q}(V_0) \to V_0.$$

The exceptional fibers over the generic points of ℓ_p and ℓ_q are smooth conics; the fibers over special points are reducible conics. This computation is similar to, but simpler than, the singularity analysis of [CTP16b, App.].

The key is that the exceptional fibers are universally CH_0-trivial, in the sense of [CTP16b, Déf. 1.2]. Applying the result on universal CH_0-triviality in [CTP16b, §1], we conclude that V fails to be universally CH_0-trivial if the same holds for $\widetilde{V_0}$. (See [ACTP16, Lemma 1.3] and [Voi15, §1] for the equivalence with integral decompositions of the diagonal.) □

In §7.4 we will show that very general double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over a $(2,2,2)$ divisor do not admit integral decompositions of the diagonal.

7.2. $d(V) = 12, h^{1,2}(V) = 9$. The first part of the second case is realized as a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2,2)$, depending on 19 parameters. Using either projection, we obtain a conic bundle over $\mathbb{P}^2$ with sextic discriminant. It is well known that the plane sextic can be chosen generally [vG05, HVA13, §9]. The main result of §2 implies that very general conic bundles over $\mathbb{P}^2$ with sextic discriminant fail to be stably rational. That is, for a very general pair $(D, D' \to D)$, where D is a plane sextic and $D' \to D$ is a non-trivial étale double cover, the corresponding conic bundle $X \to \mathbb{P}^2$ fails to be stably rational. It follows that for very general D, *every* double cover $D' \to D$ is associated with a conic bundle that is not stably rational. In particular, this applies to the very general divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

The second part of the second case is a double cover V of a hypersurface $\mathbb{F}(1,2) \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ branched over an anticanonical divisor B of bidegree $(2,2)$. This depends on 18 parameters. Again, either projection induces a conic bundle $V \to \mathbb{P}^2$ with sextic discriminant curve D. The Mori-Mukai classification [MM82] shows this is a specialization of the first part.

The surface B is a lattice polarized K3 surface of type

$$\Phi := \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & 4 \\ f_2 & 4 & 2 \end{array}$$

and the general such surface arises as a complete intersection of forms of bidegree $(1,1)$ and $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

The branch curve of $\pi_i|B : B \to \mathbb{P}^2$ coincides with the locus where $\pi_i|V : V \to \mathbb{P}^2$ fails to be smooth, i.e., the discriminant curve D. This is a sextic plane curve that is not of general moduli—the associated K3 double cover has Picard rank two. The technique of §2 does not immediately apply in this case.

It is easy to use degeneration techniques to reduce this to cases where there is no integral decomposition of the diagonal. Consider a quartic surface $B_0 \subset \mathbb{P}^3$ with nodes n_1 and n_2 and minimal resolution $\widetilde{B_0}$. Its Picard group takes the form

$$\begin{array}{c|ccc} & h & R_1 & R_2 \\ \hline h & 4 & 0 & 0 \\ R_1 & 0 & -2 & 0 \\ R_2 & 0 & 0 & -2 \end{array}$$

where h is the pullback of the hyperplane class and the R_i are the exceptional divisors. The lattice Φ embeds into this lattice

$$f_1 = h - R_1, \quad f_2 = h - R_2.$$

Projection from the nodes n_1 and n_2 gives a morphism

$$\widetilde{B_0} \to \mathbb{P}^2 \times \mathbb{P}^2$$

with image $B_\circ$ a complete intersection of hypersurfaces of degrees $(1,1)$ and $(2,2)$. This extends to a birational map

$$\mathbb{P}^3 \dashrightarrow \mathbb{F}(1,2) := \mathbb{P}(\Omega^1_{\mathbb{P}^2}(1)) \subset \mathbb{P}^2 \times \mathbb{P}^2$$

onto the divisor of bidegree $(1,1)$. To summarize:

Lemma 7. *A double cover of $\mathbb{P}^3$ branched over a general quartic surface with two nodes is birational to a double cover of the complete flag variety $\mathbb{F}(1,2)$ along a special anticanonical divisor.*

By [Voi15, Th. 1.1] a very general such quartic double solid fails to admit an integral decomposition of the diagonal. Thus the same holds for very general Fano threefolds $V \to \mathbb{F}(1,2)$ and so these fail to be stably rational.

7.3. $d(V) = 14, h^{1,2}(V) = 9$**.** The third case is the double cover of $\mathbb{P}^3$ blown up at a point, with anticanonical branch locus B meeting the exceptional divisor transversally. These conic bundles were addressed in §6.2 as singular quartic double solids.

7.4. $d(V) = 12, h^{1,2}(V) = 8$**.** The fourth case is a double cover

$$V \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

branched over a divisor of degree $(2,2,2)$. These depend on $27-1-9 = 17$ parameters. For each projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ we obtain a conic bundle, with discriminant D of bidegree $(4,4)$. Note that this is not general; it has equation

$$D = \{\det(M) = 0\}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix},$$

with

$$M_{11}, M_{12}, M_{22} \in \Gamma(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2)).$$

This may be interpreted geometrically: the K3 double cover

$$B \to \mathbb{P}^1 \times \mathbb{P}^1$$

has Picard group

$$\Pi := \begin{array}{c|ccc} & E_1 & E_2 & E_3 \\ \hline E_1 & 0 & 2 & 2 \\ E_2 & 2 & 0 & 2 \\ E_3 & 2 & 2 & 0 \end{array}$$

Regarding D as a curve on B, we may write $D \equiv 4(E_1 + E_2)$. The curves E_3 and $2(E_1 + E_2) - E_3$ are conjugate under the involution associated with the first two factors, which fixes D. Thus $E_1 + E_2 - E_3$ restricts to a two-torsion divisor η on D, which classifies the double cover $D' \to D$.

Given that D is not general in its linear series, the techniques of §2 do not apply directly. Clearly the monodromy cannot act transitively on the non-trivial two-torsion of D, as there is a distinguished element η. Keeping track of what happens to η as $D \rightsquigarrow D_1 \cup D_2$ can be delicate.

The quickest proof that the very general $V \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ fails to admit a decomposition of the diagonal is via degeneration of the branch locus. Let $B_0 \subset \mathbb{P}^3$ denote a quartic surface with nodes n_1, n_2, n_3 and minimal resolution $\widetilde{B_0}$. Let h denote the pullback of the polarization and R_1, R_2, R_3 the exceptional divisors over n_1, n_2, n_3:

	h	R_1	R_2	R_3
h	4	0	0	0
R_1	0	-2	0	0
R_2	0	0	-2	0
R_3	0	0	0	-2

Note that $R_0 = h - R_1 - R_2 - R_3$ is also a smooth rational curve in $\widetilde{B_0}$. We can embed Π naturally into this lattice:

$$E_1 = h - R_2 - R_3, \quad E_2 = h - R_1 - R_3, \quad E_3 = h - R_1 - R_2.$$

These reflect elliptic fibrations induced by pencils of planes in $\mathbb{P}^3$ passing through two of the three nodes. Note that

$$E_i = R_0 + R_i, \quad i = 1, 2, 3,$$

which means that each elliptic fibration admits a fiber of Kodaira type I_2 containing R_0 as a component.

The connection between $B_0 \subset \mathbb{P}^3$ and $(2,2,2)$ K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ goes further. There is a birational map

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

where the map onto each factors is given by the pencil of planes through a pair of nodes of B_0. This maps B_0 birationally onto a $(2,2,2)$ nodal K3 surface $B_\circ$, as R_0 is in the fiber of each of the elliptic fibrations. Conversely, general nodal $(2,2,2)$ surfaces $B_\circ \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ yield quartic surfaces with three nodes. To summarize:

Lemma 8. *General double solids $V_0 \to \mathbb{P}^3$ branched over a quartic surface with three nodes yield double covers $V_\circ \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over a nodal $(2,2,2)$ surface, and* vice versa.

Voisin [Voi15, Th. 1.1] has shown that a double solid branched over a very general quartic surface with $r \leq 7$ nodes fails to admit an integral decomposition of the diagonal. Thus the same holds for a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over a very general $(2,2,2)$ surface. We conclude that such threefolds fail to be stably rational.

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