# LOG MINIMAL MODEL PROGRAM FOR THE MODULI SPACE OF STABLE CURVES: THE FIRST FLIP

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Abstract. We give a geometric invariant theory (GIT) construction of the log canonical model $\overline{M}_g(\alpha)$ of the pairs $(\overline{M}_g, \alpha\delta)$ for $\alpha \in (7/10 - \epsilon, 7/10]$ for small $\epsilon \in \mathbb{Q}_+$. We show that $\overline{M}_g(7/10)$ is isomorphic to the GIT quotient of the Chow variety bicanonical curves; $\overline{M}_g(7/10 - \epsilon)$ is isomorphic to the GIT quotient of the asymptotically-linearized Hilbert scheme of bicanonical curves. In each case, we completely classify the (semi)stable curves and their orbit closures. Chow semistable curves have ordinary cusps and tacnodes as singularities but do not admit elliptic tails. Hilbert semistable curves satisfy further conditions, e.g., they do not contain elliptic bridges. We show that there is a small contraction $\Psi : \overline{M}_g(7/10 + \epsilon) \to \overline{M}_g(7/10)$ that contracts the locus of elliptic bridges. Moreover, by using the GIT interpretation of the log canonical models, we construct a small contraction $\Psi^+ : \overline{M}_g(7/10 - \epsilon) \to \overline{M}_g(7/10)$ that is the Mori flip of $\Psi$.

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## 1. Introduction

Our inspiration is to understand the canonical model of the moduli space $\overline{M}_g$ of stable curves of genus $g$. This is known to be of general type for $g = 22$ and $g \geq 24$ [Far06, HM82, EH86]. In these cases, we can consider the canonical ring

$$\oplus_{n\geq 0}\Gamma(n(K_{\overline{\mathcal{M}}_g})).$$

which is finitely generated by a fundamental conjecture of birational geometry, recently proven in [BCHM06]. Then the corresponding projective variety

$$\mathsf{Proj} \oplus_{n\geq 0} \Gamma(n(K_{\overline{\mathcal{M}}_g}))$$

is birational to $\overline{M}_g$ and is called its *canonical model*.

There has been significant recent progress in understanding canonical models of moduli spaces. For moduli spaces $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g \geq 12$, the canonical model exists and is equal to the first Voronoi compactification [SB06]. Unfortunately, no analogous results are known for $\overline{M}_g$, even for $g \gg 0$.

Our approach is to approximate the canonical models with *log canonical models*. Consider $\alpha \in [0, 1] \cap \mathbb{Q}$ so that $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is an effective $\mathbb{Q}$-divisor. We have the graded ring

$$\oplus_{n\geq 0}\Gamma(n(K_{\overline{\mathcal{M}}_g} + \alpha\delta))$$

and the resulting projective variety

$$\overline{M}_g(\alpha) := \mathsf{Proj}\left(\oplus_{n\geq 0}\Gamma(n(K_{\overline{\mathcal{M}}_g} + \alpha\delta))\right).$$

Our previous paper [HHar] describes $\overline{M}_g(\alpha)$ explicitly for large values of $\alpha$. For simplicity we assume that $g \geq 4$: Small genera cases have been considered in [Has05, HL07b, HL07a]. For $9/11 < \alpha \leq 1$, $\overline{M}_g(\alpha)$ is equal to $\overline{M}_g$. The first critical value is $\alpha = 9/11$: $\overline{M}_g(9/11)$ is the coarse moduli space of the moduli stack $\overline{\mathcal{M}}_g^{ps}$ of pseudostable curves [Sch91]. A pseudostable curve may have cusps but they are not allowed to have *elliptic tails*, i.e., genus one subcurves meeting the rest of the curve in one point. Furthermore, there is a divisorial contraction

$$T : \overline{M}_g \to \overline{M}_g(9/11)$$

induced by the morphism $\mathcal{T} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$ of moduli stacks which replaces an elliptic tail with a cusp. Furthermore, $\overline{M}_g(\alpha) \simeq \overline{M}_g(9/11)$ provided $7/10 < \alpha \leq 9/11$.

This paper addresses what happens when $\alpha = 7/10$. Given a sufficiently small positive $\epsilon \in \mathbb{Q}$, we construct a small contraction and its flip:

$$\begin{array}{ccc} \overline{M}_g(\frac{7}{10}+\epsilon) & & \overline{M}_g(\frac{7}{10}-\epsilon) \\ & \searrow^{\Psi} \quad {}^{\Psi^+}\swarrow & \\ & \overline{M}_g(\frac{7}{10}) & \end{array}$$

The resulting spaces arise naturally as geometric invariant theory (GIT) quotients and admit partial modular descriptions. We construct $\overline{M}_g(7/10)$ as a GIT quotient of the Chow variety of bicanonical curves; it parametrizes equivalence classes of *c-semistable curves*. We defer the formal definition, but these have nodes, cusps, and tacnodes as singularities. The flip $\overline{M}_g(7/10-\epsilon)$ is a GIT quotient of the Hilbert scheme of bicanonical curves; it parametrizes equivalence classes of *h-semistable curves*, which are c-semistable curves not admitting certain subcurves composed of elliptic curves (see Definition 2.6).

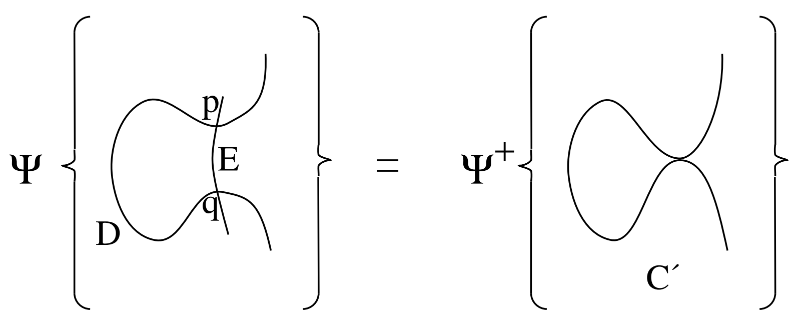


FIGURE 1. Geometry of the flip

We may express the flip in geometric terms (Figure 1): Let $C = D \cup_{p,q} E$ denote an *elliptic bridge*, where $D$ is smooth of genus $g-2$, $E$ is smooth of genus one, and $D$ meets $E$ at two nodes $p$ and $q$. Let $C'$ be a tacnodal curve of genus $g$, with normalization $D$ and conductor $\{p, q\}$. In passing from $\overline{M}_g(7/10+\epsilon)$ to $\overline{M}_g(7/10-\epsilon)$, we replace $C$ with $C'$. Note that the descent data for $C'$ includes the choice of an isomorphism of tangent spaces

$$\iota : T_pD \xrightarrow{\sim} T_qD;$$

the collection of such identifications is a principal homogeneous space for $\mathbb{G}_m$. When $C$ is a generic elliptic bridge, the fiber $(\Psi^+)^{-1}(\Psi(C)) \simeq \mathbb{P}^1$; see Proposition 6.5 for an explicit interpretation of the endpoints.

Here we offer a brief summary of the contents of this paper; a more detailed roadmap can be found in Section 2.5. Section 3 is devoted to a general discussion of the GIT of Chow points and Hilbert points. The main applications are the analysis of tautological classes and polarizations on the Hilbert scheme, the resulting formulas for Hilbert-Mumford indices (Proposition 3.17), and cycle maps (Corollary 3.14). We also recall various formulations of the Hilbert-Mumford one-parameter-subgroup criterion.

Section 4 is a brief review of the basin-of-attraction techniques used in this paper. These are important for analyzing when distinct curves are identified in the GIT quotients.

Section 5 discusses, in general terms, how to obtain contractions of the moduli space of stable curves from GIT quotients of Hilbert schemes. The resulting models of the moduli space depend on the choice of linearization; we express the polarizations in terms of tautological classes.

Section 6 summarizes basic properties of c-semistable curves: embedding theorems and descent results for tacnodal curves. Section 7 offers a preliminary analysis of the GIT of the Hilbert scheme and the Chow variety of bicanonically embedded curves of genus $g \geq 4$. Then in Section 8 we enumerate the curves with positive-dimensional automorphism groups. Section 9 applies this to give a GIT construction of the flip $\Psi^+ : \overline{M}_g(7/10 - \epsilon) \to \overline{M}_g(7/10)$.

Section 10 offers a detailed orbit closure analysis, using basins of attractions and a careful analysis of the action of the automorphism group on tangent spaces. The main application is a precise description of the semistable and stable bicanonical curves, proven in Section 11.

Throughout, we work over an algebraically closed field $k$, generally of characteristic zero. However, Sections 3 and 6 are valid in positive characteristic.

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## 2. Statement of results and strategy of proof

2.1. **Stability notions for algebraic curves.** In this paper, we will use four stability conditions: Deligne-Mumford stability [DM69], Schubert pseudostability [Sch91], c-(semi)stability, and h-(semi)stability. We recall the definition of *pseudostability*, which is obtained from Deligne-Mumford stability by allowing ordinary cusps and prohibiting elliptic tails:

**Definition 2.1.** [Sch91] A complete curve is *pseudostable* if

(1) it is connected, reduced, and has only nodes and ordinary cusps as singularities;
(2) admits no *elliptic tails*, i.e., connected subcurves of arithmetic genus one meeting the rest of the curve in one node;
(3) the canonical sheaf of the curve is ample.

The last condition means that each subcurve of genus zero meets the rest of the curve in at least three points.

Before formulating the notions of c- and h-(semi)stability, we need the following definition:

**Definition 2.2.** An *elliptic bridge* is a connected subcurve of arithmetic genus one meeting the rest of the curve in two nodes.

In our stability analysis, we will require additional technical definitions:

**Definition 2.3.** An *open elliptic chain* of length $r$ is a two-pointed projective curve $(C', p, q)$ such that

- $C' = E_1 \cup_{a_1} \cdots \cup_{a_{r-1}} E_r$ where each $E_i$ is connected of genus one, with nodes, cusps or tacnodes as singularities;
- $E_i$ intersects $E_{i+1}$ at a single tacnode $a_i$, for $i = 1, \dots, r-1$;
- $E_i \cap E_j = \emptyset$ if $|i - j| > 1$;
- $p, q \in C'$ are smooth points with $p \in E_1$ and $q \in E_r$;

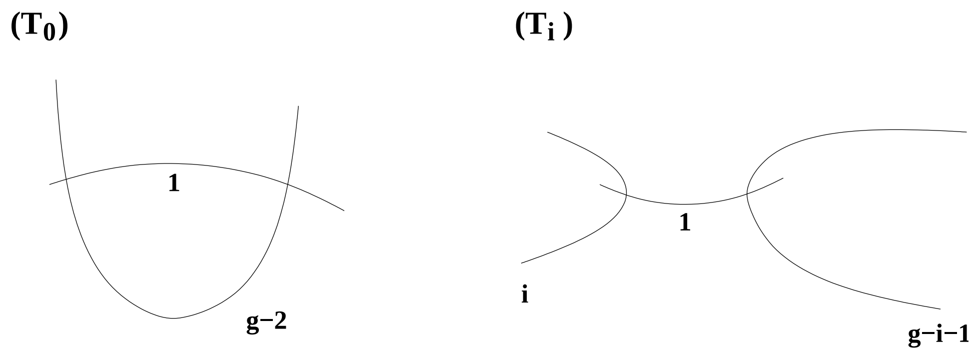


FIGURE 2. Generic elliptic bridges

- $\omega_{C'}(p+q)$ is ample.

An open elliptic chain of length $r$ has arithmetic genus $2r-1$.

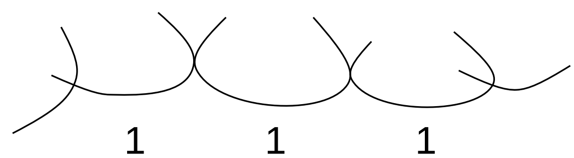


FIGURE 3. Generic elliptic chain of length three

**Definition 2.4.** Let $C$ be a projective connected curve of arithmetic genus $g \geq 3$, with nodes, cusps, and tacnodes as singularities. We say $C$ *admits an open elliptic chain* if there is an open elliptic chain $(C', p, q)$ and a morphism $\iota : C' \to C$ such that

- $\iota$ is an isomorphism over $C' \setminus \{p, q\}$ onto its image;
- $\iota(p), \iota(q)$ are nodes of $C$; we allow the case $\iota(p) = \iota(q)$, in which case $C$ is said to be a *closed elliptic chain*.

$C$ admits a *weak elliptic chain* if there exists $\iota : C' \to C$ as above with the second condition replaced by

- $\iota(p)$ is a tacnode of $C$ and $\iota(q)$ is a node of $C$; or
- $\iota(p) = \iota(q)$ is a tacnode of $C$, in which case $C$ is said to be a *closed weak elliptic chain*.

Now we are in a position to formulate our main stability notions:

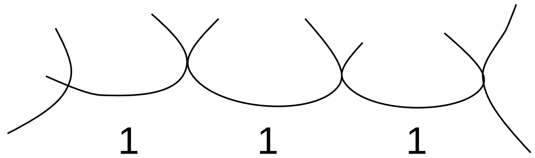


FIGURE 4. Generic weak elliptic chain

**Definition 2.5.** A complete curve $C$ is said to be *c-semistable* if

(1) $C$ has nodes, cusps and tacnodes as singularities;
(2) $\omega_C$ is ample;
(3) a connected genus one subcurve meets the rest of the curve in at least two points (not counting multiplicity).

It is said to be *c-stable* if it is c-semistable and has no tacnodes or elliptic bridges.

**Definition 2.6.** A complete curve $C$ of genus $g$ is said to be *h-semistable* if it is c-semistable and admits no elliptic chains. It is said to be *h-stable* if it is h-semistable and admits no weak elliptic chains.

**Remark 2.7.** A curve is c-stable if and only if it is pseudostable and has no elliptic bridges.

Table 1 summarizes the defining characteristics of the stability notions.

2.2. **Construction of the small contraction $\Psi$.** We start with some preliminary results. Recall from [HHar] that $\mathcal{T} : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ps}$ denotes the functorial contraction and $T : \overline{M}_g \to \overline{M}_g^{ps} = \overline{M}_g(9/11)$ the induced morphism on coarse moduli spaces, which contracts the divisor $\Delta_1$.

**Lemma 2.8.** *For $\alpha < 9/11$, $(\overline{\mathcal{M}}_g^{ps}, \alpha\delta^{ps})$ and $(\overline{M}_g^{ps}, \alpha\Delta^{ps})$ are log terminal and*

$$\overline{M}_g(\alpha) \simeq \mathrm{Proj}\left(\oplus_{n\geq 0}\Gamma(n(K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps}))\right).$$

*Proof.* Since $g > 3$, the locus in $\overline{M}_g^{ps}$ parametrizing curves with nontrivial automorphisms has codimension $\geq 2$ [HM82, §2]. (Of course, we have already collapsed $\delta_1$.) Thus the coarse moduli map $q : \overline{\mathcal{M}}_g^{ps} \to \overline{M}_g^{ps}$ is unramified in codimension one and

$$q^*(K_{\overline{M}_g^{ps}} + \alpha\Delta^{ps}) = K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps} \tag{2.1}$$

TABLE 1. Stability notions

| | singularity | genus zero subcurve meets the rest in ... | genus one subcurve meets the rest in ... | elliptic chain | weak elliptic chain |
|---|---|---|---|---|---|
| stable | nodes | $\geq 3$ points | – | – | – |
| pseudostable | nodes, cusps | $\geq 3$ points | $\geq 2$ points | – | – |
| c-semistable | nodes, cusps, tacnodes | $\geq 3$ points counting multipilicity | $\geq 2$ points | – | – |
| c-stable | nodes, cusps | $\geq 3$ points | $\geq 3$ points | – | – |
| h-semistable | nodes, cusps, tacnodes | $\geq 3$ points counting multipilicity | $\geq 3$ points counting multipilicity | not admitted | – |
| h-stable | nodes cusps, tacnodes | $\geq 3$ points counting multipilicity | $\geq 3$ points counting multipilicity | not admitted | not admitted |

for each $\alpha$. We have the log discrepancy equation [HHar, §4]

$$K_{\overline{\mathcal{M}}_g} + \alpha\delta = \mathcal{T}^*(K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps}) + (9 - 11\alpha)\delta_1 \tag{2.2}$$

and the pull back

$$\mathcal{T}^*(K_{\overline{\mathcal{M}}_g^{ps}} + 7/10\delta^{ps}) = K_{\overline{\mathcal{M}}_g} + 7/10\delta - 13/10\delta_1 \sim 10\lambda - \delta - \delta_1,$$

where $\sim$ designates proportionality.

Since $\overline{\mathcal{M}}_g$ is smooth and $\delta$ is normal crossings, the pair $(\overline{\mathcal{M}}_g, \alpha\delta + (11\alpha - 9)\delta_1)$ is log terminal. The discrepancy equation implies that $(\overline{\mathcal{M}}_g^{ps}, \alpha\delta^{ps})$ is log terminal for $\alpha \in [7/10, 9/11)$. Applying the ramification formula [KM98, 5.20] to (2.1) (or simply applying [HHar, A.13]), we find that $(\overline{M}_g^{ps}, \alpha\Delta^{ps})$ is also log terminal.

Since $\Delta_1$ is $T$-exceptional, for each Cartier divisor $L$ on $\overline{M}_g^{ps}$ and $m \geq 0$ we have $\Gamma(\overline{M}_g, T^*L + m\Delta_1) \simeq \Gamma(\overline{M}_g^{ps}, L)$. This implies that

$$\begin{aligned} \overline{M}_g(\alpha) &= \operatorname{Proj} \oplus_{n\geq 0} \Gamma(\overline{M}_g, n(K_{\overline{\mathcal{M}}_g} + \alpha\delta)) \\ &= \operatorname{Proj} \oplus_{n\geq 0} \Gamma\left(\overline{M}_g, n(T^*(K_{\overline{M}_g^{ps}} + \alpha\delta^{ps}) + (9 - 11\alpha)\delta_1)\right) \\ &\simeq \operatorname{Proj} \oplus_{n\geq 0} \Gamma\left(\overline{M}_g^{ps}, n(K_{\overline{M}_g^{ps}} + \alpha\delta^{ps})\right). \end{aligned}$$

□

We shall construct the contractions by using the powerful results of [GKM02]:

**Proposition 2.9.** *For* $\alpha \in (7/10, 9/11] \cap \mathbb{Q}$*, there there exists a birational contraction*

$$\Psi : \overline{M}_g(\alpha) \to \overline{M}_g(7/10).$$

*It contracts the codimension-two strata* $T_i, i = 0, 2, \ldots, \lfloor (g-1)/2 \rfloor$*, where*

(1) $T_0 = \{E \cup_{p,q} D \mid g(E) = 1, g(D) = g - 2\}$*;*

(2) $T_i = \{C_1 \cup_p E \cup_q C_2 \mid g(C_1) = i, g(E) = 1, g(C_2) = g - 1 - i\}, \quad 2 \leq i \leq \lfloor (g-1)/2 \rfloor$*,*

*by collapsing the loci* $\overline{M}_{1,2} \subset T_i$ *corresponding to varying* $(E, p, q)$*.*

**Remark 2.10.** We shall see in Corollary 2.15 that $\Psi$ is an isomorphism away from $T_\bullet := \cup T_i$.

*Proof.* Recall that $K_{\overline{M}_g^{ps}} + \alpha\Delta^{ps}$ is ample provided $7/10 < \alpha \leq 9/11$; this is part of the assertion that $\overline{M}_g(\alpha) = \overline{M}_g^{ps}$ for $7/10 < \alpha \leq 9/11$ [HHar, Theorem 1.2]. However, $K_{\overline{\mathcal{M}}_g^{ps}} + 7/10\Delta^{ps}$ is nef but not ample [HHar, §4]. Indeed, the pull-back to $\overline{M}_g$

$$10\lambda - \delta - \delta_1$$

can be analyzed using the classification of one-dimensional boundary strata by Faber [Fab96] and Gibney-Keel-Morrison [GKM02]. It is 'F-nef', in the sense that it intersects all these strata nonnegatively, and is therefore nef by [GKM02, 6.1]. Later on, we will list the strata meeting it with degree zero.

We apply Kawamata basepoint freeness [KM98, 3.3]:

> Let $(X, D)$ be a proper Kawamata log terminal pair with $D$ effective. Let $M$ be a nef Cartier divisor such that $aM - K_X - D$ is nef and big for some $a > 0$. Then $|bM|$ has no basepoint for all $b \gg 0$.

For our application, $M$ is a Cartier multiple of $K_{\overline{M}_g^{ps}} + 7/10\Delta^{ps}$ and $D = (7/10 - \epsilon)\Delta^{ps}$ for small positive $\epsilon \in \mathbb{Q}$. The resulting morphism is denoted $\Psi$.

We claim that $\Psi$ is birational. To establish the birationality, we show that each curve $B \subset \overline{M}_g$ meeting the interior satisfies

$$B.(10\lambda - \delta - \delta_1) > 0.$$

The Moriwaki divisor

$$A := (8g+4)\lambda - g\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} 4i(g-i)\delta_i$$

meets each such curve nonnegatively [Mor98, Theorem B]. We can write

$$10\lambda - \delta - \delta_1 = (1/g)\,A + (2-4/g)\lambda + (2-4/g)\delta_1 + \sum_{i=2}^{\lfloor g/2 \rfloor} (-1+4i(g-i)/g)\delta_i.$$

Each of these coefficients is positive: Clearly $1/g, 2-4/g > 0$ and since $2i/g \leq 1$,

$$-1+4i(g-i)/g = -1+4i-(2i/g)2i \geq -1+4i-2i > 0.$$

Thus we have

$$B.(10\lambda - \delta - \delta_1) \geq (2-4/g)\lambda.B > 0,$$

where the last inequality reflects the fact that the Torelli morphism is nonconstant along $B$.

We verify the image of $\Psi$ equals $\overline{M}_g(7/10)$. The log discrepancy formula (2.2) implies

$$\text{Image}(\Psi) = \textbf{Proj} \oplus_{n\geq 0} \Gamma(n(K_{\overline{\mathcal{M}}_g} + 7/10\delta - 13/10\delta_1)).$$

However, since $\Delta_1$ is $(\Psi \circ T)$-exceptional adding it does not change the space of global sections, whence

$$\text{Image}(\Psi) = \textbf{Proj} \oplus_{n\geq 0} \Gamma(n(K_{\overline{\mathcal{M}}_g} + 7/10\delta)) = \overline{M}_g(7/10).$$

Finally, we offer a preliminary analysis of the locus contracted by $\Psi$. The main ingredient is the enumeration of one-dimensional boundary strata in [GKM02] (see also [HHar, §4]). We list the ones orthogonal to $10\lambda - \delta - \delta_1$; any stratum swept out by these classes is necessarily contracted by $\Psi$. In the second and third cases $X_0$ denotes a varying 4-pointed curve of genus zero parametrizing the stratum.

(1) Families of elliptic tails, which sweep out $\delta_1$ and correspond to the extremal ray contracted by $\mathcal{T}$.
(2) Attach a 2-pointed curve of genus $0$ and a 2-pointed curve $(D, p, q)$ of genus $g-2$ to $X_0$ and stabilize. Contracting this and the elliptic tail stratum collapses $T_0$ along the $\overline{M}_{1,2}$'s corresponding to fixing $(D, p, q)$ and varying the other components.

(3) Attach a 1-pointed curve $(C_1, p)$ of genus $i > 1$, a 1-pointed curve $(C_2, q)$ of genus $g - 1 - i > 1$, and a 2-pointed curve of genus $0$ to $X_0$ and stabilize. Contracting this and the elliptic tail stratum collapses $T_i$ along the $\overline{M}_{1,2}$'s corresponding to fixing $(C_1, p), (C_2, q)$ and varying the other components.

Thus the codimension-two strata $T_0, T_2, \ldots, T_{\lfloor (g-1)/2 \rfloor}$ are all contracted by $\Psi$. □

2.3. **Construction of the flip $\Psi^+$.** Consider the Chow variety of degree $4g-4$ curves of genus $g$ in $\mathbb{P}^{3g-4}$. Let $\mathsf{Chow}_{g,2}$ denote the closure of the bicanonically embedded smooth curves of genus $g$. Similarly, let $\mathsf{Hilb}_{g,2}$ denote the closure of these curves in the Hilbert scheme.

**Proposition 2.11.** *The cycle class map*

$$\varpi : \mathsf{Hilb}_{g,2} \to \mathsf{Chow}_{g,2} \tag{2.3}$$

*induces a morphism of GIT quotients*

$$\mathsf{Hilb}^{ss}_{g,2} /\!\!/ \mathrm{SL}_{3g-3} \to \mathsf{Chow}^{ss}_{g,2} /\!\!/ \mathrm{SL}_{3g-3},$$

*where the Hilbert scheme has the asymptotic linearization introduced in §3.6.*

This is a special case of Corollary 3.14, which applies quite generally to cycle-class maps from Hilbert schemes for Chow varieties. (See §3.6 for background information on the cycle class map.) Let $\overline{M}_g^{hs}$ and $\overline{M}_g^{cs}$ denote the resulting GIT quotients $\mathsf{Hilb}^{ss}_{g,2} /\!\!/ \mathrm{SL}_{3g-3}$ and $\mathsf{Chow}^{ss}_{g,2} /\!\!/ \mathrm{SL}_{3g-3}$, and

$$\Psi^+ : \overline{M}_g^{hs} \to \overline{M}_g^{cs} \tag{2.4}$$

the morphism of Proposition 2.11.

**Theorem 2.12.** *Let $\epsilon \in \mathbb{Q}$ be a small positive number. There exist isomorphisms*

$$\overline{M}_g(7/10) \simeq \overline{M}_g^{cs} \tag{2.5}$$

*and*

$$\overline{M}_g(7/10 - \epsilon) \simeq \overline{M}_g^{hs} \tag{2.6}$$

*such that the induced morphism*

$$\Psi^+ : \overline{M}_g(7/10 - \epsilon) \to \overline{M}_g(7/10)$$

*is the flip of $\Psi$.*

We thus obtain a modular/GIT interpretation of the flip:

$$\begin{array}{ccc} \overline{M}_g(\frac{7}{10}+\epsilon) \simeq \overline{M}_g^{ps} & & \overline{M}_g(\frac{7}{10}-\epsilon) \simeq \overline{M}_g^{hs} \\ & \searrow^{\Psi} \quad {}^{\Psi^+}\swarrow & \\ & \overline{M}_g(\frac{7}{10}) \simeq \overline{M}_g^{cs} & \end{array}$$

2.4. **Stability results on bicanonical curves.** For c-semistable curves, $\omega_C^{\otimes 2}$ is very ample and has no higher cohomology (Proposition 6.1). The image in $\mathbb{P}^{3g-4}$ is said to be *bicanonically embedded.*

**Theorem 2.13.** *The semistable locus* $\mathsf{Chow}^{ss}_{g,2}$ *(resp. stable locus* $\mathsf{Chow}^{s}_{g,2}$*) corresponds to bicanonically embedded c-semistable (resp. c-stable) curves.*

Unlike in $\overline{M}_g$ and $\overline{M}_g^{ps}$, nonisomorphic curves may be identified in the quotient $\mathsf{Chow}_{g,2}/\!\!/\mathrm{SL}_{3g-3}$. For example, if a c-semistable curve $C = D \cup_{p,q} E$ consists of a genus $g-2$ curve $D$ meeting in two nodes $p, q$ with an elliptic curve $E$, then it is identified with any tacnodal curve obtained by replacing $E$ with a tacnode. In Section 10, we shall give a complete classification of strictly semistable curves and the curves in their orbit closure.

**Theorem 2.14.** *The semistable locus* $\mathsf{Hilb}^{ss}_{g,2}$ *(resp. stable locus* $\mathsf{Hilb}^{s}_{g,2}$*) with respect to the asymptotic linearization corresponds to bicanonically embedded h-semistable (resp. h-stable) curves.*

One difference from the case of Chow points is that tacnodal curves may well be Hilbert stable. For instance, when $g \geq 4$ *irreducible* bicanonical h-semistable curves are necessarily h-stable. When $g = 3$, a bicanonical h-semistable curve is Hilbert strictly semistable if and only if it has a tacnode [HL07a]. When $g = 4$, every h-semistable curve is h-stable and the moduli functor is thus separated.

Since c-stable curves are h-stable (see Proposition 3.13) and pseudostable (see Remark 2.7), we have

**Corollary 2.15.** $\Psi$ *and* $\Psi^+$ *are isomorphisms over the locus of c-stable curves. Thus* $\Psi$ *is a small contraction with exceptional locus* $T_\bullet$ *and* $\Psi^+$ *is a small contraction with exceptional locus* Tac*, the h-semistable curves with tacnodes.*

Thus the geometry of the flip is as indicated in Figure 1: $\Psi^+(C') = \Psi(C)$ precisely when $C$ is the 'pseudostable reduction' of $C'$.

2.5. **Detailed roadmap for the GIT analysis.** The proof of Theorems 2.13 and 2.14 is rather intricate, so we give a bird's eye view for the reader's convenience.

(1) The following implications are straightforward:

- From the definitions, it is clear that:

$$\text{h-semistable} \Rightarrow \text{c-semistable}$$

- General results on linearizations of Chow and Hilbert schemes (Proposition 3.13) imply

$$\text{Hilbert semistable} \Rightarrow \text{Chow semistable}$$

and

$$\text{Chow stable} \Rightarrow \text{Hilbert stable.}$$

(2) We next prove that non c-semistable (resp. non h-semistable) curves are Chow unstable (resp. Hilbert unstable). The main tool is the stability algorithm (Proposition 3.7).

- Non c-semistable curves can be easily destabilized by one-parameter subgroups (§7). We obtain

$$\text{Chow semistable} \Rightarrow \text{c-semistable.}$$

- We show that if a curve $C$ admits an *open rosary* of even length (see Definition 8.1), then there is a 1-PS $\rho$ coming from the automorphism group of the rosary such that the $m$*th Hilbert point* $[C]_m$ (Definition 3.5) is unstable with respect to $\rho$ for all $m \geq 2$ (Proposition 10.1 and Proposition 10.7).
- If $C$ admits an elliptic chain, then it is contained in the *basin of attraction* $A_\rho([C_0]_m)$ (see Definition 4.1) of a curve $C_0$ admitting an open rosary of even length such that $\mu([C_0]_m, \rho) < 0$. Hence such curves are Hilbert unstable (Propositions 10.3 and 10.8) and we obtain

$$\text{Hilbert semistable} \Rightarrow \text{h-semistable.}$$

(3) We prove "c-semistable $\Rightarrow$ Chow semistability", and use it to establish "h-semistability $\Rightarrow$ Hilbert semistability".

- The only possible Chow semistable replacement of a c-stable curve is itself (see Theorem 9.1). Thus c-stable curves are Chow stable and hence Hilbert stable.
- We show that any strictly c-semistable curve $C$ is contained in a basin of attraction of a distinguished c-semistable curve $C^\star$ with one-parameter isomorphism such that $\mu(Ch(C^\star), \rho) = 0$ (see Proposition 11.5). Indeed, we choose $C^\star$ so that it has closed orbit in the locus of c-semistable points (cf. Proposition 11.6).
- If $C$ is strictly c-semistable, its pseudo-stabilization $D$ has elliptic bridges. For any such $D$, there is a distinguished strictly c-semistable curve $C^\star$ such that its basins of attraction contain every c-semistable replacement for $D$. Futhermore, *every* possible Chow-semistable replacement for $D$ is contained in some basin of attraction $A_{\rho'}(Ch(C^\star))$ with $\mu(Ch(C^\star), \rho') = 0$. Since one of these must be Chow semistable, every one is Chow semistable (see Lemma 4.3).
- The Hilbert semistable curves form a subset of the set of Chow semistable curves. We first identify the Chow semistable curves admitting one-parameter subgroups that are Hilbert-destabilizing. Then we show that any curve that is Hilbert unstable but Chow semistable arises in the basin of attraction of such a curve. These basins of attraction consist of the curves that are c-semistable but not h-semistable. Thus the h-semistable curves are Hilbert semistable (§11.3).

## 3. GIT of Chow varieties and Hilbert schemes

Let $\mathbb{P}^N = \mathbb{P}(V)$ for some $(N+1)$-dimensional vector space $V$. Throughout this section, let $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ be a one-parameter subgroup and $x_0, \dots, x_N$ be homogeneous coordinates that diagonalize the $\rho$-action so that

$$\rho(t).x_i = t^{r_i} x_i,\ i = 0, \dots, N,\ r_0 \geq \cdots \geq r_N = 0.$$

We have the associated one-parameter subgroup $\rho^\circ : \mathbb{G}_m \to \mathrm{SL}(V)$

$$\rho^\circ(t).x_i = t^{r_i - (r_0 + \ldots + r_N)/(N+1)} x_i.$$

Given $x \in \mathbb{P}(V)$, the *Hilbert-Mumford index* is given by (cf. [MFK94, 2.1]):

$$\mu(x, \rho) = \max\{-r_i + (r_0 + \ldots + r_N)/(N+1) : x_i \neq 0\}.$$

We say that $x$ is (semi)stable with respect to $\rho^\circ$ if $\mu(x, \rho) > (\geq)\, 0$. A fundamental theorem of GIT is that $x$ is GIT (semi)stable if and only if it is (semi)stable with respect to every 1-PS of SL$(V)$. We will sometimes abuse terminology and say that $x$ is (semi)stable with respect to $\rho$ when it is (semi)stable with respect to $\rho^\circ$.

3.1. **GIT of Chow points.** We briefly recall Mumford's interpretation of Hilbert-Mumford criterion for Chow stability of projective varieties [Mum77]. Let $X \subset \mathbb{P}(V)$ be a projective variety and $\rho$ be a one-parameter subgroup of GL$(V)$ with weights $r_0 \geq r_1 \geq \cdots \geq r_N = 0$. Let $\mathcal{I}_\rho$ be the ideal sheaf of $\mathcal{O}_X[t]$ such that

$$\mathcal{I}_\rho \cdot \mathcal{O}_X(1)[t] = \left( \begin{array}{c} \text{the } \mathcal{O}_X\text{-submodule of } \mathcal{O}_X(1)[t] \text{ generated by} \\ t^{r_i} x_i, \quad i = 0, \dots, N. \end{array} \right)$$

**Definition 3.1.** The *Hilbert-Samuel multiplicity* $e_\rho(X)$ is the normalized leading coefficient of $P(n) := \chi(\mathcal{L}^n / \mathcal{I}_\rho^n \mathcal{L}^n)$ where $\mathcal{L}$ is the invertible $\mathcal{O}_X[t]$-module $\mathcal{O}_X(1)[t]$.

Then the Hilbert-Mumford criterion can be translated in terms of $e_\rho(X)$ as follows:

**Theorem 3.2.** [Mum77, Theorem 2.9] *The Chow point of $X$ is stable (resp. semistable) if and only if*

$$e_\rho(X) < (\text{resp.} \leq) \frac{\dim(X) + 1}{N + 1} \deg(X) \sum r_i$$

*for any one-parameter subgroup $\rho : \mathbb{G}_m \to$ GL$(V)$ with weights $r_0 \geq r_1 \geq \cdots \geq r_N = 0$.*

We shall make frequent use of the following lemma which describes how the Hilbert-Samuel multiplicity is affected by the singular points:

**Lemma 3.3.** [Sch91, Lemma 1.4] *Let $X$ be a reduced curve in $\mathbb{P}(V)$ and $\nu : \widetilde{X} \to X$ be its normalization.*

(1) $e_\rho(X) = \sum_{P \in \tilde{X}} e_\rho(\widetilde{X})_P$, *where $e_\rho(\widetilde{X})_P$ denotes the normalized leading coefficient of* $\dim_k \mathcal{O}_{\tilde{X} \times \mathbb{A}^1} / \mathcal{I}^m_{P \times \{0\}}$.
(2) *Suppose that $\nu(\nu^* x_i) + r_i \geq a$ for all $i$ where $\nu$ is the natural valuation of $\mathcal{O}_{P, \tilde{X}}$. Then $e_\rho(\widetilde{X}) \geq a^2$.*

We shall also use the following lemma that addresses the case in which $X$ is degenerate and $\rho$ acts on $X$ trivially.

**Lemma 3.4.** [Sch91, Lemma 1.2] *Let $X$ be an $r$-dimensional variety in $\mathbb{P}^N$. Let $\rho$ be a 1-PS of* $\mathrm{GL}_{N+1}(k)$ *such that $\rho(t) \cdot x_i = t^{r_i} x_i$, $r_0 \geq \cdots \geq r_N = 0$. Suppose that $x_j, x_{j+1}, \ldots, x_N$ vanish on $X$ and $r_0 = r_1 = \cdots = r_{j-1}$. Then*

$$e_\rho(X) = (r+1) r_0 \deg(X).$$

3.2. **GIT of Hilbert points.** Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a projective variety with Hilbert polynomial $P(t)$. Choose an integer $m$ sufficiently large so that

- $\mathcal{O}_X(m)$ has no higher cohomology;
- the natural map
$$\mathrm{Sym}^m V^* \to \Gamma(\mathcal{O}_X(m))$$
is surjective.

**Definition 3.5.** The $m$*th Hilbert point* $[X]_m$ *of* $X$ is defined

$$[X]_m := [\mathrm{Sym}^m V^* \to \Gamma(\mathcal{O}_X(m))] \in \mathrm{Gr}(P(m), \mathrm{Sym}^m V) \hookrightarrow \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V).$$

Note that $X$ is determined by $[X]_m$ provided $X$ is cut out by forms of degree $m$.

**Definition 3.6.** $X$ is said to be $m$-*Hilbert stable (resp. semistable)* if $[X]_m$ is GIT stable (semistable) with respect to the natural $\mathrm{SL}(V)$ action on $\mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V)$.

We refer the reader to [HHL07] for detailed discussion of an algorithm (and a Macaulay 2 implementation) using Gröbner basis to determine whether a variety is $m$-Hilbert (semi)stable with respect to a given one-parameter subgroup. We sketch the main results here.

For any given $\nu \in \mathbb{R}^{N+1}$, $\prec_\nu$ denotes the monomial order defined by declaring $x^a \prec_\nu x^b$ if

(1) $\deg x^a < \deg x^b$;
(2) $\deg x^a = \deg x^b$ and $\nu.a < \nu.b$;
(3) $\deg x^a = \deg x^b$, $\nu.a = \nu.b$ and $x^a \prec_{\mathrm{Lex}} x^b$ in the lexicographic order.

In particular, given a one-parameter subgroup $\rho$ with the weight vector $w = (r_0, \ldots, r_N)$, the monomial order $\prec_\rho$ means the graded lexicographic order associated to the weight $w$. Given a monomial $x^a = x_0^{a_0} \cdots x_N^{a_N}$, the $\rho$-weight is defined

$$wt_\rho(x^a) := w.a = r_0a_0 + \ldots + r_Na_N.$$

For each polynomial $f$, let $\mathrm{in}_{\prec_\rho}(f)$ denote the largest term of $f$ with respect to $\prec_\rho$. For an ideal $I \subset \mathrm{Sym}V^*$, we let $\mathrm{in}_{\prec_\rho}(I) := \langle \mathrm{in}_{\prec_\rho}(f) \,|\, f \in I\rangle$. Let $I \subset \mathrm{Sym}V^*$ be a homogeneous ideal with graded pieces $I_m = I \cap \mathrm{Sym}^m V^*$. The monomials $\{x^{a(1)}, \ldots, x^{a(P(m))}\}$ of degree $m$ not contained in $\mathrm{in}_{\prec_\rho}(I)$ form a basis for $\mathrm{Sym}^m V^*/I_m$.

We reformulate Gieseker's stability criterion for Hilbert points [Gie82, pp. 8] in these terms:

**Proposition 3.7.** *The Hilbert-Mumford index of* $[X]_m$ *with respect to a one-parameter subgroup* $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ *with weights* $r_0, r_1, \ldots, r_N$ *is given by*

$$\mu([X]_m, \rho) = \frac{mP(m)}{N+1}\sum r_i - \sum_{j=1}^{P(m)} wt_\rho(x^{a(j)}) \tag{3.1}$$

*where* $a(1), \ldots, a(P(m))$ *index the monomials of degree* $m$ *not contained in* $\mathrm{in}_{\prec_\rho}(I)$*. In particular,* $[X]_m \in \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V)$ *is stable (resp. semistable) under the natural* $\mathrm{SL}(V)$*-action if and only if for any one-parameter subgroup* $\rho$ *we have*

$$\sum_{j=1}^{P(m)} wt_\rho(x^{a(j)}) < (\textit{resp.} \leq) \frac{mP(m)}{N+1}\sum r_i.$$

3.3. **Polarizations on Hilbert schemes.** Let $\mathrm{Hilb}$ be the connected component of the Hilbert scheme containing $X$, $\mathcal{X} \subset \mathbb{P}(V) \times \mathrm{Hilb}$ the universal family, $\pi : \mathcal{X} \to \mathrm{Hilb}$ the natural projection, and $\mathcal{O}_{\mathcal{X}}(1)$ the polarization.

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}(V)$ is said to be $M$-*regular* in the sense of Castelnuovo and Mumford [Mum66, ch. 14] if $H^i(\mathcal{F}(M-i)) = 0$ for each $i > 0$. Suppose that the ideal sheaf $\mathcal{I}_X$ is $M$-regular. It follows that for each $m \geq M$

- $\Gamma(\mathcal{I}_X(m)) \otimes V^* \to \Gamma(\mathcal{I}_X(m+1))$ is surjective;
- $H^i(\mathcal{I}_X(m-i)) = 0$ for each $i > 0$;

and also

- $\mathrm{Sym}^{m-1}V^* \to \Gamma(\mathcal{O}_X(m-1))$ is surjective;
- $H^i(\mathcal{O}_X(m-1-i)) = 0$ for $i > 0$.

In particular, $\mathcal{O}_X$ is $(M-1)$-regular. Conversely, if $\mathcal{O}_X$ is $(M-1)$-regular and $M \geq 0$ then $\mathcal{I}_X$ is $M$-regular. [Eis05, pp. 68]

There exists an $M \gg 0$ such that *every* $[X] \in \mathrm{Hilb}$ has $M$-regular ideal sheaf [Mum66, ch.14]. Then for each $m \geq M$ we get a closed embedding [Mum66, ch. 15]

$$\begin{array}{ccc} \mathrm{Hilb} & \subset & \mathrm{Gr}(P(m), \mathrm{Sym}^m V) \subset \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V) \\ [X] & \mapsto & [X]_m \end{array}$$

The universal quotient bundle $Q \to \mathrm{Gr}(P(m), \mathrm{Sym}^m V)$ satisfies

$$Q|\mathrm{Hilb} = \pi_* \mathcal{O}_{\mathcal{X}}(m),$$

and on taking determinants we find

$$\Lambda_m := \mathcal{O}_{\mathrm{Gr}}(1)|\mathrm{Hilb} = \det(\pi_* \mathcal{O}_{\mathcal{X}}(m)).$$

3.4. **Tautological classes on the Hilbert scheme.** Recall the *tautological divisor classes* developed in [Fog69] and [KM76, Theorem 4]: There exist Cartier divisors $L_0, \dots, L_{r+1}$ on $\mathrm{Hilb}$ such that

$$\det(\mathbb{R}^\bullet \pi_* \mathcal{O}_{\mathcal{X}}(m)) = \sum_{i=0}^{r+1} \binom{m}{i} L_i, \tag{3.2}$$

where $r$ is the dimension of subschemes parametrized by $\mathrm{Hilb}$. That is, the determinant of cohomology of $\mathcal{O}_{\mathcal{X}}(m)$ can be expressed as a polynomial in the tautological class. This is a relative version of the Hilbert polynomial of $\mathcal{X}$ over $\mathrm{Hilb}$. It follows that the polarizations introduced above satisfy:

$$\Lambda_m = \det(\pi_* \mathcal{O}_{\mathcal{X}}(m)) = \sum_{i=0}^{r+1} \binom{m}{i} L_i. \tag{3.3}$$

Using these formulas, we extend our definition:

**Definition 3.8.** For each $m \in \mathbb{Z}$, write

$$\Lambda_m = \det(\mathbb{R}^\bullet \pi_* \mathcal{O}_{\mathcal{X}}(m)) = \sum_{i=0}^{r+1} \binom{m}{i} L_i.$$

In many situations the tautological divisors satisfy a dependence relation:

**Proposition 3.9.** *Let* $\mathrm{Hilb}$ *denote a connected component of the Hilbert scheme parametrizing subschemes in* $\mathbb{P}(V)$ *of dimension* $r$ *and* $L_0, \dots, L_{r+1}$ *the tautological divisors on* $\mathrm{Hilb}$. *Let* $\mathrm{Hilb}^{\bullet,1} \subset \mathrm{Hilb}$ *denote an open subset corresponding to subschemes* $X$ *where the following hold:*

- $\mathcal{O}_X(1)$ *has no higher cohomology;*

- *the restriction map* $V^* \to \Gamma(\mathcal{O}_X(1))$ *is an isomorphism.*

*Over* $\mathsf{Hilb}^{\bullet,1}$ *we have the relation* $L_0 + L_1 = 0$.

In particular, if $r = 1$ then Equation 3.3 takes the form

$$\Lambda_m = L_0 + mL_1 + \frac{m(m-1)}{2}L_2 = (m-1)(L_1 + \frac{m}{2}L_2). \tag{3.4}$$

*Proof.* Let $\pi : \mathcal{X} \to \mathsf{Hilb}^{\bullet,1}$ be the universal family embedded in $\mathbb{P}(V) \times \mathsf{Hilb}^{\bullet,1}$. Our first assumption implies $\pi_*\mathcal{O}_{\mathcal{X}}(1)$ is locally free and

$$\Lambda_1 = \det(\pi_*\mathcal{O}_{\mathcal{X}}(1)) = L_0 + L_1.$$

The second assumption implies we have a trivialization (cf. [Vie95, pp.44])

$$\Gamma(\mathcal{O}_{\mathbb{P}(V)}(1)) \otimes \mathcal{O}_{\mathsf{Hilb}^{\bullet,1}} \simeq \pi_*\mathcal{O}_{\mathcal{X}}(1).$$

In particular, it follows that $L_0 + L_1 = 0$. □

3.5. **Hilbert points and Hilbert schemes.** We have seen that $\mathsf{Hilb}$ admits an embedding into $\mathrm{Gr}(P(m), \mathrm{Sym}^m V)$ for $m \gg 0$. In practice, we are usually interested in subsets of $\mathsf{Hilb}$, that exclude degenerate subschemes with very high Castelnuovo-Mumford regularity:

**Proposition 3.10.** *Let* $\mathsf{Hilb}^{\circ,m} \subset \mathsf{Hilb}$ *denote the open subset parametrizing* $[X] \in \mathsf{Hilb}$ *satisfying:*

- $\mathcal{O}_X(m)$ *has no higher cohomology;*
- $\mathrm{Sym}^m V^* \to \Gamma(\mathcal{O}_X(m))$ *is surjective.*

*Let* $\pi : \mathcal{X} \to \mathsf{Hilb}^{\circ,m}$ *denote the universal family restricted to this subset. Then we have*

$$\det(\pi_*\mathcal{O}_{\mathcal{X}}(m)) = \Lambda_m|\mathsf{Hilb}^{\circ,m} = \sum_{i=0}^{r+1} \binom{m}{i} L_i|\mathsf{Hilb}^{\circ,m}$$

*and there exists a morphism*

$$\begin{array}{rcl} \phi_m : \mathsf{Hilb}^{\circ,m} & \to & \mathrm{Gr}(P(m), \mathrm{Sym}^m V) \subset \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V) \\ {[X]} & \mapsto & [X]_m \end{array}$$

*such that* $\phi_m^*\mathcal{O}(1) = \Lambda_m$.

**Remark 3.11.** $\mathsf{Hilb}^{\circ,m}$ contains the open subset parametrizing subschemes $X$ with $m$-regular ideal sheaf $\mathcal{I}_X$ and $(m-1)$-regular structure sheaf $\mathcal{O}_X$. In particular, $\mathsf{Hilb}^{\circ,m} = \mathsf{Hilb}$ for $m \gg 0$.

*Proof.* The first assertion is just an application of Equation 3.2 in the special situation when $\mathbb{R}^i\pi_*\mathcal{O}_{\mathcal{X}}(m) = 0$ for $i > 0$. The morphism $\phi_m$ is just the classifying map for the surjection of locally free sheaves

$$\mathrm{Sym}^m V^* \otimes \mathcal{O}_{\mathrm{Hilb}^{\circ,m}} \twoheadrightarrow \pi_*\mathcal{O}_{\mathcal{X}}(m).$$

Again, if $Q \to \mathrm{Gr}(P(m), \mathrm{Sym}^m V)$ is the universal quotient bundle then $\phi_m^* Q = \pi_*\mathcal{O}_{\mathcal{X}}(m)$, and taking determinants gives the equality of line bundles. □

Applying the functorial properties of the Hilbert-Mumford index [MFK94, 2.1] we obtain:

**Corollary 3.12.** *Retain the notation of Proposition 3.10. Suppose that* $[X] \in \mathrm{Hilb}^{\circ,m}$ *and* $\rho : \mathbb{G}_m \to \mathrm{GL}(V)$ *is a one-parameter subgroup as before such that*

$$\lim_{t\to 0} \rho(t).[X] \in \mathrm{Hilb}^{\circ,m}.$$

*Then we have*

$$\mu([X]_m, \rho) = \mu^{\Lambda_m}([X], \rho). \tag{3.5}$$

3.6. **Chow stability and Hilbert stability.** We compare the geometric invariant theory of the Hilbert points $[X]_m, m \gg 0$ with that of the Chow point $\mathrm{Ch}(X)$.

Let $\mathrm{Chow} \subset \mathbb{P}(\otimes^{r+1}\mathrm{Sym}^d V)$ denote the corresponding *Chow variety*, i.e., the image of the Hilbert scheme under the morphism [MFK94, §5.4]

$$\begin{array}{rcl} \varpi : \mathrm{Hilb} & \to & \mathrm{Chow} \subset \mathbb{P}(\otimes^{r+1}\mathrm{Sym}^d V) \\ [X] & \mapsto & \mathrm{Ch}(X) \end{array}.$$

This is equivariant under the natural actions of $\mathrm{SL}(V)$. By [KM76, Theorem 4], we obtain the proportionality

$$\varpi^*\mathcal{O}_{\mathrm{Chow}}(1) \sim L_{r+1} \tag{3.6}$$

and thus

$$\lim_{m\to\infty} \frac{\Lambda_m}{\binom{m}{r+1}} \sim \varpi^*\mathcal{O}_{\mathrm{Chow}}(1). \tag{3.7}$$

In other words, the sequence $\{[\Lambda_m]\}$ converges to the pull back of the Chow polarization in the projectivized Néron-Severi group of the Hilbert scheme.

Let

$$\mathrm{Chow}^s \subset \mathrm{Chow}^{ss} \subset \mathrm{Chow}$$

denote the locus of points stable and semistable under the $\mathrm{SL}(V)$-action. For each $m \gg 0$, let

$$\mathrm{Hilb}^{s,m} \subset \mathrm{Hilb}^{ss,m} \subset \mathrm{Hilb}$$

the locus of points stable and semistable under the $\mathrm{SL}(V)$-action linearized by $\Lambda_m$. The ample cone of $\mathrm{Hilb}$ admits a finite decomposition into locally-closed cells, such that the semistable locus is constant for linearizations taken from a given cell [DH98, Theorem 0.2.3(i)]. In particular, $\mathrm{Hilb}^{s,m}$ and $\mathrm{Hilb}^{ss,m}$ are constant for $m \gg 0$; these are loci of the points *stable and semistable with respect to the asymptotic linearization*. While the linearization is not well-defined, its locus of semistable points is!

Moreover, applying functoriality of stability [Rei89, Theorem 2.1], we find

**Proposition 3.13.** *Let $X \subset \mathbb{P}^N$ be a variety. If $X$ is Chow stable then $X$ is $m$-Hilbert stable for $m \gg 0$. If $X$ is $m$-Hilbert semistable for $m \gg 0$ then $X$ is Chow semistable.*

**Corollary 3.14.** *Assume the cycle map induces an $\mathrm{SL}(V)$-equivariant map*

$$\varpi : \mathrm{Hilb}^{ss,m} \to \mathrm{Chow}^{ss},$$

*which is the case for $m \gg 0$. Then we obtain a natural morphism of GIT quotients*

$$\varpi : \mathrm{Hilb}^{ss,m}/\!\!/\mathrm{SL}(V) \to \mathrm{Chow}^{ss}/\!\!/\mathrm{SL}(V).$$

3.7. **Filtered Hilbert polynomials.**

**Definition 3.15.** Given a graded ideal $I \subset \mathrm{Sym} V^*$, the *filtered Hilbert function* is defined

$$H_{\mathrm{Sym} V^*/I,\rho}(m) = \sum wt_\rho(x^a)$$

where the sum is taken over the monomials of degree $m$ not contained in $in_{\prec_\rho} I$. For a closed subscheme $X \subset \mathbb{P}(V)$, we define

$$H_{X,\rho} = H_{\mathrm{Sym} V^*/I_X,\rho}.$$

**Proposition 3.16.** *The filtered Hilbert function $H_{X,\rho}(m)$ is a polynomial $P_{X,\rho}(m)$ for $m \geq M$, the Castelnuovo-Mumford regularity of $\mathcal{O}_X$. This polynomial is called the* filtered Hilbert polynomial*.*

*Proof.* Since $m \geq M$ we have an embedding

$$\mathrm{Hilb} \subset \mathrm{Gr}(P(m), \mathrm{Sym}^m V) \hookrightarrow \mathbb{P}(\bigwedge^{P(m)} \mathrm{Sym}^m V)$$

and Proposition 3.10 implies

$$\mu([X]_m, \rho) = \mu^{\Lambda_m}([X], \rho).$$

Equation 3.3 gives

$$\Lambda_m = \sum_{i=0}^{r+1} \binom{m}{i} L_i$$

Fixing the point and the one-parameter subgroup, $\mu$ is a homomorphism in the line bundle variable [MFK94, 2.2] and we have

$$\mu^{\Lambda_m}([X], \rho) = \sum_{i=0}^{r} \binom{m}{i} \mu^{L_i}([X], \rho). \tag{3.8}$$

The result follows from Equation 3.1. □

3.8. **Hilbert schemes of curves.** In this section, we assume $\mathrm{Hilb}$ parametrizes schemes of pure dimension one. Here Equation 3.2 takes the form

$$\Lambda_m = L_0 + mL_1 + \binom{m}{2} L_2.$$

**Proposition 3.17.** *Let* $\mathrm{Hilb}^\bullet \subset \mathrm{Hilb}$ *denote the open subset parametrizing* $[X] \in \mathrm{Hilb}$ *satisfying:*

- $X$ *is connected of pure dimension one;*
- $V^* \to \Gamma(\mathcal{O}_X(1))$ *is an isomorphism;*
- $\mathcal{O}_X$ *is 2-regular.*

*Then for each* $m \geq 2$ *we have*

(3.9)

$$\mu([X]_m, \rho) = \mu^{\Lambda_m}([X], \rho) = (m-1)\left[(3-m)\mu^{\Lambda_2}([X], \rho) + (m/2 - 1)\mu^{\Lambda_3}([X], \rho)\right].$$

*Proof.* Proposition 3.9 gives the relation $L_0 + L_1 = 0$ over $\mathrm{Hilb}^\bullet$, and Equation 3.4 gives

$$\Lambda_m | \mathrm{Hilb}^\bullet = (m-1)(L_1 + \frac{m}{2} L_2).$$

Under our regularity hypothesis, Proposition 3.10 applies for each $m \geq 2$ and

$$\mu^{\Lambda_m}([X], \rho) = (m-1)\left[\mu^{L_1}([X], \rho) + \frac{m}{2}\mu^{L_2}([X], \rho)\right] \tag{3.10}$$

is a polynomial for $m \geq 2$ (see Proposition 3.16). One can obtain (3.9) by expressing $\mu^{L_1}([X], \rho)$ and $\mu^{L_2}([X], \rho)$ in terms of $\mu^{\Lambda_2}([X], \rho)$ and $\mu^{\Lambda_3}([X], \rho)$ and plugging them in (3.10). ☐

## 4. Basin of attraction and equivalences

**Definition 4.1.** Let $X$ be a variety with $\mathbb{G}_m$ acting via $\rho : \mathbb{G}_m \to \mathrm{Aut}(X)$ with fixed points $X^\rho$. For each $x^\star \in X^\rho$, the *basin of attraction* is defined

$$A_\rho(x^\star) := \left\{ x \in X \mid \lim_{t \to 0} \rho(t).x = x^\star \right\}.$$

When $X$ is smooth and projective this can be interpreted via the Białynicki-Birula decomposition [BB73, Theorem 4.3]: Consider the decomposition $X_i^\rho, i \in I$ of the fixed points into connected components. Then there is a unique locally closed $\rho$-invariant decomposition $X = \cup_{i \in I} X_i$ and morphisms $\gamma_i : X_i \to X_i^\rho$ such that

- $(X_i)^\rho = X_i^\rho$ for each $i \in I$;
- $\gamma_i$ is an affine bundle;
- for each $x \in X_i^\rho$, the tangent space $T_x X_i \subset T_x X$ is the subspace over which $\rho$ acts with nonnegative weights.

For $x^\star \in X_i^\rho$ we have $A_\rho(x^\star) = \gamma_i^{-1}(x^\star)$.

The importance of this decomposition for the analysis of semistable points is clear from the following proposition which is well known to experts. Given a point on a projective variety $x \in X \subset \mathbb{P}^N$, let $x^* \in \mathbb{A}^{N+1}$ denote an affine lift, i.e., a point in the affine cone over $X$ lying over $x$.

**Proposition 4.2.** *Suppose that $G$ is a reductive linear algebraic group acting on a projective variety $X$ and $L$ is a $G$-linearized ample line bundle. Suppose $x_1, x_2 \in X$ be semistable points mapping to the same point in the GIT quotient $X /\!\!/ G$. Then there exists a semistable point $x_0 \in X$ with the following properties:*

- *the orbit $Gx_0^*$ is closed, or equivalently, the stabilizer $G_{x_0^*} \subset G$ is reductive;*
- *there exists $g \in G$, one-parameter subgroups $\rho_1, \rho_2$ of $G_{x_0^*}$, and lifts $x_1^*$ and $x_2^*$ of $x_1$ and $x_2$ such that*

$$x_1^* \in A_{\rho_1}(x_0^*) \quad g \cdot x_2^* \in A_{\rho_2}(x_0^*).$$

*Proof.* Since $x_1$ and $x_2$ are identified in the GIT quotient, any homogeneous invariant vanishing on $x_1$ automatically vanishes on $x_2$, and vice versa. Consider the orbit closures $\overline{Gx_1}$ and $\overline{Gx_2}$ in $X$. Their orbit closures meet [Ses77, Proposition 7, pp. 254]:

$$\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset,$$

and moreover there exist $x_1^*$ and $x_2^*$ lying over $x_1$ and $x_2$ in the affine cone over $X$ such that

$$\overline{Gx_1^*} \cap \overline{Gx_2^*} \neq \emptyset.$$

(This is essentially the fact that invariants separate orbit closures in affine space, e.g., [MFK94, Corollary 1.2, pp. 29].) Pick $y_0^* \in \overline{Gx_1^*} \cap \overline{Gx_2^*}$ generating a *closed* orbit of the intersection.

Recall Matsushima's Criterion [Mat60, BB63]: Suppose $G$ is a reductive algebraic group and $H \subset G$ a closed subgroup; the homogeneous space $G/H$ is affine if and only if $H$ is reductive. This gives the equivalence of the two conditions on $x_0^*$.

We apply [Kem78, Theorem 1.4] to the closed $G$-invariant set $S = \overline{Gy_0^*} = Gy_0^*$: There exist one-parameter subgroups $\rho_1$ and $\rho_2'$ such that

$$x_0^* := \lim_{t \to 0} \rho_1(t) \cdot x_1^* \in Gy_0^* \quad \lim_{t \to 0} \rho_2'(t) \cdot x_2^* \in Gy_0^*.$$

Clearly there exists $g \in G$ such that

$$g \cdot \lim_{t \to 0} \rho_2'(t) \cdot x_2^* = x_0^*.$$

Setting $\rho_2 = g\rho_2' g^{-1}$, we obtain the desired result. □

Also, as far as stability is concerned, the points in a basin of attraction are all equivalent if the attracting point is strictly semistable with respect to the 1-PS:

**Lemma 4.3.** *Let $G$, $X$, $L$ be as in Proposition 4.2. Let $x \in X$ and suppose there exists an $x_0 \in X$ and a one-parameter subgroup $\rho$ of $G$ such that $x \in A_\rho(x_0)$. If $\mu^L(x, \rho) = 0$ then $x_0$ is semistable with respect to $L$ if and only if $x$ is semistable with respect to $L$.*

*Proof.* Assume that $X$ is embedded in $\mathbb{P}^N$ by sections of $L$ and $x^* \in \mathbb{A}^{N+1} \setminus \{0\}$ be an affine lift of $x$. Since $\mu^L(x, \rho) = 0$, $\rho(t).x^*$ has a specialization, say $x_0^* \neq 0$, which corresponds to $x_0 \in \mathbb{P}^N$. Let $s$ be a $G$-invariant section of $L$. Then $s(x_0^*) = s(\rho(t).x^*) = s(x^*)$ and it follows that $s(x_0) \neq 0$ if and only if $s(x) \neq 0$. □

Let $x_0$ be a point in $X$ and $\rho : \mathbb{G}_m \to G$ be a one-parameter subgroup fixing $x_0$. It follows directly from the definition of the Hilbert-Mumford index that if $\mu^L(x_0, \rho) < 0$ then every point in the basin of attraction $A_\rho(x_0)$ is unstable. This observation can be used to classify unstable points in certain situations; our approach is similar to [Tha96, §4]:

**Proposition 4.4.** *Let $G$, $X$, $L$ be as in Proposition 4.2 and $M$ a second $G$-linearized ample line bundle. Let $x_1 \in X$ be semistable with respect to $L$ but unstable with respect to $L \otimes M^\epsilon$ for each rational $\epsilon > 0$. Then there exists a point $x_0 \in X$ having the following properties:*

(1) *$x_0$ is strictly semistable with respect to $L$;*
(2) *there exists a one-parameter subgroup $\rho : \mathbb{G}_m \to G_{x_0}$ such that*
$$x_1 \in A_\rho(x_0);$$
(3) *$\mu^{L\otimes M^\epsilon}(x_0, \rho) < 0$.*

That is, every strictly semistable point that becomes unstable after perturbing $L$ can be destabilized by a one-parameter subgroup acting via automorphisms of a point strictly semistable with respect to $L$.

*Proof.* Let $\rho$ be a one-parameter subgroup with $\mu^{L\otimes M^\epsilon}(x_1, \rho) < 0$, which exists by the Hilbert-Mumford criterion. Let $x_0 = \lim_{t\to 0} \rho(t)\cdot x_1$ denote the corresponding limit point in $X$. Clearly, $x_1 \in A_\rho(x_0)$ and $\mu^{L\otimes M^\epsilon}(x_0, \rho) < 0$. □

For the convenience of the reader, we recall the standard Semistable Replacement Theorem:

**Theorem 4.5.** *Retain the assumptions of Lemma 4.3. Assume that $G$ is reductive so the GIT quotient scheme $X^{ss}/\!\!/G$ exists. Let $B$ be a smooth curve, $0 \in B$ a closed point, and $f : B \setminus \{0\} \to X^{ss}$ be a regular morphism. Then there exists a covering $\alpha : B' \to B$ branched only over $0$ and $\gamma : B' \setminus \{0'\} \to G$, $0' = \alpha^{-1}(0)$, such that*

- *there is a regular morphism $f' : B' \to X^{ss}$;*
- *$f(\alpha(b')) = \gamma(b').f'(b')$ for all $b' \neq 0'$.*

**Definition 4.6.** Two c-semistable curves $C_1$ and $C_2$ are said to be *c-equivalent*, denoted $C_1 \sim_c C_2$, if there exists a curve $C^\star$ (which we may assume has reductive automorphism group) and one-parameter subgroups $\rho_1, \rho_2$ of $\mathrm{Aut}(C^\star)$

with $\mu(Ch(C^\star),\rho_i) = 0$ such that the basins of attraction $A_{\rho_1}(Ch(C^\star))$ and $A_{\rho_2}(Ch(C^\star))$ contain Chow-points of curves isomorphic to $C_1$ and $C_2$ respectively.

We define *h-equivalence*, denoted $\sim_h$, in an analogous way. Lemma 4.3 shows that these equivalence relations respect the semistable and unstable loci. Proposition 4.2 shows that for GIT-semistable curves $C_1 \sim_c C_2$ if and only if $Ch(C_1)$ and $Ch(C_2)$ yield the same point of $\overline{M}_g^{cs}$; the analogous statement holds for h-equivalence.

## 5. Computations over the moduli space of stable curves

Let $\pi : \overline{\mathcal{C}_g} \to \overline{\mathcal{M}}_g$ denote the universal curve over the moduli stack of curves of genus $g$. For each $n \geq 1$ we have the vector bundle

$$E_n = \pi_* \omega_\pi^n,$$

of rank

$$r(n) = \begin{cases} g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n > 1. \end{cases}$$

Write

$$\lambda_n = c_1(E_n)$$

and use $\lambda$ to designate $\lambda_1$.

Consider the multiplication maps

$$\mathrm{Sym}^m E_n \to E_{mn} \tag{5.1}$$

for each $m > 1$. We have the Chern-class identities

$$\begin{aligned} c_1(\mathrm{Hom}(\mathrm{Sym}^m(E_n), E_{mn})) &= \mathrm{rk}(\mathrm{Sym}^m(E_n))c_1(E_{mn}) - \mathrm{rk}(E_{mn})c_1(\mathrm{Sym}^m(E_n)) \\ &= \binom{m + r(n) - 1}{m}\lambda_{mn} - r(mn)\binom{m + r(n) - 1}{m-1}\lambda_n \\ &= \frac{(m + r(n) - 1)!}{m!\ r(n)!}(r(n)\lambda_{mn} - r(mn)m\lambda_n) \\ &\sim r(n)\lambda_{mn} - r(mn)m\lambda_n, \end{aligned}$$

where $\sim$ designates proportionality. These divisor classes were introduced by Viehweg [Vie89, §1.4] and Cornalba-Harris [CH88, §2] and their significance is explained by the following fact:

**Proposition 5.1.** *Assume that* $n \geq 2$ *when* $g = 2$*. Consider the Hilbert scheme* $\mathsf{Hilb}$ *of degree* $2n(g-1)$ *curves of genus* $g$ *in* $\mathbb{P}^{r(n)-1}$*. Let* $\mathsf{Hilb}_{g,n} \subset \mathsf{Hilb}$ *denote the closure of the* $n$*-pluricanonically embedded smooth curves of genus* $g$*. Suppose that* $\Lambda_m$ *(introduced in §3.2) is well-defined and ample on* $\mathsf{Hilb}_{g,n}$*.*

*Consider the open subsets*

$$V_{g,n}^{s,m} \subset \mathsf{Hilb}_{g,n}^{s,m} \subset \mathsf{Hilb}_{g,n}, \quad V_{g,n}^{ss,m} \subset \mathsf{Hilb}_{g,n}^{ss,m} \subset \mathsf{Hilb}_{g,n}$$

*corresponding to* $n$*-canonically embedded Deligne-Mumford stable curves that are GIT stable and GIT semistable with respect to* $\Lambda_m$*. Let*

$$\mathcal{U}_{g,n}^{s,m} \subset \mathcal{U}_{g,n}^{ss,m} \subset \overline{\mathcal{M}}_g$$

*denote their images in moduli. Then* $\Lambda_m$ *descends to a multiple of* $r(n)\lambda_{mn} - r(mn)m\lambda_n$ *along* $\mathcal{U}_{g,n}^{ss,m}$*. This restricts to an ample divisor on the coarse moduli space* $U_{g,n}^{s,m}$*.*

*Proof.* (cf. [Vie95, §1.6]) We illustrate how $\Lambda_m$ descends to $\mathcal{U}_{g,n}^{ss,m}$. Let $\varpi : \mathcal{X} \to \mathsf{Hilb}_{g,n}$ denote the universal family. The multiplication map (5.1) on moduli is obtained by descent from the multiplication map over $\mathsf{Hilb}_{g,n}$

$$\mathrm{Sym}^m(\varpi_* \mathcal{O}_{\mathcal{X}}(1)) \to \varpi_* \mathcal{O}_{\mathcal{X}}(m).$$

As in Proposition 3.9, we have a trivialization

$$\Gamma(\mathcal{O}_{\mathbb{P}^{r(n)-1}}(1)) \otimes \mathcal{O}_{V_{g,n}^{ss,m}} \simeq \varpi_* \mathcal{O}_{\mathcal{X}}(1)|V_{g,n}^{ss,m}.$$

Thus the divisor class

$$c_1\left(\mathrm{Hom}(\mathrm{Sym}^m(\varpi_* \mathcal{O}_{\mathcal{X}}(1)), \varpi_* \mathcal{O}_{\mathcal{X}}(m))|V_{g,n}^{ss,m}\right)$$

is proportional to

$$c_1(\varpi_* \mathcal{O}_{\mathcal{X}}(m)|V_{g,n}^{ss,m}) = \Lambda_m|V_{g,n}^{ss,m}.$$

As for the ampleness, the coarse moduli space $U_{g,n}^{s,m}$ of $\mathcal{U}_{g,n}^{s,m}$ can be identified with an open subset of the GIT quotient

$$\mathsf{Hilb}_{g,n}^{ss,m} /\!\!/ \mathrm{SL}_{r(n)}.$$

$\Lambda_m$ descends to a polarization of this quotient. $\square$

Mumford [Mum77, Theorem 5.10] showed that the Grothendieck-Riemann-Roch formula gives

$$\lambda_n = (6n^2 - 6n + 1)\lambda - \binom{n}{2}\delta, \quad n > 1. \tag{5.2}$$

We therefore find

$$r(n)\lambda_{mn} - r(mn)m\lambda_n = \begin{cases} \lambda + (m-1)((4g+2)m - g + 1)\lambda - \frac{gm}{2}\delta) & \text{if } n = 1, \\ (m-1)(g-1)((6mn^2 - 2mn - 2n + 1)\lambda - \frac{mn^2}{2}\delta) & \text{if } n > 1. \end{cases} \tag{5.3}$$

Asymptotically as $m \to \infty$, we obtain the proportionality

$$\lim_{m\to\infty} (r(n)\lambda_{mn} - r(mn)m\lambda_n) \sim \begin{cases} (4g+2)\lambda - \frac{g}{2}\delta \text{ if } n = 1 \\ (6n-2)\lambda - \frac{n}{2}\delta \text{ if } n > 1. \end{cases}$$

Combining Proposition 5.1 and Equation 3.7, we obtain:

**Proposition 5.2.** *Assume that* $n \geq 2$ *when* $g = 2$*. Consider the Chow variety* $\mathrm{Chow}$ *of degree* $2n(g-1)$ *curves of genus* $g$ *in* $\mathbb{P}^{r(n)-1}$*. Let* $\mathrm{Chow}_{g,n} \subset \mathrm{Chow}$ *denote the closure of the* $n$*-pluricanonically embedded curves of genus* $g$*.*

*Consider the open subsets*

$$V^{s,\infty}_{g,n} \subset \mathrm{Chow}^s_{g,n} \subset \mathrm{Chow}_{g,n}, \quad V^{ss,\infty}_{g,n} \subset \mathrm{Chow}^{ss}_{g,n} \subset \mathrm{Chow}_{g,n}$$

*corresponding to* $n$*-canonically embedded Deligne-Mumford stable curves that are Chow stable and Chow semistable respectively. Let*

$$\mathcal{U}^{s,\infty}_{g,n} \subset \mathcal{U}^{ss,\infty}_{g,n} \subset \overline{\mathcal{M}}_g$$

*denote their images in moduli. Then the polarization descends to a multiple of*

$$(4g+2)\lambda - \frac{g}{2}\delta \text{ if } n = 1$$

*or*

$$(6n-2)\lambda - \frac{n}{2}\delta \text{ if } n > 1.$$

*The restriction to the coarse moduli space* $U^{s,\infty}_{g,n}$ *is ample.*

**Remark 5.3** (Application to polarizations on $\overline{M}_g$)**.** Mumford has proven that $\mathcal{U}^{s,\infty}_{g,n} = \overline{\mathcal{M}}_g$ for each $n \geq 5$ [Mum77, Theorem 5.1]. Proposition 3.13 then guarantees that $\mathcal{U}^{s,m}_{g,n} = \overline{\mathcal{M}}_g$ for all $m \gg 0$. Proposition 5.2 then implies that $a\lambda - \delta$ is ample for $a > 11.2$ [Mum77, Corollary 5.18]. Cornalba and Harris [CH88] established the sharp result: $a\lambda - \delta$ is ample if and only if $a > 11$.

We are primarily interested in situations where not all Deligne-Mumford stable curves have stable Hilbert/Chow points. Here GIT yields alternate birational models of the moduli space.

**Theorem 5.4.** *Retain the notation of Propositions 5.1 and 5.2 with the convention that $m = \infty$ in the Chow case. Suppose that*

- *the complement to the Deligne-Mumford stable curves in the GIT-semistable locus $\mathsf{Hilb}^{ss,m}_{g,n}$ (resp. $\mathsf{Chow}^{ss}_{g,n}$) has codimension $\geq 2$;*
- *there exist Deligne-Mumford stable curves in the GIT-stable locus $\mathsf{Hilb}^{s,m}_{g,n}$ (resp $\mathsf{Chow}^{s}_{g,n}$).*

*Then there exists a birational contraction*

$$F : \overline{M}_g \dashrightarrow \mathsf{Hilb}^{ss,m}_{g,n} /\!\!/ \mathrm{SL}_{r(n)} \text{ (resp. } \mathsf{Chow}^{ss}_{g,n} /\!\!/ \mathrm{SL}_{r(n)})$$

*regular along the Deligne-Mumford stable curves with GIT-semistable Hilbert (resp. Chow) points.*

*If $\mathcal{L}_m$ is the polarization on the GIT quotient induced by $\Lambda_m$ then the moving divisor*

$$F^*\mathcal{L}_m \sim r(n)\lambda_{mn} - r(mn)m\lambda_n \pmod{\mathrm{Exc}(F)}, \tag{5.4}$$

*where $\mathrm{Exc}(F) \subset \mathrm{Pic}(\overline{M}_g)$ is the subgroup generated by $F$-exceptional divisors.*

A rational map of proper normal varieties is said to be a *birational contraction* if it is birational and its inverse has no exceptional divisors. Note that Propositions 5.1 and 5.2 cover the case where $F$ is an isomorphism.

*Proof.* Our assumptions can be written in the notation of Propositions 5.1 and 5.2:

- $V^{ss,m}_{g,n} \subset \mathsf{Hilb}^{ss,m}_{g,n}$ (resp. $V^{ss,\infty}_{g,n} \subset \mathsf{Chow}^{ss}_{g,n}$) has codimension $\geq 2$;
- $V^{s,m}_{g,n} \neq \emptyset$.

The GIT quotient morphism

$$V^{s,m}_{g,n} \to \mathcal{U}^{s,m}_{g,n}$$

identifies the stack-theoretic quotient $[V^{s,m}_{g,n}/\mathrm{SL}_{r(n)}]$ with $\mathcal{U}^{s,m}_{g,n}$. This gives a birational map

$$\mathsf{Hilb}^{ss,m}_{g,n} /\!\!/ \mathrm{SL}_{r(n)} \text{ (resp. } \mathsf{Chow}^{ss}_{g,n} /\!\!/ \mathrm{SL}_{r(n)}) \dashrightarrow \overline{M}_g;$$

we define $F$ as its inverse.

We establish that $F$ is regular along $U^{ss,m}_{g,n}$: We have an $\mathrm{SL}_{r(n)}$-equivariant morphism

$$\mathrm{Hilb}^{ss,m}_{g,n} \to \mathrm{Hilb}^{ss,m}_{g,n}/\!\!/\mathrm{SL}_{r(n)},$$

which descends to

$$\mathcal{U}^{ss,m}_{g,n} \to \mathrm{Hilb}^{ss,m}_{g,n}/\!\!/\mathrm{SL}_{r(n)}.$$

Recall the universal property of the coarse moduli space: Any morphism from a stack to a scheme factors through its coarse moduli space. In our context, this gives

$$U^{ss,m}_{g,n} \to \mathrm{Hilb}^{ss,m}_{g,n}/\!\!/\mathrm{SL}_{r(n)}(\text{resp. } \mathrm{Chow}^{ss}_{g,n}/\!\!/\mathrm{SL}_{r(n)}).$$

Furthermore, the total transform of $\overline{M}_g \setminus U^{ss,m}_{g,n}$ is contained in the complement $\mathrm{Hilb}^{ss,m}_{g,n} \setminus V^{ss,m}_{g,n}$ (resp. $\mathrm{Chow}^{ss}_{g,n} \setminus V^{ss,\infty}_{g,n}$), which has codimension $\geq 2$. Thus any divisorial components of $\overline{M}_g \setminus U^{ss,m}_{g,n}$ are $F$-exceptional divisors. Similary, $F^{-1}$ has no exceptional divisors: These would give rise to divisors in the complement to $V^{ss,m}_{g,n}$ in the semistable locus.

We now analyze $F^*\mathcal{L}_m$ in the rational Picard group of $\overline{M}_g$. (Since $\overline{M}_g$ has quotient singularities, its Weil divisors are all $\mathbb{Q}$-Cartier.) If $\mathcal{L}^a_m$ is very ample on the GIT quotient then $F^*\mathcal{L}^a_m$ induces $F$, i.e., $F^*\mathcal{L}^a_m$ has no fixed components and is generated by global sections over $U^{ss,m}_{g,n}$. Now $F^*\mathcal{L}_m$ is proportional to $r(n)\lambda_{mn} - r(mn)m\lambda_n$ over $U^{ss,m}_{g,n}$ and Formula (5.4) follows. □

## 6. Properties of c-semistable and h-semistable curves

### 6.1. **Embedding c-semistable curves.**

**Proposition 6.1.** *If* $g \geq 3$ *and* $C$ *is a c-semistable curve of genus* $g$ *over* $k$*, then* $H^1(C, \omega_C^{\otimes n}) = 0$ *and* $\omega_C^{\otimes n}$ *is very ample for* $n \geq 2$.

**Remark 6.2.** For the rest of this paper, when we refer to the Chow or Hilbert point of a c-semistable curve $C$ it is with respect to its bicanonical embedding in $\mathbb{P}(\Gamma(C, \omega_C^{\otimes 2})^*)$.

*Proof.* Our argument follows [DM69, Theorem 1.2].

By Serre Duality, $H^1(C, \omega_C^{\otimes n})$ vanishes if $H^0(C, \omega_C^{\otimes 1-n})$ vanishes. The restriction of $\omega_C^{\otimes 1-n}$ to each irreducible component $D \subset C$ has negative degree because $\omega_C$ is ample. It follows that $\Gamma(D, \omega_C^{\otimes 1-n}|D) = 0$, hence $\Gamma(C, \omega_C^{\otimes 1-n}) = 0$.

To show that $\omega_C^{\otimes n}$ is very ample for $n \geq 2$, it suffices to prove for all $x, y \in C$ that

$$\mathrm{Hom}(\mathfrak{m}_x\mathfrak{m}_y, \omega_C^{\otimes -n}) = 0, \quad n \geq 1. \tag{6.1}$$

Let $\pi : C' \to C$ denote the partial normalization of any singularities at $x$ and $y$. When $x$ is singular, a local computation gives

$$\mathrm{Hom}(\mathfrak{m}_x, \mathcal{L}) \simeq \Gamma(C', \pi^*\mathcal{L}).$$

If $x$ is a cusp and $x' \in C'$ its preimage then

$$\mathrm{Hom}(\mathfrak{m}_x^2, \mathcal{L}) \simeq \Gamma(C', \pi^*\mathcal{L}(2x')).$$

If $x$ is a node or tacnode and $x_1, x_2 \in C'$ the preimage points then

$$\mathrm{Hom}(\mathfrak{m}_x^2, \mathcal{L}) \simeq \Gamma(C', \pi^*\mathcal{L}(x_1 + x_2)).$$

Thus in each case we can express

$$\mathrm{Hom}(\mathfrak{m}_x\mathfrak{m}_y, \omega_C^{\otimes -n}) = \Gamma(C', \mathcal{M})$$

for a suitable invertible sheaf $\mathcal{M}$ on $C'$. Moreover, we have an inclusion

$$\pi^*\omega_C^{-n} \hookrightarrow \mathcal{M}$$

with cokernel $Q$ supported in $\pi^{-1}\{x, y\}$ of length $\ell(Q) \leq 2$. For instance, if both $x$ and $y$ are smooth then

$$\mathcal{M} = \omega_C^{-n}(x + y);$$

if both $x$ and $y$ are singular and $x \neq y$ then

$$\mathcal{M} = \pi^*\omega_C^{-n}.$$

Suppose that for each irreducible component $D' \subset C'$, the degree $\deg \mathcal{M}|D' < 0$. Then $\Gamma(C', \mathcal{M}) = 0$ and the desired vanishing follows. We therefore classify situations where

$$\deg \mathcal{M}|D' = -n \deg \pi^*\omega_C|D' + \ell(Q|D') \geq 0,$$

which divide into the following cases:

(a) $\deg \pi^*\omega_C|D' = 1$, $n = 1$, $\ell(Q|D') = 1$;
(b) $\deg \pi^*\omega_C|D' = 1$, $n = 1, 2$, $\ell(Q|D') = 2$;
(c) $\deg \pi^*\omega_C|D' = 2$, $n = 1$, $\ell(Q|D') = 2$.

We write $D = \pi(D') \subset C$.

We enumerate the various possibilities. We use the assumption that $C$ is c-semistable and thus has no elliptic tails. In cases (a) and (b), $D$ is necessarily isomorphic to $\mathbb{P}^1$ and meets the rest of $C$ in either three nodes or in one node and one tacnode. After reordering $x$ and $y$, we have the following subcases:

(a1) $x = y \in D$ a node or tacnode of $C$;
(a2) $x \in D$ a node or tacnode of $X$ and $y \in D$ a smooth point of $C$;
(a3) $x \in D$ a smooth point of $C$ and $y \notin D$.
(b1) $x, y \in D$ smooth points of $C$.

In case (c), $D$ may have arithmetic genus zero or one:

(c1) $D \simeq \mathbb{P}^1$ with $x, y \in D$ smooth points of $C$;
(c2) $D$ of arithmetic genus one with $x, y \in D$ smooth points of $C$;
(c3) $D$ of arithmetic genus one, $x = y$ a node or cusp of $D$, and $D' \simeq \mathbb{P}^1$.

In subcase (c1), $D$ meets the rest of $C$ in either four nodes, or in two nodes and one tacnode, or in two tacnodes. In subcases (c2) and (c3), $D$ meets the rest of $C$ in two nodes. Except in case (c3), $\pi : D' \to D$ is an isomorphism.

For subcases (b1), (c1), and (c2), $\pi$ is an isomorphism. Moreover, $Q$ is supported along $D$ so $\mathcal{M}$ has negative degree along any other irreducible components of $C$. There are other components because the genus of $C$ is at least three. Thus elements of $\Gamma(C, \mathcal{M})$ restrict to elements of $\Gamma(D, \mathcal{M}|D)$ that vanish at the points where $D$ meets the other components, i.e., in at least two points. Since $\deg \mathcal{M}|D = 0$ or $1$, we conclude $\Gamma(C, \mathcal{M}) = 0$.

For subcase (c3), $\pi$ is not an isomorphism but $Q$ is still supported along $D'$. As before, $\mathcal{M}$ has negative degree along other irreducible components of $C'$, and elements of $\Gamma(C', \mathcal{M})$ restrict to elements of $\Gamma(D', \mathcal{M}|D')$ vanishing where $D'$ meets the other components. There are at least two such points but $\deg \mathcal{M}|D' = 0, 1$, so we conclude that $\Gamma(C', \mathcal{M}) = 0$.

In case (a), we have $\deg \mathcal{M}|D = 0$. Subcases (a1) and (a2) are similar to (b1) and (c1): $Q$ is supported along $D'$ so elements in $\Gamma(C', \mathcal{M})$ restrict to elements of $\Gamma(D', \mathcal{M}|D')$ vanishing at the points where $D'$ meets the other components. There is at least one such point, e.g., the singularity not lying over $x$, hence $\Gamma(C', \mathcal{M}) = 0$.

Subcase (a3) is more delicate. If $D'$ is the unique component such that $\deg(\mathcal{M}|D') \geq 0$ then the arguments of the previous cases still apply. However, the support of $Q$ might not be confined to a single component. We suppose there are two components $D_1'$ and $D_2'$ as described in (a3), such that $\deg(\mathcal{M}|D_i') \geq 0$. Since the genus of $C$ is $> 2$, $C$ cannot just be the union of $D_1'$ and $D_2'$; there is at least one additional component meeting each $D_i'$ at some point $z_i$, and the restriction of $\mathcal{M}$ to this component has negative degree. Thus elements of $\Gamma(C', \mathcal{M})$ restrict to elements of $\Gamma(D_i', \mathcal{M}|D_i')$ vanishing at $z_i$, which are necessarily zero. □

**Corollary 6.3.** *Let $C \subset \mathbb{P}^{3g-4}$ be a c-semistable bicanonical curve.*

- *$\mathcal{O}_C$ is 2-regular.*
- *The Hilbert scheme is smooth at $[C]$.*
- *Let $p_1, \ldots, p_n$ denote the singularities of $C$ and $\mathrm{Def}(C, p_i), i = 1, \ldots, n$ their versal deformation spaces. Then there exists a neighborhood $U$ of $[C]$ in the Hilbert scheme such that*

$$U \to \prod_i \mathrm{Def}(C, p_i)$$

*is smooth.*

*Proof.* Proposition 6.1 yields

$$H^1(C, \mathcal{O}_C(1)) = H^1(C, \omega_C^{\otimes 2}) = 0$$

which gives the regularity assertion. This vanishing also implies [Kol96, I.6.10.1]

$$H^1(C, \mathrm{Hom}(I_C/I_C^2, \mathcal{O}_C)) = 0;$$

since the singularities of $C$ are local complete intersections we have

$$\mathrm{Ext}^1(I_C/I_C^2, \mathcal{O}_C) = H^1(C, \mathrm{Hom}(I_C/I_C^2, \mathcal{O}_C)) = 0$$

thus the Hilbert scheme is unobstructed at $[C]$ (see [Kol96, I.2.14.2]). The assertion about the map onto the versal deformation spaces is [Kol96, I.6.10.4]. □

**Corollary 6.4.** *Let $C \subset \mathbb{P}^{3g-4}$ be a bicanonical c-semistable curve and $C^\star$ denote the curve to which $\rho(t).C$ specializes. If $C^\star$ is a bicanonical c-semistable curve then*

$$\mu([C]_m, \rho) = (m-1)\left[(3-m)\mu([C]_2, \rho) + (m/2-1)\mu([C]_3, \rho)\right].$$

*Thus $[C]_m$ is stable (resp. strictly semistable, resp. unstable) with respect to $\rho$ for each $m \geq 2$ if and only if $\mu([C]_3, \rho) \geq 2\mu([C]_2, \rho) > 0$ (resp. $\mu([C]_3, \rho) =$*

$\mu([C]_2, \rho) = 0$, *resp.* $\mu([C]_3, \rho) \leq 2\mu([C]_2, \rho) < 0$.*) The Chow point* $Ch(C)$ *is stable (resp. strictly semistable, resp. unstable) with respect to* $\rho$ *if and only if* $\mu([C]_3, \rho) - 2\mu([C]_2, \rho) > 0$ *(resp.* $= 0$*, resp.* $< 0$*.)*

*Proof.* By Proposition 6.1, a bicanonical c-semistable is 2-regular and the assertion on the Hilbert points follows immediately from Equation (3.9). Equations (3.6) and (3.8) allow us to interpret the Hilbert-Mumford index of the Chow point in terms of the leading coefficient of $\mu([C]_m, \rho)$ as a polynomial in $m$. □

6.2. **Basic properties of tacnodal curves.** Let $C$ be a curve with a tacnode $r$, i.e., a singularity with two smooth branches intersecting with simple tangency. Let $\nu : D \to C$ be the partial normalization of $C$ at $r$ and $\nu^{-1}(r) = \{p, q\} \subset D$ the conductor. The descent data from $(D, p, q)$ to $(C, r)$ consists of a choice of isomorphism

$$\iota : T_pD \tilde{\to} T_qD$$

identifying the tangent spaces to the branches. Functions on $C$ pull back to functions $f$ on $D$ satisfying $f(p) = f(q)$ and $\iota(f'(p)) = f'(q)$.

Varying the descent data gives a one-parameter family of tacnodal curves:

**Proposition 6.5.** *Let* $D$ *be a reduced curve and* $p, q \in D$ *distinct smooth points with local parameters* $\sigma_p$ *and* $\sigma_q$*. Each invertible linear transformation* $T_pD \to T_qD$ *can be expressed*

$$\iota(t) : \frac{\partial}{\partial \sigma_p} \mapsto t \frac{\partial}{\partial \sigma_q}$$

*for some* $t \neq 0$*; let* $\mathbb{G}_m \simeq \mathrm{Isom}(T_pD, T_qD)$ *denote the corresponding identification. Then there exists a family* $\mathcal{C} \to \mathbb{G}_m$*, a section* $r : \mathbb{G}_m \to \mathcal{C}$*, and a morphism*

$$\begin{array}{ccc} D \times \mathbb{G}_m & \xrightarrow{\nu} & \mathcal{C} \\ & \searrow \quad \swarrow & \\ & \mathbb{G}_m & \end{array}$$

*such that*

(1) $\nu$ *restricts to an isomorphism*

$$D \setminus \{p, q\} \times \mathbb{G}_m \tilde{\to} \mathcal{C} \setminus r;$$

(2) *for each* $t \in \mathbb{G}_m$*,* $r_t \in C_t$ *is a tacnode and* $\nu_t$ *its partial normalization;*

(3) *the descent data from* $(D, p, q)$ *to* $(C_t, r_t)$ *is given by* $\iota(t)$*.*

*Every tacnodal curve normalized by* $(D, p, q)$ *occurs as a fiber of* $\mathcal{C} \to \mathbb{G}_m$.

If $D$ is projective of genus $g-2$ then each $C_t$ has genus $g$.

We sketch the construction of $\mathcal{C}$: $\iota(t)$ tautologically yields an identification over $\mathbb{G}_m$

$$\iota : T_{p\times\mathbb{G}_m} D \times \mathbb{G}_m/\mathbb{G}_m \xrightarrow{\sim} T_{q\times\mathbb{G}_m} D \times \mathbb{G}_m/\mathbb{G}_m \tag{6.2}$$

which is the descent data from $D \times \mathbb{G}_m$ to $\mathcal{C}$. Fiber-by-fiber, we get the universal family of tacnodal curves normalized by $(D, p, q)$.

We will extend $\mathcal{C} \to \mathbb{G}_m$ to a family of tacnodal curves $\mathcal{C}' \to \mathbb{P}^1$. First, observe that the graph construction gives an open embedding

$$\mathbb{G}_m \simeq \mathrm{Isom}(T_p D, T_q D) \subset \mathbb{P}(T_p D \oplus T_q D) \simeq \mathbb{P}^1,$$

where $t = 0$ corresponds to $[1, 0]$ and $t = \infty$ corresponds to $[0, 1]$. However, the identification (6.2) fails to extend over all of $\mathbb{P}^1$; indeed, it is not even defined at $p \times [0, 1]$ and its inverse is not defined at $q \times [1, 0]$. We therefore blow up

$$\mathcal{D}' = \mathrm{Bl}_{p\times[0,1],q\times[1,0]} D \times \mathbb{P}^1$$

and consider the sections

$$\mathfrak{p}, \mathfrak{q} : \mathbb{P}^1 \to \mathcal{D}'$$

extending $p \times \mathbb{G}_m$ and $q \times \mathbb{G}_m$. Now (6.2) extends to an identification

$$\iota' : T_{\mathfrak{p}} \mathcal{D}'/\mathbb{P}^1 \xrightarrow{\sim} T_{\mathfrak{q}} \mathcal{D}'/\mathbb{P}^1.$$

**Proposition 6.6.** *Retain the notation of Proposition 6.5. There exists an extension*

$$\begin{array}{ccccc} & \mathcal{C} & \subset & \mathcal{C}' & \\ & \downarrow & & \downarrow & \\ \mathrm{Isom}(T_p D, T_q D) \simeq & \mathbb{G}_m & \subset & \mathbb{P}^1 & \simeq \mathbb{P}(T_p D \oplus T_q D) \end{array}$$

*where* $\mathcal{C}' \to \mathbb{P}^1$ *denotes the family of curves obtained from* $\mathcal{D}'$ *and* $\iota'$ *by descent,* $r' : \mathbb{P}^1 \to \mathcal{C}'$ *the tacnodal section, and* $\nu' : \mathcal{D}' \to \mathcal{C}'$ *the resulting morphism. The new fiber* $(C'_0, r'(0))$ *(resp.* $(C'_\infty, r'(\infty))$*) is normalized by* $(D'_0 = D \cup_q \mathbb{P}^1, p, \mathfrak{q}(0))$ *(resp.* $(D'_\infty = D \cup_p \mathbb{P}^1, \mathfrak{p}(\infty), q)$.*)*

We say that the tacnodes in the family $\{C'_t, r_t\}_{t\in\mathbb{P}^1}$ are *compatible*, and that two curves are *compatible* if one can be obtained from the other by replacing some tacnodes by compatible tacnodes.

## 7. Unstable bicanonical curves

In this section, we show that if a curve is not c-semistable then it has unstable Chow point:

**Proposition 7.1.** *If* $\mathsf{Ch}(C) \in \mathsf{Chow}_{g,2}$ *is Chow semistable then* $C \subset \mathbb{P}^{3g-4}$ *is c-semistable.*

We prove this by finding one-parameter subgroups destabilizing curves that are not c-semistable. Many statements in this section are fairly direct generalizations of results in [Mum77] and [Sch91].

7.1. **Badly singular curves are Chow unstable.** A Chow semistable bicanonical curve $C$ cannot have a triple point, since $\frac{d}{N+1} = \frac{4g-4}{3g-3} < \frac{3}{2}$ and this implies that $C$ is Chow unstable by Proposition 3.1 of [Mum77]. We need to show that among the double points, only nodes, ordinary cusps and tacnodes are allowed.

**Lemma 7.2.** *If* $C$ *has a non-ordinary cusp, then it is Chow unstable.*

*Proof.* Suppose that $C$ has a non-ordinary cusp at $p$. Let $\nu : \widetilde{C} \to C$ be the normalization, $p' = \nu^{-1}(p)$ and assume $p = [1, 0, \dots, 0]$. Recall that the singularity at $p$ is determined by the vanishing sequence $\left(a_i(\nu^*|\omega_C^{\otimes 2}|, p')\right)_{i=1}^{N+1}$ which is the strictly increasing sequence determined by the condition

$$\{a_i(\nu^*|\omega_C^{\otimes 2}|, P) \mid i = 1, 2, \dots, N+1\} = \{\mathrm{ord}_{p'}(\sigma) \mid \sigma \neq 0 \in \nu^*|\omega_C^{\otimes 2}|\}.$$

$C$ has a cusp at $p$ if and only if the vanishing sequence $(a_i(\nu^*|\omega_C^{\otimes 2}|, p'))$ is of the form $(0, 2, \geq 3)$, and it has an ordinary cusp if it is of the form $(0, 2, 3, \geq 4)$.

Hence if $C$ has a non-ordinary cusp at $p$, then we can choose coordinates $x_0, \dots, x_N$ such that $\mathrm{ord}_{p'} x_0 = 0$, $\mathrm{ord}_{p'} x_1 = 2$, $\mathrm{ord}_{p'} x_2 = 4$, and $\mathrm{ord}_{p'} x_i \geq 5, i = 3, 4, \dots, N$. Let $\rho : \mathbb{G}_m \to \mathrm{GL}_{N+1}(k)$ be the one-parameter subgroup such that $\rho(t).x_i = t^{r_i} x_i$, where the weights are:

$$(r_0, r_1, \dots, r_N) = (5, 3, 1, 0, \dots, 0).$$

Then $\mathrm{ord}_{p'} x_i + r_i \geq 5$ for all $i$, and it follows from Lemma 3.3 that

$$e_\rho(C) = e_\rho(\widetilde{C}) \geq e_\rho(\widetilde{C})_{p'} \geq 5^2 = 25,$$

while $\frac{2d}{N+1} \sum r_i = \frac{2 \cdot 4(g-1)}{3(g-1)} \cdot 9 = 24$. The assertion now follows from Theorem 3.2. $\square$

**Lemma 7.3.** *Suppose* $C$ *has a singularity at* $p$ *such that*

$$\widehat{\mathcal{O}}_{C,p} \simeq k[x,y]/(y^2 - x^{2s}),\ s \geq 3.$$

*Then* $C$ *is unstable.*

*Proof.* Let $\nu : \widetilde{C} \to C$ be the normalization, $\nu^{-1}(p) = \{p_1, p_2\}$. Since the two branches of $C$ agree to order $s$ at $p$, we may choose coordinates $x_0, \dots, x_N$ such that

$$(\operatorname{ord}_{p_i} x_0, \dots, \operatorname{ord}_{p_i} x_N) = (0, 1, 2, \geq 3), \quad i = 1, 2.$$

Let $\rho$ be the one-parameter subgroup of $\mathrm{GL}_{N+1}(k)$ with weights $(r_0, \dots, r_N) = (3, 2, 1, 0, \dots, 0)$. Then we have

$$\operatorname{ord}_{p_i} x_j + r_j \geq 3, \quad i = 1, 2 \text{ and } j = 0, 1, \dots, N,$$

and by Lemma 3.3,

$$e_\rho(C) = e_\rho(\widetilde{C}) \geq e_\rho(\widetilde{C})_{p_1} + e_\rho(\widetilde{C})_{p_2} \geq 2 \cdot 3^2 = 18,$$

which is strictly greater than $\frac{2d}{N+1} \sum r_i = \frac{2 \cdot 4(g-1)}{3(g-1)} \cdot 6 = 16.$

□

**Lemma 7.4.** *If* $C$ *has a multiple component,* $C$ *is Chow unstable.*

*Proof.* Let $C_1$ be a component of $C$ with multiplicity $n \geq 2$. Choose a smooth non-flex point $p \in C_1^{red}$ such that $p$ does not lie in any other component. Since $p$ is smooth on $C_1^{red}$, we may choose coordinates $x_0, \dots, x_N$ such that

$$(\operatorname{ord}_p x_0, \dots, \operatorname{ord}_p x_N) = (0, 1, 2, \geq 3).$$

Let $\rho$ be the one-parameter subgroup of $\mathrm{GL}_{N+1}(k)$ with weights $(r_0, \dots, r_N) = (3, 2, 1, \dots, 0)$. Then we have

$$\operatorname{ord}_p x_i + r_i \geq 3.$$

This yields the inequality

$$e_\rho(C) \geq n \cdot e_\rho(C_1) \geq 2 \cdot 3^2 = 18$$

whereas $\dfrac{2d}{N+1} \sum r_i = \dfrac{8}{3} \cdot 6 = 16.$ □

7.2. **Polarizations on semistable limits of bicanonical curves.** We first prove that the semistable limit of a one parameter family of smooth bicanonical curves is bicanonical:

**Proposition 7.5.** *Let* $\mathcal{C} \to \operatorname{Spec} k[[t]]$ *be a family of Chow semistable curves of genus* $g$ *such that the generic fibre* $\mathcal{C}_\eta$ *is smooth. If* $\Phi : \mathcal{C} \to \mathbb{P}^{3g-4}_{k[[t]]}$ *is an embedding such that* $\Phi_\eta^*(\mathcal{O}(1)) = \omega^{\otimes 2}_{\mathcal{C}_\eta/k[[t]]}$ *then* $\mathcal{O}_{\mathcal{C}}(1) = \omega^{\otimes 2}_{\mathcal{C}/k[[t]]}$.

By [Mum77, 4.15], nonsingular bicanonical curves are Chow stable. Hence any Chow semistable curve is a limit of nonsingular bicanonical curves and Proposition 7.5 implies that if $C$ is not bicanonical, then $\mathrm{Ch}(C) \notin \mathrm{Chow}^{ss}_{g,2}$. In particular, a Chow semistable curve does not have a smooth rational component meeting the rest of the curve in $< 3$ points. Mumford proved the statement for the $n$-canonical curves for $n \geq 5$, and his argument can be easily modified to suit our purpose. It is an easy consequence of (ii) of the following proposition, which, in Mumford's words, says that the degrees of the components of $C$ are roughly in proportion to their *natural* degrees.

**Proposition 7.6** (Proposition 5.5, [Mum77])**.** *Let* $C \subset \mathbb{P}^{3g-4}$ *be a connected curve of genus* $g$ *and degree* $4g-4$*. Then*

(i) $C$ *is embedded by a non-special complete linear system.*

(ii) *Let* $C = C_1 \cup C_2$ *be a decomposition of* $C$ *into two sets of components such that* $W = C_1 \cap C_2$ *and* $w = \#W$ *(counted with multiplicity). Then*

$$|\deg C_1 - 2\deg_{C_1} \omega_C| \leq \frac{w}{2}.$$

Mumford's argument goes through in the bicanonical case except for the proof of $H^1(C_1, \mathcal{O}_{C_1}(1)) = 0$. If $H^1(C_1, \mathcal{O}_{C_1}(1)) \neq 0$, then by Clifford's theorem we have

$$h^0(C_1, \mathcal{O}_{C_1}(1)) \leq \frac{\deg(C_1)}{2} + 1$$

and the Chow semistability of $C$ forces

$$w + 2\deg C_1 \leq \frac{2\deg C}{3g-3} h^0(C_1, \mathcal{O}_{C_1}(1)).$$

Combining the two, we obtain

$$\deg C_1 \leq 4 - \frac{3}{2}w.$$

If $w \neq 0$, then $\deg C_1 \leq 2$, hence $C_1$ is rational and $H^1(C_1, \mathcal{O}_{C_1}(1)) = 0$. If $w = 0$, then $\deg C_1 \leq 4$ which is absurd since $C_1 = C$ and $\deg C_1 = 4g - 4$.

We need to justify our use of Clifford's theorem here, as Chow semistable bicanonical curves have cusps and tacnodes. We shall sketch the proof of Gieseker and Morrison [Gie82] and highlight the places where modifications are required to accommodate the worse singularities ([Gie82] assumes that $C$ has only nodes).

**Theorem 7.7** (Clifford's Theorem). *Let $C \subset \mathbb{P}^N$ be a reduced curve with nodes, cusps and tacnodes. Let $L$ be a line bundle generated by sections. If $H^1(C, L) \neq 0$, then there is a subcurve $C_1 \subset C$ such that*

$$h^0(C, L) \leq \frac{\deg_{C_1} L}{2} + 1.$$

*Sketch proof.* Suppose that $H^1(C, L) \neq 0$ and $\varphi \neq 0 \in \mathrm{Hom}(L, \omega_C)$. Let $C_1$ be the union of components where $\varphi$ does not vanish entirely and $p_1, \dots, p_w$ be the intersection points of $C_1$ and $\overline{C - C_1}$. Assume that $p_i$'s are ordered so that $p_1, \dots, p_\ell$ are tacnodes. Then we have

$$\omega_C|_{C_1}(-2\sum_{i=1}^{\ell} p_i - \sum_{i=\ell+1}^{w} p_i) = \omega_{C_1}.$$

We claim that $\varphi$ restricts to give a homomorphism from $L_{C_1}$ to $\omega_{C_1}$: Let $p_i$ be a tacnode and let $D \not\subset C_1$ be the irreducible component containing $p_i$. Since $\varphi$ vanishes entirely on $D$, $\varphi$ must vanish to order $\geq 2$ at $p_i$ on $C_1$. Likewise, $\varphi$ must vanish at each node. It follows that $\varphi|_{C_1}$ factors through $\omega_C|_{C_1}(-2\sum_{i=1}^{\ell} p_i - \sum_{i=\ell+1}^{w} p_i)$. Let $s_1, \dots, s_r$ be a basis of $\mathrm{Hom}(L_{C_1}, \omega_{C_1})$ such that $s_1 = \varphi$, and let $t_1, \dots, t_p$ be a basis for $H^0(C, L)$ such that $t_1$ does not vanish at the support of $s_1$ and at any singular points. It is shown in [Gie82] that

$$\begin{array}{lllll} [s_1, t_1], & [s_1, t_2], & [s_1, t_3], & \dots, & [s_1, t_p], \\ & [s_2, t_1], & [s_3, t_1], & \dots, & [s_r, t_1] \end{array}$$

are linearly independent sections of $H^0(C_1, \omega_{C_1})$, which implies that $p + r - 1 \leq p_a(C_1) + 1$. Combining it with the Riemann-Roch gives the desired inequality. □

7.3. **Elliptic subcurves meeting the rest of the curve in one point.** Let $C$ be a Deligne-Mumford stable curve with an elliptic tail $E \subset C$. Then $\omega_C^{\otimes 2}$ fails to be very ample along $E$ and thus $C$ does not admit a bicanonical embedding. In particular, $C$ does not arise in GIT quotients of the Chow variety/Hilbert scheme of bicanonical curves.

Here, we focus on curves with an elliptic subcurve meeting the rest of the curve in a *tacnode*.

**Proposition 7.8.** *Let* $C = E \cup_p R \cup_q D$ *be a bicanonical curve consisting of a rational curve* $E$ *with one cusp, a rational curve* $R$ *and a genus* $g-2$ *curve* $D$ *such that* $p$ *is a tacnode and* $q$ *is a node. Then* $C$ *is Chow unstable with respect to a one-parameter subgroup coming from its automorphism group.*

*Proof.* Restricting $\omega_C^{\otimes 2}$ we get

$$\omega_C^{\otimes 2}|E \simeq \mathcal{O}_E(4p), \quad \omega_C^{\otimes 2}|R \simeq \mathcal{O}_R(2), \quad \omega_C^{\otimes 2}|D \simeq \omega_D^{\otimes 2}(2q).$$

Since $h^0(\omega_C^{\otimes 2}|D) = 3(g-2)-3+2 = 3g-7$, we can choose coordinates so that

$$E \cup_p R \subset \{x_6 = x_7 = \cdots = x_{3g-4} = 0\}$$

and $D \subset \{x_0 = x_1 = x_2 = x_3 = 0\}$. $E$ and $R$ can be parametrized by

$$[s,t] \mapsto [s^4, s^2t^2, st^3, t^4, 0, \dots, 0]$$

and

$$[u,v] \mapsto [0,0,uv,u^2,v^2,0,\dots,0].$$

The cusp is at $[1,0,\dots,0]$, $p = [0,0,0,1,0,\dots,0]$, and $q = [0,0,0,0,1,0,\dots,0]$. Let $\rho$ be the 1-PS with weight $(0,2,3,4,2,\dots,2)$. We have

$$e_\rho(C) \geq e_\rho(E)_p + e_\rho(R)_p + e_\rho(R)_q + e_\rho(D).$$

On $E$ (and $R$), we have $\nu_p(x_i) + r_i \geq 4$ for all $i$ where $\nu_p$ is the valuation of $\mathcal{O}_{E,p}$ (and $\mathcal{O}_{R,p}$ respectively) and $r_i$ are weights of $\rho$. By Lemma 3.3, $e_\rho(E)_p \geq 4^2$ and $e_\rho(R)_p \geq 4^2$. On $R$, $\nu_q(x_i) + r_i \geq 2$ and $e_\rho(R)_q \geq 2^2$. Since $\rho$ acts trivially on $D$ with weight 2, we use Lemma 3.4 and obtain

$$e_\rho(D) = 2 \cdot 2 \cdot \deg D = 4(4g-10).$$

Combining them all, we obtain

$$e_\rho(C) \geq 36 + 16g - 40 > 2 \cdot \frac{4}{3} \sum_{i=1}^{3g-4} r_i = 16g - \frac{40}{3}.$$

□

**Corollary 7.9.** *Let* $C' = E' \cup_p D$ *be a bicanonical curve consisting of a genus one curve* $E'$ *and a genus* $g-2$ *curve* $D$ *meeting in one tacnode* $p$. *Then* $C'$ *is Chow unstable.*

*Proof.* In view of Proposition 7.8, it suffices to show that $C'$ is in the basin of attraction of $E \cup_p R \cup_q D$ with respect to $\rho$. Consider the induced action on the local versal deformation space of the cusp $[1, 0, \ldots, 0]$ which is given by

$$y^2 = x^3 + ax + b$$

where $y = x_2/x_0$ and $x = x_1/x_0$. The $\mathbb{G}_m$ action is given by

$$t.(a, b) = (t^4 a, t^6 b)$$

and the basin of attraction contains arbitrary smoothing of the cusp. On the other hand, the local versal deformation space of the tacnode $p = [0, 0, 0, 1, 0, \ldots, 0]$ is given by

$$y^2 = x^4 + ax^2 + bx + c$$

where $x = x_2/x_3$ so that $\mathbb{G}_m$ acts on $(a, b, c)$ with weight $(-2, -3, -4)$ and the basin of attraction does not contain any smoothings of the tacnode. At the node $q = [0, 0, 0, 0, 1, 0, \ldots, 0]$, the local versal deformation space is

$$xy = c_0$$

where $x$ may be taken to be $x_2/x_4$ and $\mathbb{G}_m$ acts with weight $+1$ on the branch of $R$ and trivially on $D$. Thus the induced action on the deformation space has weight $+1$, and the basin of attraction contains arbitrary smoothing of the node. □

7.4. **Hilbert unstable curves.** Let $C$ be a bicanonical curve. By Proposition 3.13, $C$ is Chow semistable if it is Hilbert semistable. Note that by definition, if $C$ does not admit an elliptic chain, then $C$ is c-semistable if and only if it is h-semistable. Combining this with Proposition 7.1, we obtain:

**Proposition 7.10.** *If a bicanonical curve is Hilbert semistable and does not admit an elliptic chain, then it is h-semistable.*

We shall have completed the implication

$$\text{Hilbert semistable} \Rightarrow \text{h-semistable}$$

once we prove that a Hilbert semistable curve does not admit an elliptic chain. We accomplish this in Proposition 10.4 and Corollary 10.9.

## 8. Classification of curves with automorphisms

In this section, we classify c-semistable curves with infinite automorphisms.

8.1. **Rosaries.**

**Definition 8.1.** An *open rosary*[1] $R_r$ of length $r$ is a two-pointed connected curve $(R_r, p, q)$ such that

- $R_r = L_1 \cup_{a_1} L_2 \cup_{a_2} \cdots \cup_{a_{r-1}} L_r$ where $L_i$ is a smooth rational curve, $i = 1, \ldots, r$;
- $L_i$ and $L_{i+1}$ meet each other in a single tacnode $a_i$, for $i = 1, \ldots, r-1$;
- $L_i \cap L_j = \emptyset$ if $|i - j| > 1$;
- $p \in L_1$ and $q \in L_r$ are smooth points.

**Remark 8.2.** An open rosary of length $r$ has arithmetic genus $r - 1$. Note that an open rosary of length $r = 2r'$ is naturally an open elliptic chain of length $r'$.

**Definition 8.3.** We say that a curve $C$ *admits an open rosary* or length $r$ if there is a 2-pointed open rosary $(R_r, p, q)$ and a morphism $\iota : R_r \to C$ such that

- $\iota$ is an isomorphism onto its image over $R_r \setminus \{p, q\}$;
- $\iota(p), \iota(q)$ are nodes of $C$; we allow the case $\iota(p) = \iota(q)$.

A *closed rosary* $C$ is a curve admitting $\iota : C' \to C$ as above with the second condition replaced by

- $\iota(p) = \iota(q)$ at a tacnode of $C$.

A *closed rosary with broken beads* is a curve expressible as a union of open rosaries.

**Remark 8.4.** If $C$ admits an open rosary of length $r \geq 2$ then $C$ admits a weak elliptic chain. If $r$ is even then $C$ admits an elliptic chain. Thus a closed rosary of even length is also a closed weak elliptic chain.

**Proposition 8.5.** *Consider the closed rosaries of genus* $r+1$. *If the genus is even then there is a unique closed rosary* $C$ *(of the given genus) and the automorphism group* $\mathrm{Aut}(C)$ *is finite. If the genus is odd then the closed rosaries depend on one modulus and the connected component of the identity* $\mathrm{Aut}(C)^\circ$ *is isomorphic to* $\mathbb{G}_m$.

*There is a unique open rosary* $(R, p, q)$ *of length* $r$. *If* $\mathrm{Aut}(R, p, q)$ *denotes the automorphisms fixing* $p$ *and* $q$ *then*

$$\mathrm{Aut}(R, p, q)^\circ \simeq \mathbb{G}_m.$$

[1]This name was suggested to us by Jamie Song.

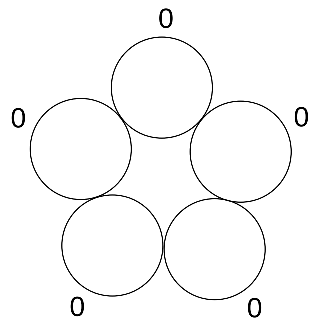


FIGURE 5. Closed rosary of genus six

*It acts on tangent spaces of the endpoints with weights satisfying*

$$wt_{\mathbb{G}_m}(T_pR) = (-1)^r wt_{\mathbb{G}_m}(T_qR).$$

*Proof.* Let $C$ be a closed $r$-rosary obtained by gluing $r$ smooth rational curves $\{[s_i, t_i]\}$ so that

$$\frac{\partial}{\partial(s_r/t_r)} = \alpha_r \frac{\partial}{\partial(t_1/s_1)}; \quad \frac{\partial}{\partial(s_i/t_i)} = \alpha_i \frac{\partial}{\partial(t_{i+1}/s_{i+1})}, \quad i = 1, 2, \dots, r-1.$$

Let $C'$ be another such rosary with the gluing data

$$\frac{\partial}{\partial(s'_r/t'_r)} = \alpha'_r \frac{\partial}{\partial(t'_1/s'_1)}; \quad \frac{\partial}{\partial(s'_i/t'_i)} = \alpha'_i \frac{\partial}{\partial(t'_{i+1}/s'_{i+1})}, \quad i = 1, 2, \dots, r-1.$$

Consider the morphism $f : \widetilde{C} \to \widetilde{C}'$ between the normalizations of $C$ and $C'$ given by $[s_i, t_i] \mapsto [\beta_i s'_i, t'_i]$. For $f$ to descend to an isomorphism from $C$ to $C'$, the following is necessary and sufficient:

$$df\left(\frac{\partial}{\partial(s_i/t_i)}\right) = \frac{\partial}{\beta_i \partial(s'_i/t'_i)} = \frac{\alpha'_i}{\beta_i}\frac{\partial}{\partial(t'_{i+1}/s'_{i+1})} = \alpha_i\beta_{i+1}\frac{\partial}{\partial(t'_{i+1}/s'_{i+1})} = df\left(\alpha_i \frac{\partial}{\partial(t_{i+1}/s_{i+1})}\right)$$

This gives rise to $\beta_i\beta_{i+1} = \alpha'_i/\alpha_i$ and $\beta_r\beta_1 = \alpha'_r/\alpha_r$. Solving for $\beta_i$, we get

$$\beta_i = \begin{cases} \frac{\alpha'_i \alpha_{i+1} \alpha'_{i+2} \cdots \alpha_r}{\alpha_i \alpha'_{i+1} \alpha_{i+2} \cdots \alpha'_r} \beta_1, & \text{if } r - i \text{ is odd} \\ \\ \frac{\alpha'_i \alpha_{i+1} \alpha'_{i+2} \cdots \alpha'_r}{\alpha_i \alpha'_{i+1} \alpha_{i+2} \cdots \alpha_r} \beta_1^{-1}, & \text{if } r - i \text{ is even} \end{cases}$$

When $r$ is odd, there is no constraint and all $r$-rosaries are isomorphic. When $r = 2k$,

$$(\beta_1\beta_2)(\beta_3\beta_4)\cdots(\beta_{2k-1}\beta_{2k}) = (\beta_2\beta_3)(\beta_4\beta_5)\cdots(\beta_{2k}\beta_1)$$

forces the condition

$$\frac{\alpha_1'\alpha_3'\cdots\alpha_{2k-1}'}{\alpha_1\alpha_3\cdots\alpha_{2k-1}} = \frac{\alpha_2'\alpha_4'\cdots\alpha_{2k}'}{\alpha_2\alpha_4\cdots\alpha_{2k}}. \tag{8.1}$$

This means that the $2k$-rosaries are parametrized by

$$\frac{\alpha_1\alpha_3\cdots\alpha_{2k-1}}{\alpha_2\alpha_4\cdots\alpha_{2k}} \in \mathbb{G}_m.$$

To describe the automorphisms we take $C' = C$. When $r$ is odd we get $\beta_i = \beta_i^{-1}$ for each $i$ which implies that $\mathrm{Aut}(C)^\circ$ is trivial. When $r = 2k$ we get a unique solution

$$\beta_1 = \beta_2^{-1} = \beta_3 = \ldots = \beta_{2k}^{-1}$$

and thus $\mathrm{Aut}(C)^\circ \simeq \mathbb{G}_m$.

The open rosary case entails exactly the same analysis, except that we omit the gluing datum

$$\frac{\partial}{\partial(s_r/t_r)} = \alpha_r \frac{\partial}{\partial(t_1/s_1)}$$

associated with the end points. Thus we get a $\mathbb{G}_m$-action regardless of the parity of $r$. Our assertion on the weights at the distinguished points $p$ and $q$ follows from the computation above of the action on tangent spaces. □

**Definition 8.6.** By *breaking the* $i$*th bead* of a rosary (open or closed), we mean replacing $L_i$ with a union $L_i' \cup L_i''$ of smooth rational curves meeting in a node such that $L_i'$ meets $L_{i-1}$ in a tacnode $a_{i-1}$ and $L_i''$ meets $L_{i+1}$ in a tacnode $a_{i+1}$ (Figure 6).

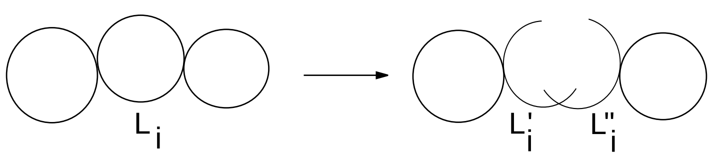


FIGURE 6. Breaking a bead of a rosary

8.2. **Classification of automorphisms.**

**Proposition 8.7.** *A c-semistable curve* $C$ *of genus* $\geq 4$ *has infinite automorphisms if and only if*

(1) $C$ *admits an open rosary of length* $\geq 2$, *or*

(2) $C$ *is a closed rosary of odd genus (possibly with broken beads).*

*Proof.* We have already seen in Proposition 8.5 that closed rosaries of odd genus have infinite automorphisms.

Let $C$ be a c-semistable curve of genus $g \geq 4$ that is not a closed rosary. For $C$ to have infinitely many automorphisms, it must have a smooth rational component, say $C_1$. To satisfy the stability condition and still give rise to infinite automorphisms, $C_1$ has to meet the rest of the curve in one node and a tacnode, or in two tacnodes. We examine each case below:

(1) $C_1$ meets the rest in one node $a_0$ and in a tacnode $a_1$: For the automorphisms of $C_1$ to extend to automorphisms of $C$, the irreducible component $C_2(\neq C_1)$ containing $a_1$ must be a smooth rational component - this follows easily from that an automorphism of $C$ lifts to an automorphism of its normalization. Also, $C_2$ has to meet the rest of the curve in one point $a_2$ other than $a_1$ since otherwise $C_1 \cup C_2$ would be an elliptic tail (or $a_1 = a_0$ and $C$ is of genus two).

(2) $C_1$ meets the rest in two tacnodes $a_0$ and $a_1$: For the automorphisms to extend to $C$, the components $C_0 \neq C_1$ containing $a_0$ and $C_2 \neq C_1$ containing $a_1$ must be smooth rational curves. Hence $C$ contains $C_0 \cup C_1 \cup C_2$ which is a rosary of length three. Moreover, $C_0$ and $C_2$ do not intersect: If they do meet, say at $a_2$, then either $C = C_0 \cup C_1 \cup C_2$ and the genus of $C$ is of genus three (if $a_2$ is a node) or $C$ is a closed rosary if $a_2$ is a tacnode.

Iterating, we eventually produce an open rosary $\iota : R_r \to C$ of length $r \geq 2$ containing $C_1$ as a bead. □

**Corollary 8.8.** *An h-semistable curve* $C$ *of genus* $\geq 4$ *has infinite automorphisms if and only if*

(1) $C$ *admits an open rosary of odd length* $\geq 3$, *or*
(2) $C$ *is a closed rosary of odd genus (possibly with broken beads).*

Let $C$ be a c-semistable curve and suppose $D$ is a Deligne-Mumford stabilization of $C$. In other words, there exists a smoothing of $C$

$$\varpi : \mathcal{C} \to T$$

such that $D = \lim_{t \to t_0} \mathcal{C}_t$ in the moduli space of stable curves. Here, a smoothing is a flat proper morphism to a smooth curve with distinguished point $(T, t_0)$ such that $\varpi^{-1}(t_0) = C$ and the generic fiber is smooth.

Our classification result (Proposition 8.7) has the following immediate consequence:

**Corollary 8.9.** *Suppose* $C$ *is a c-semistable curve with infinite automorphism group. Then the Deligne-Mumford stabilization and the pseudo-stabilization of* $C$ *admit an elliptic bridge.*

Indeed, $C$ necessarily admits a tacnode which means that its stabilization contains a connected subcurve of genus one meeting the rest of the curve in two points.

## 9. Interpreting the flip via GIT

We will eventually give a complete description of the semistable and stable points of $\mathsf{Chow}_{g,2}$ and $\mathsf{Hilb}_{g,2}$. For our immediate purpose, the following partial result will suffice:

**Theorem 9.1.** *If* $C$ *is c-stable, i.e., a pseudostable curve admitting no elliptic bridges, then* $\mathsf{Ch}(C) \in \mathsf{Chow}^s_{g,2}$. *Thus the Hilbert point* $[C]_m \in \mathsf{Hilb}^{s,m}_{g,2}$ *for* $m \gg 0$.

*Proof.* The GIT-stable loci $\mathsf{Chow}^s_{g,2}$ and $\mathsf{Hilb}^{s,m}_{g,2}, m \gg 0$, contain the nonsingular curves by [Mum77, 4.15]. (We discussed the relation between Chow and asymptotic Hilbert stability in Section 3.6.)

Recall that Proposition 6.1 guarantees that c-semistable curves admits bicanonical embeddings. In particular, this applies to pseudostable curves without elliptic bridges.

Suppose that $C$ is a singular pseudostable curve without elliptic bridges. Assume that $\mathsf{Ch}(C)$ is *not* in $\mathsf{Chow}^s_{g,2}$. If $\mathsf{Ch}(C)$ is strictly semistable then it is c-equivalent to a semistable curve $C'$ with infinite automorphism group. It follows that $C$ is a pseudo-stabilization of $C'$, and we get a contradiction to Corollary 8.9. Suppose $\mathsf{Ch}(C)$ is unstable and let $C'$ denote a semistable replacement (see Theorem 4.5). By uniqueness of the pseudo-stabilization, $C'$ is not pseudostable but has $C$ as its pseudo-stabilization. It follows that $C'$ has a tacnode. However, the pseudo-stabilization of such a curve necessarily contains an elliptic bridge. □

With our current partial understanding of the GIT of bicanonical curves, we are ready to prove Theorem 2.12. Our main task is to establish Isomorphisms (2.5) and (2.6). Proposition 2.9 established the existence of a birational contraction morphism $\Psi : \overline{M}_g^{ps} \to \overline{M}_g(7/10)$. The first step here is to show that $\overline{M}_g^{cs}$ and $\overline{M}_g^{hs}$ are birational contractions of $\overline{M}_g$ and small contractions of $\overline{M}_g^{ps}$. In particular, we may identify the divisor class groups of these GIT quotients with the divisor class group of $\overline{M}_g^{ps}$ (which in turn is a subgroup of the divisor class group of $\overline{M}_g$). Furthermore, we obtain

$$\Gamma(\overline{M}_g^{ps}, n(K_{\overline{M}_g^{ps}} + \alpha\delta^{ps})) \simeq \Gamma(\overline{M}_g^{hs}, n(K_{\overline{M}_g^{hs}} + \alpha\delta^{hs})) \simeq \Gamma(\overline{M}_g^{cs}, n(K_{\overline{M}_g^{cs}} + \alpha\delta^{cs}))$$

and Lemma 2.8 gives

$$\begin{aligned} \overline{M}_g(7/10) &\simeq \mathrm{Proj}\left(\oplus_{n\geq 0}\Gamma(n(K_{\overline{M}_g^{cs}} + 7/10\delta^{cs}))\right) \\ \overline{M}_g(7/10-\epsilon) &\simeq \mathrm{Proj}\left(\oplus_{n\geq 0}\Gamma(n(K_{\overline{M}_g^{hs}} + (7/10-\epsilon)\delta^{hs}))\right). \end{aligned} \tag{9.1}$$

The second step is to compute the induced polarizations of $\overline{M}_g^{cs}$ and $\overline{M}_g^{hs}$ in the divisor class group of $\overline{M}_g^{ps}$. This will show that $K_{\overline{M}_g^{cs}} + 7/10\delta^{cs}$ (resp. $K_{\overline{M}_g^{hs}} + (7/10-\epsilon)\delta^{hs}$) is ample on $\overline{M}_g^{cs}$ (resp. $\overline{M}_g^{hs}$). Isomorphisms (2.5) and (2.6) then follow from (9.1).

To realize our GIT quotients as contractions of $\overline{M}_g$, we apply Theorem 5.4 in the bicanonical case. Consider the complement of the Deligne-Mumford stable curves $V_{g,2}^{ss,\infty}$ in the GIT-semistable locus $\mathrm{Chow}_{g,2}^{ss}$; we must show this has codimension $\geq 2$. Since $\varpi(\mathrm{Hilb}_{g,2}^{ss,m}) \subset \mathrm{Chow}_{g,2}^{ss}$ and $\varpi|\mathrm{Hilb}_{g,2}^{s,m}$ is an isomorphism where $\varpi$ denotes the cycle class map from $\mathrm{Hilb}$ to $\mathrm{Chow}$, the analogous statement for the Hilbert scheme follows immediately.

Proposition 7.1 implies $\mathrm{Chow}_{g,2}^{ss} \setminus V_{g,2}^{ss,\infty}$ parametrizes

- pseudostable curves that are not Deligne-Mumford stable, i.e., those with cusps; and
- c-semistable curves with tacnodes.

The cuspidal pseudostable curves have codimension two in moduli; the tacnodal curves have codimension three. Indeed, a generic tacnodal curve of genus $g$ is determined by a two-pointed curve $(C', p, q)$ of genus $g-2$ and an isomorphism $T_pC' \simeq T_qC'$. We conclude there exist rational contractions $F^{cs} : \overline{M}_g \dashrightarrow \overline{M}_g^{cs}$ and $F^{hs} : \overline{M}_g \dashrightarrow \overline{M}_g^{hs}$.

It remains to show that we have small contractions $G^{cs} : \overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{cs}$ and $G^{hs} : \overline{M}_g^{ps} \dashrightarrow \overline{M}_g^{hs}$. To achieve this, we must establish that $\Delta_1$ is the unique exceptional divisor of $F^{cs}$ (resp. $F^{hs}$). The exceptional locus of $F^{cs}$ (resp. $F^{hs}$)

lies in the complement to the GIT-stable curves in the moduli space

$$\overline{M}_g \setminus U^{s,\infty}_{g,2} \quad (\text{resp. } \overline{M}_g \setminus U^{s,m}_{g,2}).$$

Chow stable points are asymptotically Hilbert stable (cf. Proposition 3.13), i.e., $U^{s,\infty}_{g,2} \subset U^{s,m}_{g,2}$ when $m \gg 0$. It suffices then to observe that $\delta_1$ is the unique divisorial component of $\overline{M}_g \setminus U^{s,\infty}_{g,2}$, which is guaranteed by Theorem 9.1.

Theorem 5.4 gives moving divisors on $\overline{M}_g$ inducing the contractions $F^{cs}$ and $F^{hs}$. We apply Equation (5.3), which in our situation takes the form

$$r(2)\lambda_{m2} - r(2m)m\lambda_2 = (m-1)(g-1)((20m-3)\lambda - 2m\delta) \sim (10 - \frac{3}{2m})\lambda - \delta;$$

this approaches $10\lambda - \delta$ as $m \to \infty$. Thus we have

$$(F^{hs})^*\mathcal{L}_m \sim (10 - \frac{3}{2m})\lambda - \delta \pmod{\delta_1}, \quad m \gg 0$$

and

$$(F^{cs})^*\mathcal{L}_\infty \sim 10\lambda - \delta \pmod{\delta_1}.$$

Using the identity

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta$$

we obtain (for $m \gg 0$)

$$(F^{hs})^*\mathcal{L}_m \sim K_{\overline{\mathcal{M}}_g} + (7/10 - \epsilon(m)))\delta \pmod{\delta_1}, \quad \epsilon(m) = 39/(200m - 30)$$

and

$$(F^{cs})^*\mathcal{L}_\infty \sim K_{\overline{\mathcal{M}}_g} + 7/10\delta \pmod{\delta_1}.$$

It follows then that

$$(G^{hs})^*\mathcal{L}_m \sim K_{\overline{M}^{ps}_g} + (7/10 - \epsilon(m)))\delta^{ps}$$

and

$$(G^{cs})^*\mathcal{L}_\infty \sim K_{\overline{M}^{ps}_g} + 7/10\delta^{ps}.$$

The proof of Theorem 2.12 will be complete if we can show that $\Psi^+$ is the flip of $\Psi$. More precisely, for small positive $\epsilon \in \mathbb{Q}$, $\Psi^+$ is a small modification of $\overline{M}^{ps}_g$ with $K_{\overline{M}^{hs}_g} + (7/10 - \epsilon)\delta^{hs}$ ample. Since $\overline{M}^{cs}_g$ and $\overline{M}^{hs}_g$ ar both small contractions of $\overline{M}^{ps}_g$, $\Psi^+$ is small as well. And the polarization we exhibited on $\overline{M}^{hs}_g$ gives the desired positivity, which completes the proof of Theorem 2.12.

## 10. Stability under one-parameter subgroups

In this section, we analyze whether c-semistable curves are GIT-semistable with respect to the one-parameter subgroups of their automorphism group. We shall also use deformation theory to classify the curves that belong to basins of attraction of such curves.

Our analysis will focus primarily on the Hilbert points. Indeed, Corollary 6.4 shows that we can recover the sign of the Hilbert-Mumford index of the Chow point from the indices of the Hilbert points. And in view of the cycle map $\varpi : \mathrm{Hilb}_{g,2} \to \mathrm{Chow}_{g,2}$, if $[C]_m \in A_\rho([C^\star]_m)$ for $m \gg 0$ then $\mathrm{Ch}(C) \in A_\rho(\mathrm{Ch}(C^\star))$.

### 10.1. **Stability analysis: Open rosaries.**

**Proposition 10.1.** *Let $C = D \cup_{a_0,a_{r+1}} R$ be a c-semistable curve of genus $g$ consisting of a genus $g-r-1$ curve $D$ meeting the genus $r$ curve $R$ in two nodes $a_0$ and $a_{r+1}$ where*

$$R := L_1 \cup_{a_1} L_2 \cup_{a_2} \cdots \cup_{a_r} L_{r+1}$$

*is a rosary of length $r+1$, and $D \cap L_1 = \{a_0\}$ and $D \cap L_{r+1} = \{a_{r+1}\}$. There is a one-parameter subgroup $\rho$ coming from the automorphisms of $C$ of the rosary $R$ such that for all $m \geq 2$,*

1. $\mu([C]_m, \rho) = 0$ *if $r$ is even;*
2. $\mu([C]_m, \rho) = -m + 1$ *if $r$ is odd.*

*In particular, $C$ is Hilbert unstable if $R$ is of even length and strictly semistable otherwise.*

An application of Corollary 6.4 then yields:

**Corollary 10.2.** *Let $C$ and $\rho$ be as in Proposition 10.1. Then $C$ is Chow strictly semistable with respect to $\rho$ and $\rho^{-1}$.*

*Proof.* Upon restricting $\omega_C$ to $D$ and each component of $L$, we get

- $\omega_C|_D \simeq \omega_D(a_0 + a_{r+1})$;
- $\omega_C|_{L_1} \simeq \omega_{L_1}(a_0 + 2a_1)$;
- $\omega_C|_{L_{r+1}} \simeq \omega_{L_{r+1}}(a_{r+1} + 2a_r)$;
- $\omega_C|_{L_i} \simeq \omega_{L_i}(2a_{i-1} + 2a_i), \quad 2 \leq i \leq r$.

Hence we may choose coordinates $x_0, \ldots, x_N$, $N = 3g-4$, such that

(1) $L_1$ is parametrized by

$$[s_1, t_1] \mapsto [s_1^2, s_1 t_1, t_1^2, 0, \dots, 0];$$

(2) $L_{r+1}$ is parametrized by

$$[s_{r+1}, t_{r+1}] \mapsto [\underbrace{0, \dots, 0}_{3r-2}, s_{r+1} t_{r+1}, s_{r+1}^2, t_{r+1}^2, 0, \dots, 0].$$

(3) For $2 \leq j \leq r$, $L_j$ is parametrized by

$$[s_j, t_j] \mapsto [\underbrace{0, \dots, 0}_{3j-5}, s_j^3 t_j, s_j^4, s_j^2 t_j^2, s_j t_j^3, t_j^4, 0, \dots, 0].$$

(4) $D$ is contained in the linear subspace

$$x_1 = x_2 = \dots = x_{3r-1} = 0$$

and $a_0 = [1, 0, \dots, 0]$ and $a_{r+1} = [\underbrace{0, \dots, 0}_{3r}, 1, 0, \dots, 0]$.

From the parametrization, we obtain a set of generators for the ideal of $L$:

$$\begin{aligned} &x_1^2 - x_0 x_2 - x_2 x_3, x_0 x_3, x_0 x_4, \dots, x_0 x_{3r}, \\ &x_i x_{i+5}, x_i x_{i+6}, \dots, x_i x_{3r}, \quad i = 1, 2, \dots, 3r-5, \end{aligned} \tag{10.1}$$

and for $j = 1, 2, \dots, r-1$,

(10.2)

$$\begin{aligned} &x_{3j-1} x_{3j+3},\ x_{3j} x_{3j+3},\ x_{3j} x_{3j+4},\ x_{3j+1}^2 - x_{3j} x_{3j+2} - x_{3j+2} x_{3j+3},\ x_{3j}^2 - x_{3j-1} x_{3j+2}, \\ &x_{3j-2} x_{3j+1} - x_{3j-1} x_{3j+2},\ x_{3j-2} x_{3j} - x_{3j-1} x_{3j+1},\ x_{3j-2} x_{3j+2} - x_{3j} x_{3j+1}. \end{aligned}$$

In Proposition 8.5 we showed that $\mathbb{G}_m$ acts on the open rosary via automorphisms. With respect to our coordinates, this is the one-parameter subgroup $\rho$ with weights:

$$\begin{cases} (2, 1, 0, 2, 3, 4, 2, 1, 0, \dots, 2, 3, 4, \underbrace{2, 2, \dots, 2}_{N-3r}), & \text{if } r \text{ is even} \\ (2, 1, 0, 2, 3, 4, 2, 1, 0, \dots, 2, 1, 0, \overbrace{2, 2, \dots, 2}), & \text{if } r \text{ is odd.} \end{cases}$$

By considering the parametrization, it is easy to see that $C$ is stable under the action of $\rho$.

Now we shall enumerate the degree two monomials in the initial ideal of $C$. From (10.1) and (10.2), we get the following monomials in $x_0, \dots, x_{3r}$:

$$\begin{aligned}
&x_0x_2, x_0x_3, x_0x_4, \dots, x_0x_{3r}, \\
&x_ix_{i+5}, x_ix_{i+6}, \dots, x_ix_{3r}, \quad i = 1, 2, \dots, 3r-5, \\
&x_{3j-1}x_{3j+3}, x_{3j}x_{3j+3}, x_{3j}x_{3j+4}, x_{3j}x_{3j+2}, \\
&x_{3j-1}x_{3j+2}, x_{3j-2}x_{3j+1}, x_{3j-2}x_{3j}, x_{3j-2}x_{3j+2}, \quad j = 1, 2, \dots, r-1.
\end{aligned} \tag{10.3}$$

The weights of these $(9r^2 - 5r)/2$ monomials sum up to give

$$\begin{cases} 18r^2 - 10r, & \text{if } r \text{ is even;} \\ 18r^2 - 19r + 7, & \text{if } r \text{ is odd.} \end{cases}$$

The total weight $\sum_{i \leq j,\, 0 \leq i,j \leq 3r} wt_\rho(x_ix_j)$ of all degree two monomials in $x_0, \dots, x_{3r}$ is

$$\begin{cases} (3r+2)(6r+2), & \text{if } r \text{ is even;} \\ (3r+2)(6r-1), & \text{if } r \text{ is odd.} \end{cases}$$

Hence the degree two monomials in $x_0, \dots, x_{3r}$ that are *not* in the initial ideal contributes, to the total weight,

$$\begin{cases} (3r+2)(6r+2) - (18r^2 - 10r) = 28r + 4, & \text{if } r \text{ is even;} \\ (3r+2)(6r-1) - (18r^2 - 19r + 7) = 28r - 9, & \text{if } r \text{ is odd.} \end{cases} \tag{10.4}$$

The rest of the contribution comes from the monomials supported on the component $D$: These are the degree two monomials in $x_0, x_{3r}, x_{3r+1}, \dots, x_N$ that vanish at $a_0$ and $a_{r+2}$. The number of such monomials is, by Riemann-Roch,

$$h^0(D, \mathcal{O}_D(2)(-a_0 - a_{r+2})) = 7(g - r - 1) - 1. \tag{10.5}$$

Since $wt_\rho(x_i) = 2$ for all $i = 0, 3r, 3r+1, \dots, N$, these monomials contribute $28g - 28r - 32$ to the sum. Combining (10.4) and (10.5), we find the sum of the weights of the degree two monomials not in $in(C)$ to be

$$\begin{cases} 28g - 28, & \text{if } r \text{ is even;} \\ 28g - 41, & \text{if } r \text{ is odd.} \end{cases}$$

On the other hand, the average weight is

$$\frac{2 \cdot P(2)}{N+1} \sum_{i=0}^{N} wt_\rho(x_i) = \begin{cases} 28g - 28, & \text{if } r \text{ is even;} \\ 28g - 42, & \text{if } r \text{ is odd.} \end{cases}$$

Hence by (3.1), we find that $\mu([C]_2, \rho) = 0$ if $r$ is even and $\mu([C]_2, \rho) = -1$ if $r$ is odd.

We enumerate the degree three monomials in the same way: The degree three monomials in $x_0, \dots, x_{3r}$ that are in the initial ideal are the multiples of (10.3) together with

$$x_{3j+1}^2 x_{3j+3}, \quad j = 0, 1, \dots, r-1, \tag{10.6}$$

that come from the linear relation

$$x_{3j+2}(x_{3j}x_{3j+3}) - x_{3j+3}(x_{3j+1}^2 - x_{3j}x_{3j+2} - x_{3j+2}x_{3j+3}) = 0.$$

From this, we find that the degree three monomials in $x_0, \dots, x_{3r}$ that are not in the initial ideal contribute

$$\begin{cases} 66r + 6, & \text{if } r \text{ is even;} \\ 66r - 25, & \text{if } r \text{ is odd.} \end{cases}$$

The contribution from $D$ is

$$6h^0(D, \mathcal{O}_D(3)(-a_0 - a_{r+2})) = 6(11g - 11r - 12) = 66g - 66r - 72.$$

Hence the grand total is

$$\sum_{j=1}^{P(3)} wt_\rho(x^{a(j)}) = \begin{cases} 66g - 66, & \text{if } r \text{ is even;} \\ 66g - 97, & \text{if } r \text{ is odd} \end{cases}$$

On the other hand, the average weight is

$$\frac{3P(3)}{N+1} \sum_{i=0}^{N} wt_\rho(x_i) = \begin{cases} 66g - 66, & \text{if } r \text{ is even;} \\ 66g - 99, & \text{if } r \text{ is odd.} \end{cases}$$

Using (3.1), we compute

$$\mu([C]_3, \rho) = \begin{cases} 0, & \text{if } r \text{ is even;} \\ -2, & \text{if } r \text{ is odd.} \end{cases}$$

Corollary 6.4 implies

$$\mu([C]_m, \rho) = \begin{cases} 0, & \text{if } r \text{ is even;} \\ -m+1, & \text{if } r \text{ is odd.} \end{cases}$$

for each $m \geq 2$. □

10.2. **Basin of attraction: Open rosaries.** Let $C$ and $R$ be as in the previous section. Let $x_i, y_i$ be homogeneous coordinates on $L_i$. We may assume that

$$a_0 = [0,1];\ a_{r+1} = [1,0];\ a_i = \begin{cases} [1,0] = \infty \text{ on } L_i \\ [0,1] = 0 \text{ on } L_{i+1} \end{cases}$$

Consider the $\mathbb{G}_m$ action associated to $\rho$. The action on $R$ is given by $(t, [x_i, y_i]) \mapsto [x_i, t^{(-1)^{i-1}} y_i]$ on each $L_i$. Hence it induces an action on the tangent space $T_{a_i} L_i$ given by

$$\left(t, \frac{\partial}{\partial(y_i/x_i)}\right) \mapsto \frac{\partial}{\partial\left(t^{(-1)^{i-1}} y_i/x_i\right)} = t^{(-1)^i} \frac{\partial}{\partial(y_i/x_i)}.$$

There is an induced $\mathbb{G}_m$ action on the Hilbert scheme and $\mathrm{Hilb}_{g,2}$. Corollary 6.3 asserts that a neighborhood of $[C]$ in the Hilbert scheme dominates the product of the versal deformation spaces. These inherit a $\mathbb{G}_m$ action as well, which we shall compute explicitly.

(A) *$\mathbb{G}_m$ action on the versal deformation spaces of nodes $a_0$ and $a_r$*: Let $z$ be a local parameter at $a_0$ on $D$. We have $x_1/y_1$ as a local parameter at $a_0$ on $L_0$ and the local equation at $a_0$ on $C$ is $z \cdot (x_1/y_1) = 0$. Hence the action on the node $a_0$ is given by $(z, x_1/y_1) \mapsto (z, t^{-1}x_1/y_1)$ and the action on the versal deformation space is

$$c_0 \mapsto t^{-1} c_0.$$

Likewise, at $a_{r+1}$, the action on the node is

$$(y_{r+1}/x_{r+1}, z') \mapsto (t^{(-1)^{r+1}} y_{r+1}/x_{r+1}, z')$$

where $z'$ is a local parameter at $a_{r+1}$ on $D$, and the action on the versal deformation space is

$$c_0 \mapsto t^{(-1)^r} c_0.$$

(B) *$\mathbb{G}_m$ action on the versal deformation space of a tacnode $a_i$*: At $a_i$, the local analytic equation is of the form $y^2 = x^4$ where $x := (y_i/x_i, x_{i+1}/y_{i+1})$ and

$y := ((y_i/x_i)^2, -(x_{i+1}/y_{i+1})^2)$ in $k[[y_i/x_i]] \oplus k[[x_{i+1}/y_{i+1}]]$ and the $\mathbb{G}_m$ action at the tacnode is given by

$$t.x = (t^{(-1)^{i-1}} y_i/x_i, x_{i+1}/(t^{(-1)^i} y_{i+1})) = t^{(-1)^{i-1}} x$$
$$t.y = t^{2(-1)^{i-1}} y.$$

Therefore the action on the versal deformation space is

$$(c_0, c_1, c_2) \mapsto (t^{4(-1)^{i-1}} c_0, t^{3(-1)^{i-1}} c_1, t^{2(-1)^{i-1}} c_2).$$

From these observations, we conclude that the basin of attraction of $C$ with respect to $\rho$ contains arbitrary smoothings of $a_{2k+1}$ but no smoothing of $a_{2k}$ for all $0 \le k < \lceil (r+1)/2 \rceil$. We have established:

**Proposition 10.3.** *Retain the notation of Proposition 10.1 and assume that* $m \gg 0$*.*

(1) *If* $r$ *is even (i.e., the length of the rosary is odd) then* $A_\rho([C]_m)$ *(resp.* $A_{\rho^{-1}}([C]_m)$*) parametrizes the curves consisting of* $D$ *and a weak elliptic chain* $C'$ *of length* $r/2$ *meeting* $D$ *in a node at* $a_0$ *and in a tacnode at* $a_{r+1}$ *(resp. in a tacnode at* $a_0$ *and in a node at* $a_{r+1}$*) (Figure 7);*
(2) *If* $r$ *is odd (i.e., the length of the rosary is even) then* $A_\rho([C]_m)$ *(resp.* $A_{\rho^{-1}}([C]_m)$*) parametrizes the curves consisting of* $D$ *and an elliptic chain* $C'$ *of length* $(r+1)/2$ *(resp. length* $(r-1)/2$*) meeting* $D$ *in a node (resp. tacnode) at* $a_0$ *and* $a_{r+1}$ *(Figure 8). When* $r = 1$*,* $A_\rho([C]_m)$ *consists of tacnodal curves normalized by* $D$*.*

It follows from Proposition 10.3 and Proposition 10.1 that

**Proposition 10.4.** *If a bicanonical curve admits an open elliptic chain then it is Hilbert unstable. In particular, a bicanonical curve with an elliptic bridge is Hilbert unstable.*

The closed case can be found in Proposition 10.8 and Corollary 10.9.

### 10.3. **Stability analysis: Closed rosaries.**

**Proposition 10.5.** *Let* $C$ *be a bicanonical closed rosary of even length* $r$*. Then* $C$ *is Hilbert strictly semistable with respect to the one-parameter subgroup* $\rho : \mathbb{G}_m \to \mathrm{SL}(3r)$ *arising from* $\mathrm{Aut}(C)$*.*

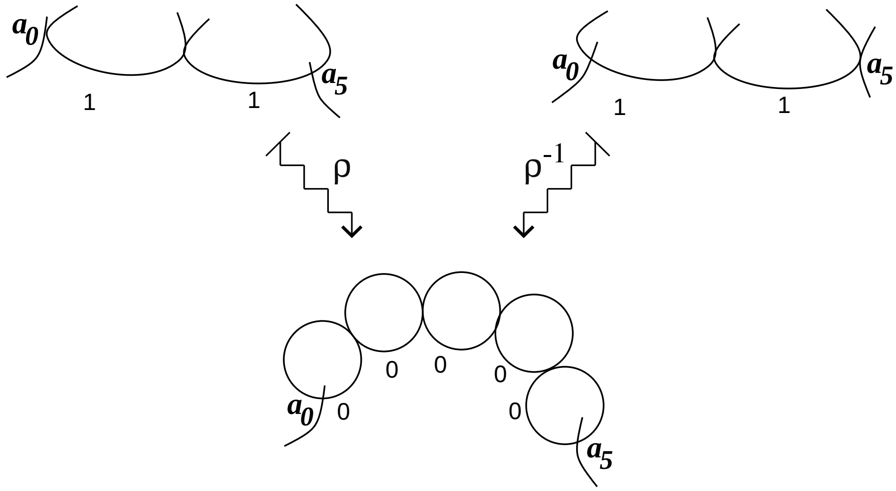


FIGURE 7. Basin of attraction of an open rosary of length five

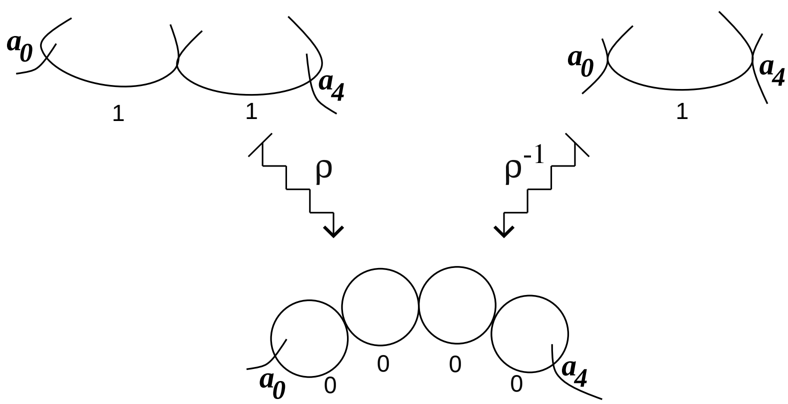


FIGURE 8. Basin of attraction of an open rosary of length four

The relevant one-parameter subgroup was introduced in Proposition 8.5.

*Proof.* Restricting $\omega_C^{\otimes 2}$ to each component $L_i$, we find that each $L_i$ is a smooth conic in $\mathbb{P}^{3g-4}$. We can choose coordinates $x_0, \dots, x_N$ such that $L_i$ is parametrized by

- $[s_i, t_i] \mapsto [\underbrace{0, \dots, 0}_{3(i-1)}, s_i^3 t_i, s_i^4, s_i^2 t_i^2, s_i t_i^3, t_i^4, 0, \dots, 0], \quad i = 1, \dots, r-1;$
- $[s_r, t_r] \mapsto [s_r t_r^3, t_r^4, 0, \dots, 0, s_r^3 t_r, s_r^4, s_r^2 t_r^2]$

The normalization of $C$ admits the automorphisms given by

$$[s_i, t_i] \mapsto [\alpha^{\mathrm{sgn}(i)} s_i,\ \alpha^{1-\mathrm{sgn}(i)} t_i], \quad \mathrm{sgn}(i) := i - 2\lfloor i/2 \rfloor,$$

for $i = 1, \ldots, r-1$ and $[s_r, t_r] \mapsto [s_r, \alpha t_r]$. The one-parameter subgroup $\rho$ associated to this automorphism has weights

$$(3, 4, 2, 1, 0, 2, \cdots, 3, 4, 2, 1, 0, 2)$$

The sum of the weights $\sum_{i=1}^{N} wt_\rho(x_i)$ is $6r$ if $r$ is even and $6r + 3$ if $r$ is odd.

From the parametrization, we obtain a set of generators for the ideal of $C$:

(10.7)

$$\begin{array}{l} x_0x_5, x_0x_6, \ldots, x_0x_{3r-4},\ x_1x_5, x_1x_6, \ldots, x_1x_{3r-4}, \\ x_ix_{i+5},\ x_ix_{i+6},\ \ldots,\ x_ix_{3r-1}, \quad i = 2, \ldots, 3r-6, \\ x_{3j-2}x_{3j+2}, x_{3j-1}x_{3j+2}, x_{3j-1}x_{3j+3}, x_{3j}x_{3j+2}, \\ x_0^2 - x_1x_2 - x_1x_{3r-1},\ x_{3j}^2 - x_{3j-1}x_{3j+1} - x_{3j+1}x_{3j+2}, \\ x_{3r-1}^2 - x_1x_{3r-2},\ x_{3j-1}^2 - x_{3j-2}x_{3j+1}, \\ x_0x_{3r-3} - x_1x_{3r-2},\ x_{3j-3}x_{3j} - x_{3j-2}x_{3j+1}, \\ x_0x_{3r-1} - x_1x_{3r-3}, x_0x_{3r-2} - x_{3r-3}x_{3r-1},\ x_{3j-3}x_{3j-1} - x_{3j-2}x_{3j}, \quad j = 1, 2, \ldots, r-1, \\ x_{3j}x_{3j+3}, x_{3j}x_{3j+4}, \quad j = 1, 2, \ldots, r-2. \end{array}$$

We have the following $(9r^2 - 11r)/2$ degree two monomials that are in the initial ideal:

(10.8)

$$\begin{array}{l} x_0^2, x_0x_5, x_0x_6, \ldots, x_0x_{3r-1}, \\ x_1x_5, x_1x_6, \ldots, x_1x_{3r-4}, x_1x_{3r-2}, \\ x_ix_{i+5},\ x_ix_{i+6},\ \ldots,\ x_ix_{3r-1}, \quad i = 2, \ldots, 3r-6, \\ x_{3j-3}x_{3j-1}, x_{3j-3}x_{3j}, x_{3j-2}x_{3j+1}, x_{3j-2}x_{3j+2}, x_{3j-1}x_{3j+1}, x_{3j-1}x_{3j+2}, \quad j = 1, 2, \ldots, r-1, \\ x_{3j-1}x_{3j+3}, x_{3j}x_{3j+4}, \quad j = 1, 2, \ldots, r-2. \end{array}$$

The sum of the weights of these monomials is $34r - 18r^2$. It follows that the sum of the weights of the monomials *not* in the initial ideal is

$$(3r-1)6r - (34r - 18r^2) = 28r$$

which is precisely $\frac{2P(2)}{N+1} \sum_{i=0}^{N} wt_\rho(x_i)$. Hence $\mu([C]_2, \rho) = 0$.

We shall now enumerate the degree three monomials in the initial ideal of $C$. Together with the monomials divisible by the monomials from (10.8), we have the initial terms

$$x_{3r-3}^2 x_{3r-1},\ x_1x_{3r-3}^2,\ x_{3j-2}x_{3j}^2, \quad j = 1, 2, \ldots, r-1 \tag{10.9}$$

that come from the Gröbner basis members

$$x_1x_{3r-3}^2 - x_{3r-1}^3,\ x_{3r-3}^2x_{3r-1} - x_{3r-2}x_{3r-1}^2,\ x_{3j-2}x_{3j}^2 - x_{3j-1}^3, \quad j = 1, 2, \dots, r-1.$$

The degree three monomials in (10.9) and the degree three monomials divisible by monomials in (10.8) have total weight $66r$. This agrees with the average weight $\frac{3P(3)}{N+1}\sum_{i=0}^{N} wt_\rho(x_i) = \frac{3\cdot 11(g-1)}{3g-3}\frac{3g-3}{6}(3+4+2+1+0+2) = 66(g-1) = 66r$. Therefore, $\mu([C]_2, \rho) = 0 = \mu([C]_3, \rho)$ and $C$ is $m$-Hilbert strictly semistable by Corollary 6.4.

Since $Aut(C) \simeq \mathbb{G}_m$, a one-parameter subgroup coming from $Aut(C)$ is of the form $\rho^a$ for some $a \in \mathbb{Z}$, and we have

$$\mu([C]_m, \rho^a) = a\,\mu([C]_m, \rho) = 0.$$

□

### 10.4. **Basin of attraction: Closed rosaries.**

**Proposition 10.6.** *Retain the notation of Proposition 10.5. Then the basin of attraction $A_\rho([C]_m)$ parametrizes the closed weak elliptic chains of length $r/2$ (Figure 9).*

*Proof.* We use the parametrization from the proof of Proposition 10.5. $C$ has tacnodes $a_i = [\underbrace{0, \dots, 0}_{3i+1}, 1, 0, \dots, 0]$, $i = 1, \dots, k$. From the parametrization, we find that the local parameters $x_{3i}/x_{3i+1}$ at $a_i$ to the two branches is acted upon by $\rho$ with weight $(-1)^{i-1}$. It follows that $\rho$ acts on the versal deformation space $(c_0, c_1, c_2)$ of the tacnode $a_i$ with weights $(4(-1)^{i-1}, 3(-1)^{i-1}, 2(-1)^{i-1})$. Hence the basin of attraction $A_\rho([C])$ has arbitrary smoothings of $a_i$ for odd $i$ but no nontrivial deformations of $a_i$ for even $i$.

□

### 10.5. **Stability analysis: Closed rosaries with a broken bead.** 

Closed rosaries with broken beads of *even* genus are unstable:

**Proposition 10.7.** *Let $r \geq 3$ be an odd number and $C_r$ the curve obtained from a closed rosary of length $r$ by breaking a bead. Then there exists a one-parameter subgroup $\rho$ of $Aut(C_r)$ with $\mu([C_r]_m, \rho) = 1 - m$ for each $m \geq 2$, so $[C_r]_m$ is unstable. Furthermore, $Ch(C_r)$ is strictly semistable with respect to $\rho$.*

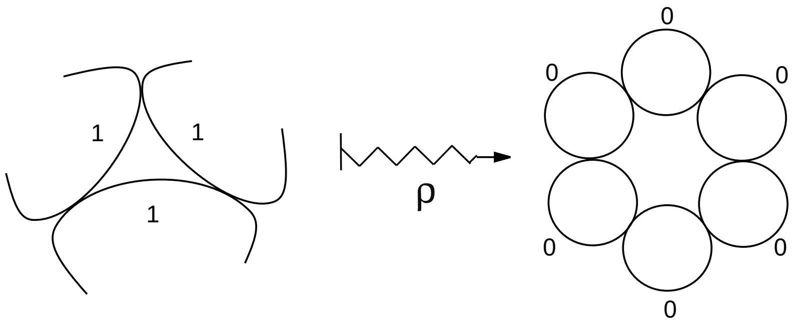


FIGURE 9. Basin of attraction of a closed rosary of length six

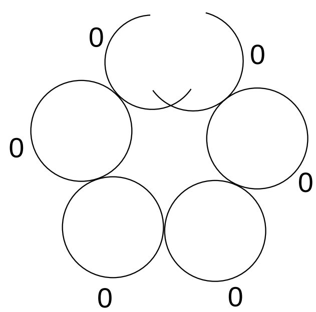


FIGURE 10. Closed rosary of genus six with a broken bead

*Proof.* Note that $C_r$ is unique up to isomorphism and it can parametrized by

$$\begin{aligned} &\bullet (s_0, t_0) &\mapsto\ & (s_0t_0, s_0^2, t_0^2, 0, \dots, 0); \\ &\bullet (s_1, t_1) &\mapsto\ & (0, 0, s_1^2, s_1t_1, t_1^2, 0, \dots, 0); \\ &\bullet (s_i, t_i) &\mapsto\ & (\underbrace{0, \dots, 0}_{3(i-1)}, s_i^3t_i, s_i^4, s_i^2t_i^2, s_it_i^3, t_i^4, 0, \dots, 0), \quad i = 2, \dots, r-1; \\ &\bullet (s_r, t_r) &\mapsto\ & (s_rt_r^3, t_r^4, 0, \dots, 0, s_r^3t_r, s_r^4, s_r^2t_r^2). \end{aligned} \tag{10.10}$$

We give the set of monomials the graded $\rho$-weighted lexicographic order, where $\rho$ is the one-parameter subgroup with the weight vector

$$(1, 0, 2, 1, 0, 2, 3, 4, 2, 1, 0, 2, 3, 4, 2, \dots, 1, 0, 2, 3, 4, 2). \tag{10.11}$$

A Gröbner basis for $C_r$ is:

$$x_0x_3, x_0x_4, \dots, x_0x_{3r-4};\ x_1x_3, x_1x_4, \dots, x_1x_{3r-4};\ x_2x_5, x_2x_6, \dots, x_2x_{3r-1};$$
$$x_0x_{3r-2} - x_{3r-3}x_{3r-1},\ x_{3r-1}^2 - x_1x_{3r-2}, x_{3r-1}^2 - x_0x_{3r-3},$$
$$x_0^2 - x_1x_2 - x_1x_{3r-1},\ x_0x_{3r-1} - x_1x_{3r-3};$$
$$x_{6j+2}x_{6j+4} - x_{6j+3}^2 + x_{6j+4}x_{6j+5}, \quad j = 0, 1, \dots, \tfrac{r}{2} - 1;$$

and for $j = 1, 2, \dots, r-2$,

$$x_{3j}x_{3j+2} - x_{3j+1}x_{3j+3};\ x_{3j+2}^2 - x_{3j+1}x_{3j+4};\ x_{3j+2}^2 - x_{3j}x_{3j+3};$$
$$x_{3j}x_{3j+4} - x_{3j+2}x_{3j+3};\ x_{3j+2}x_{3j+4} - x_{3j+3}^2 + x_{3j+4}x_{3j+5};$$
$$x_{3j}x_{3j+5}, x_{3j}x_{3j+6}, \dots, x_{3j}x_{3r-1};\ x_{3j+1}x_{3j+5}, x_{3j}x_{3j+6}, \dots, x_{3j+1}x_{3j-1};$$
$$x_{3j+2}x_{3j+5}, x_{3j+2}x_{3j+6}, \dots, x_{3j+2}x_{3j-1},$$

together with the following degree three polynomials

$$\begin{aligned} &x_1x_{3r-3}^2 - x_{3r-1}^3, x_{3r-3}^2x_{3r-1} - x_{3r-2}x_{3r-1}^2, \\ &x_{3j+1}x_{3j+3}^2 - x_{3j+2}^3; \quad j = 1, 2, \dots, r-2. \end{aligned} \tag{10.12}$$

The degree two initial monomials are:

$$\begin{aligned} &x_0^2, x_0x_3, x_0x_4, \dots, x_0x_{3r-1};\ x_1x_3, x_1x_4, \dots, x_1x_{3r-4}, x_1x_{3r-2}; \\ &x_2x_4, x_2x_5, x_2x_6, \dots, x_2x_{3r-1}; \end{aligned} \tag{10.13}$$

and for $j = 1, 2, \dots, r-2$,

$$\begin{aligned} &x_{3j}x_{3j+2}, x_{3j}x_{3j+3}, \dots, x_{3j}x_{3r-1}; x_{3j+1}x_{3j+4}, x_{3j}x_{3j+5}, \dots, x_{3j+1}x_{3r-1}; \\ &x_{3j+2}x_{3j+4}, x_{3j+2}x_{3j+5}, \dots, x_{3j+2}x_{3r-1}. \end{aligned} \tag{10.14}$$

The sum of the weights of the monomials in (10.13) is $27r-33$, whereas the monomials in (10.14) contribute $18r^2-58r+43$ to the total weight of the monomials in the initial ideal.

The total weight of all degree two monomials is $18r^2 - 3r - 3$. Hence the weights of all degree two monomials not in the initial ideal sum up to

$$18r^2 - 3r - 3 - (18r^2 - 58r + 43) - (27r - 33) = 28r - 13$$

On the other hand, the average weight $\frac{2P(2)\sum r_i}{N+1}$ is $28r - 14$. It follows from Proposition 3.7 that $\mu([C_r]_2, \rho) = (-(28r - 13) + 28r - 14) = -1$.

The degree three monomials divisible by the ones in the lists (10.13), (10.14) contribute $27r^3 + \frac{27}{2}r^2 - \frac{159}{2}r^2 + 26$ to the total weight of the monomials in the initial ideal. On the other hand, the monomials

$$x_1x_{3r-3}^2,\ x_{3r-3}^2x_{3r-1};\ x_{3j+2}^3, \quad j = 1, 2, \dots, r-2 \tag{10.15}$$

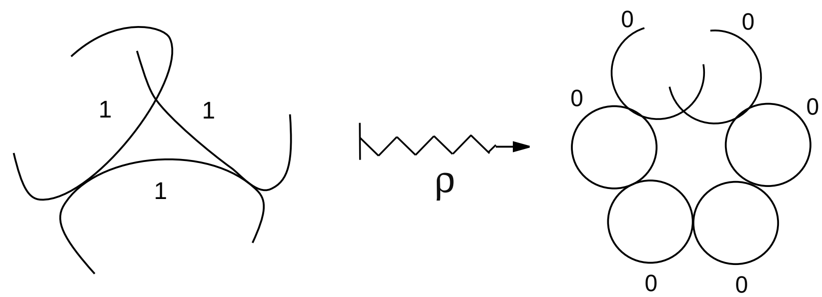


FIGURE 11. Basin of attraction of a closed rosary with a broken bead

coming from the degree three Gröbner basis members (10.12) contribute $6r+2$. The sum of the weights of all degree three monomials is $27r^3+\frac{27}{2}r^2-\frac{15}{2}r-3$. Hence the total weight of the degree three monomials not in the initial ideal is

$$27r^3+\frac{27}{2}r^2-\frac{15}{2}r-3-\left(27r^3+\frac{27}{2}r^2-\frac{159}{2}r^2+26\right)-(6r+2)=66r-31.$$

On the other hand, the average weight is

$$\frac{3P(3)}{N+1}(6r-3)=66r-33.$$

By Proposition 3.7, the Hilbert-Mumford index is $\mu([C_r]_3,\rho)=-(66r-31)+66r-33)=-2$. Since $\mu([C_r]_2,\rho)=2\mu([C_r]_3,\rho)<0$, it follows from Corollary 6.4 that $C_r$ is $m$-Hilbert unstable for all $m\geq 2$. Indeed, we find that

$$\mu([C_r]_m,\rho)=1-m$$

for each $m\geq 2$ and $\mu(Ch(C_r),\rho)=0$. □

10.6. **Basin of attraction: Closed rosary with a broken bead.**

**Proposition 10.8.** *Let* $C_r$ *and* $\rho$ *be as in Proposition 10.7. Then the basin of attraction* $A_\rho([C]_m)$ *parametrizes closed elliptic chains* $(C',p,q)$ *of length* $(r+1)/2$ *such that* $\iota(p)=\iota(q)$ *(Figure 11).*

**Corollary 10.9.** *A closed elliptic chain is Hilbert unstable.*

*Proof.* At the node, the local analytic equation is given by

$$\frac{x_0}{x_2}\cdot\frac{x_3}{x_2}=0$$

and $\mathbb{G}_m$ acts on the local parameters $x := x_0/x_2$ and $y := x_3/x_2$ with weight $-1$. Hence $\mathbb{G}_m$ acts on the local versal deformation space (defined by $xy = c_0$) with weight $-2$. At the adjacent tacnode, $\mathbb{G}_m$ acts on the tangent space to the two branches with positive weights: The tangent lines are traced by $x_3/x_4$ and $x_6/x_4$. In fact, $\mathbb{G}_m$ acts on the local versal deformation of the tacnode (defined by $y^2 = x^4 + c_2x^2 + c_1x + c_0$) with a positive weight vector $(2, 3, 4)$. Similar analysis reveals that $\mathbb{G}_m$ acts on the subsequent tacnode with a negative weight vector $(-2, -3, -4)$. Using the symmetry of the rosary, we can conclude that $\mathbb{G}_m$ acts on the local versal deformation space of the tacnodes with weight vector alternating between $(2, 3, 4)$ and $(-2, -3, -4)$. The assertion now follows. □

## 11. Proofs of semistability and applications

Our main goal is a complete description of all c-equivalence and h-equivalence classes. Throughout, each c-semistable curve $C$ is embedded bicanonically (cf. Proposition 6.1)

$$C \hookrightarrow \mathbb{P}^{3g-4},$$

and we consider the corresponding Chow points $\mathrm{Ch}(C) \in \mathrm{Chow}_{g,2}$ and Hilbert points $[C]_m \in \mathrm{Hilb}_{g,2}, m \gg 0$. To summarize:

- If $C$ is c-stable (resp. h-stable), then the equivalence class of $C$ is trivial: It coincides with the $\mathrm{SL}_{3g-3}$ orbit of $\mathrm{Ch}(C)$ (resp. $[C]_m$ for $m \gg 0$).
- If $C$ is strictly c- or h-semistable its equivalence class is nontrivial. We shall identify the unique closed orbit curve and describe all equivalent curves.
- Since closed orbit curves are separated in a good quotient [Ses72, 1.5], we have a complete classification of curves identified in the quotient spaces $\mathrm{Hilb}_{g,2}/\!\!/\mathrm{SL}_{3g-3}$ and $\mathrm{Chow}_{g,2}/\!\!/\mathrm{SL}_{3g-3}$.

### 11.1. Elliptic bridges and their replacements.

**Definition 11.1.** An *elliptic bridge of length* $k$ is a two-pointed curve $(C', p, q)$ (Figure 12) such that

- $C' = E_1 \cup_{a_1} \cdots \cup_{a_{k-1}} E_k$ consists of connected genus-one curves $E_1, \dots, E_k$ such that $E_i$ meets $E_{i+1}$ in a node $a_i$, $i = 1, 2, \dots, k-1$;
- $E_i \cap E_j \neq \emptyset$ if $|i - j| \neq 1$;
- $p \in E_1$ and $q \in E_k$ are smooth points.

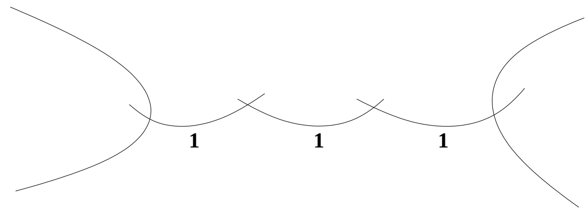


FIGURE 12. A generic elliptic bridge of length three

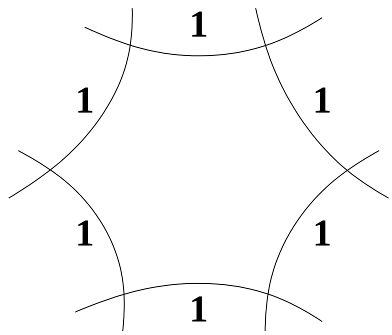


FIGURE 13. A generic closed elliptic bridge of length six and genus seven

An ordinary elliptic bridge is an elliptic bridge of length one.

Let $C$ be a strictly c-semistable curve that is pseudostable, i.e., $C$ has no tacnodes. Let $E_1, \dots, E_N$ be the genus-one subcurves of $C$ arising as components of elliptic bridges.

**Lemma 11.2.** *Every c-semistable curve $C'$ admitting $C$ as a pseudostable reduction can be obtained from the following procedure:*

(1) *Fix a subset*
$$\{E_i\}_{i \in I} \subset \{E_1, \dots, E_N\}$$
*of the genus-one subcurves arising in elliptic bridges.*

(2) *Choose a subset of the nodes of $C$ lying on $\cup_{i \in I} E_i$ consisting of points of the following types:*
   - *If $E_i \cap E_{i'} \neq \emptyset$ for some distinct $i, i' \in I$ then the node where they intersect must be included.*
   - *Nodes where the $E_i, i \in I$ meet other components may be included.*

(3) *Replace each of these nodes by a smooth $\mathbb{P}^1$ (for any point of our subset) or by a chain of two smooth $\mathbb{P}^1$'s (only for points of the first type). Precisely,*

*let $Z$ denote the curve obtained by normalizing our set of nodes and then joining each pairs of glued points with a $\mathbb{P}^1$ or a chain of two $\mathbb{P}^1$'s with one component meeting each glued point.*

(4) *Let $E_i'$ denote the proper transform of $E_i, i \in I$, which are pairwise disjoint in $Z$. Replace each $E_i'$ with a tacnode. Precisely, write*

$$D = Z \setminus \cup_{i \in I} E_i'$$

*and consider a morphism*

$$\nu : Z \to C'$$

*such that*

- $\nu|D$ *is an isomorphism and* $\nu|\overline{D} \to C'$ *is the normalization;*
- *for* $i \in I$, $\nu$ *contracts* $E_i'$ *to a tacnode of* $C'$.

*The generic curve $C'$ produced by this procedure does not admit components isomorphic to $\mathbb{P}^1$ containing a node of $C'$. We introduce $\mathbb{P}^1$'s in Step (3) only to separate two adjacent contracted elliptic components.*

When enumerating the c-semistable curves, it is convenient to use a graph that is similar to the dual graph: We use a single line to denote a node and a double line to denote a tacnode (e.g. Figure 14).

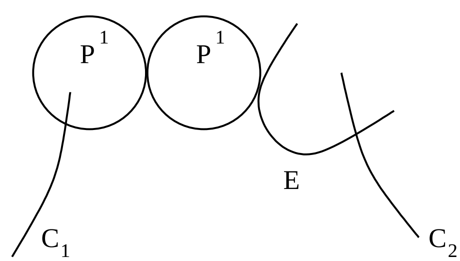


FIGURE 14. A configuration corresponding to $C_1 - \mathbb{P}^1 = \mathbb{P}^1 = E - C_2$

**Example 11.3.** Let $C$ be the elliptic bridge of length one

$$C_1 - E - C_2.$$

The possible $Z$ are

$$C_1 - E - C_2, \quad C_1 - \mathbb{P}^1 - E - C_2, \quad C_1 - E - \mathbb{P}^1 - C_2, \quad C_1 - \mathbb{P}^1 - E - \mathbb{P}^1 - C_2$$

and the possible $C'$ are

$$C_1 - E - C_2, \quad C_1 = C_2, \quad C_1 - \mathbb{P}^1 = C_2, \quad C_1 = \mathbb{P}^1 - C_2, \quad C_1 - \mathbb{P}^1 = \mathbb{P}^1 - C_2.$$

The first two are the generic c-semistable configurations.

If $C$ is an elliptic bridge of length two

$$C_1 - E_1 - E_2 - C_2$$

then the possible $Z$ are

$$\begin{gathered} C_1 - E_1 - E_2 - C_2, \quad C_1 - \mathbb{P}^1 - E_1 - E_2 - C_2, \quad C_1 - E_1 - \mathbb{P}^1 - E_2 - C_2 \\ C_1 - E_1 - E_2 - \mathbb{P}^1 - C_2, \quad C_1 - \mathbb{P}^1 - E_1 - \mathbb{P}^1 - E_2 - C_2, \\ C_1 - E_1 - \mathbb{P}^1 - \mathbb{P}^1 - E_2 - C_2, \quad C_1 - E_1 - \mathbb{P}^1 - E_2 - \mathbb{P}^1 - C_2, \\ C_1 - \mathbb{P}^1 - E_1 - \mathbb{P}^1 - \mathbb{P}^1 - E_2 - C_2, \quad C_1 - \mathbb{P}^1 - E_1 - \mathbb{P}^1 - E_2 - \mathbb{P}^1 - C_2, \\ C_1 - E_1 - \mathbb{P}^1 - \mathbb{P}^1 - E_2 - \mathbb{P}^1 - C_2, \quad C_1 - \mathbb{P}^1 - E_1 - \mathbb{P}^1 - \mathbb{P}^1 - E_2 - \mathbb{P}^1 - C_2. \end{gathered}$$

The generic c-semistable configurations are

$$C_1 - E_1 - E_2 - C_2, \quad C_1 = E_2 - C_2, \quad C_1 - E_1 = C_2, \quad C_1 = \mathbb{P}^1 = C_2.$$

*Proof.* (of Lemma 11.2) Our hypotheses give a flat family

$$(\dagger) \qquad \mathcal{C}' \to B := \operatorname{Spec} k[[t]]$$

whose generic fibre $\mathcal{C}'_\eta$ is smooth and the special fibre $\mathcal{C}'_0$ is $C'$. Furthermore, after a base change

$$\begin{array}{ccc} B & \to & B_1 = \operatorname{Spec} k[[t_1]] \\ t & \mapsto & t_1^s \end{array}$$

there exists a birational modification over $B_1$

$$\psi : \mathcal{C} \dashrightarrow \mathcal{C}' \times_B B_1$$

such that $\mathcal{C}_0$ is $C$. In other words, $\mathcal{C} \to B_1$ is the *pseudostable reduction* of $\mathcal{C}' \to B$; we replace each tacnode by an elliptic bridge and contract any rational component that meets the rest of the curve in fewer than three points.

Let $\mathcal{Z}$ be the normalization of the graph of $\psi$, with $\pi_1$ and $\pi_2$ the projections to $\mathcal{C}$ and $\mathcal{C}' \times_B B_1$ respectively:

$$\begin{array}{ccc} & \mathcal{Z} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{C} & & \mathcal{C}' \times_B B_1 \end{array}$$

By [Sch91, 4.4], $\mathcal{Z}$ is flat over $B$ and $Z = \mathcal{Z}_0$ is reduced. An argument similar to [Sch91, 4.5-4.8] yields

- The exceptional locus of $\pi_2$ is a disjoint union of connected genus-one subcurves
$$\sqcup_{i\in I} E'_i \subset Z$$
that arise as proper transforms of components of elliptic bridges in $C$. Each component is mapped to a tacnode of $C'$.
- The exceptional locus of $\pi_1$ is a union of chains of rational curves of length one or two
$$\sqcup \mathbb{P}^1 \sqcup (\mathbb{P}^1 \cup \mathbb{P}^1) \subset Z$$
that arise as proper transforms of rational components of $C'$ meeting the rest of the curve in two points (either two tacnodes or one node and one tacnode). Each component is mapped to a node of $C$ contained in an elliptic bridge.

This yields the schematic description for the possible combinatorial types of $C'$.

We analyze the generic curves arising from our procedure. Suppose there is a component isomorphic to $\mathbb{P}^1$ meeting the rest of the curve in a node and a tacnode. Corollary 6.3 implies we can smooth the node to get a c-semistable curve. The smoothed curve also arises from our procedure. □

**Remark 11.4.** Lemma 11.2 yields a bijection between subsets
$$\{E_i\}_{i\in I} \subset \{E_1, \dots, E_N\}$$
and generic configurations of the locus of curves arising from our procedure. Indeed, there is a unique generic configuration contracting the curves $\{E_i\}_{i\in I}$.

**Proposition 11.5.** *Let $C$ be strictly c-semistable without tacnodes and $E_1, \dots, E_N$ the genus-one subcurves of $C$ arising as components of elliptic bridges. Let $C^\star$ be the curve obtained from $C$ by replacing each $E_i$ with an open rosary $(R_i, p_i, q_i)$ of length two. Then there exists a one-parameter subgroup*
$$\rho : \mathbb{G}_m \to \mathrm{Aut}(C^\star)$$
*such that $\mathrm{Ch}(C) \in A_\rho(\mathrm{Ch}(C^\star))$ and $\mu(\mathrm{Ch}(C^\star), \rho) = 0$.*

*If $C'$ is another c-semistable curve with pseudostable reduction $C$ then there exists a one-parameter subgroup*
$$\rho' : \mathbb{G}_m \to \mathrm{Aut}(C^\star)$$

*such that* $\mathsf{Ch}(C') \in A_{\rho'}(\mathsf{Ch}(C^\star))$ *and* $\mu(\mathsf{Ch}(C^\star), \rho') = 0$.

*Proof.* The assmption that $C$ is strictly c-semistable without tacnodes ensures it contains an elliptic chain of length one, i.e., an elliptic bridge.

The analysis of Proposition 8.5 makes clear that our description of $C^\star$ determines it uniquely up to isomorphism. Furthermore, we have

$$\mathrm{Aut}(C^\star)^\circ \simeq \mathbb{G}_m^N$$

with basis $\{\rho_1, \ldots, \rho_N\}$; here $\rho_i$ denotes the one-parameter subgroup acting trivially on $R_j, j \neq i$ and with weight $1$ on the tangent spaces $T_{p_i} R_i$ and $T_{q_i} R_i$. As explained in § 10.2, it acts with negative weights on the versal deformation space of the tacnode of $R_i$.

Consider the one-parameter subgroup

$$\rho = \prod_{i=1}^{N} \rho_i^{-1},$$

which acts with positive weights on each of the tacnodes. The basin of attraction analysis of Proposition 10.3 shows that $A_\rho(\mathsf{Ch}(C))$ parametrizes those curves obtained by replacing each open rosary of $C^\star$ with an elliptic bridge/chain of length one. This includes our original curve $C$.

Now for any one-parameter subgroup

$$\rho' = \prod_{i=1}^{N} \rho_i^{-e_i},$$

we can compute

$$\mu(\mathsf{Ch}(C^\star), \rho') = -\sum_{i=1}^{N} e_i \, \mu(\mathsf{Ch}(C^\star), \rho_i) = 0$$

using Corollary 10.2. In particular, we have

$$\mu(\mathsf{Ch}(C^\star), \rho) = 0.$$

Section 10.2 gives the action of $\rho'$ on the versal deformations of the singularities of $C^\star$. It acts with weights $(2e_i, 3e_i, 4e_i)$ on the versal deformation space of the tacnode on $R_i$. At a node ($p_i$ or $q_i$) lying on a single open rosary $R_i$ of length two, it acts with weight $-e_i$. For nodes on two open rosaries $R_i$ and $R_j$, it acts with weight $-(e_i + e_j)$.

Restrict attention to one-parameter subgroups with weights $e_i \neq 0$ for each $i, j = 1, \dots, N$. These naturally divide up into $2^N$ equivalence classes, depending on the signs of the $e_i$. Let $I \subset \{1, \dots, N\}$ denote those indices with $e_i < 0$. Just as in the proof of Proposition 10.3, the basin of attraction $A_{\rho'}(Ch(C^\star))$ does not contain smoothings of the tacnodes in $R_i, i \in I$ but does contain all smoothings of the remaining tacnodes. Choosing the negative $e_i$ suitably large in absolute value, we can assume each $-(e_i + e_j) > 0$, so the nodes where two rosaries meet are smoothed provided at least one of the adjacent tacnodes is *not* smoothed.

Thus $A_{\rho'}(Ch(C^\star))$ consists of the c-semistable curves obtained by smoothing all the tacnodes *not* indexed by I, as well as the nodes on the rosaries containing one of the remaining tacnodes (indexed by I). The generic member of the basin equals the generic configuration indexed by I, as described in Remark 11.4. It follows that *each* curve $C'$ enumerated in Lemma 11.2 appears in the the basin of attraction of $Ch(C^\star)$ for a suitable one-parameter subgroup $\rho'$. □

11.2. **Chow semistability of c-semistable curves.** Here we prove that bicanonical c-semistable curves are Chow semistable. By Theorem 9.1, it suffices to consider curves that are not c-stable.

Let $C'$ denote a strictly c-semistable curve, with tacnodes and/or elliptic bridges. Assume that $C'$ is Chow unstable and let

$$\mathcal{C}' \to B := \operatorname{Spec} k[[t]]$$

be a smoothing. Let $C''$ be a Chow semistable reduction of this family (see Theorem 4.5) and $C$ the pseudostable reduction.

Reversing the steps outlined in Lemma 11.2, we see that $C$ is obtained by replacing each tacnode of $C'$ (or $C''$) with an elliptic bridge and then pseudostabilizing. Let $C^\star$ denote the curve obtained from $C$ in Proposition 11.5, which guarantees that $Ch(C')$ and $Ch(C'')$ are contained in basins of attraction $A_{\rho'}(Ch(C^\star))$ and $A_{\rho''}(Ch(C^\star))$ respectively. Moreover, since

$$\mu(Ch(C^\star), \rho') = \mu(Ch(C^\star), \rho'') = 0$$

Lemma 4.3 implies that $C'$ (resp. $C''$) is Chow semistable iff $C$ is Chow semistable. This contradicts our assumption that $C'$ is Chow unstable.

Next, we give a characterizaztion of the closed orbit curves in c-equivalence classes of strictly semistable curves:

**Proposition 11.6.** *A strictly c-semistable curve has a closed orbit if and only if*

- *each tacnode is contained in an open rosary;*
- *each open rosary has length two; and*
- *there are no elliptic bridges other than length two rosaries.*

Since each length-two rosary has one tacnode and contributes a $\mathbb{G}_m$-factor to $\mathrm{Aut}(C)$, we have:

**Corollary 11.7.** *If $C$ is a strictly c-semistable curve with closed orbit then*

$$\mathrm{Aut}(C)^{\circ} \simeq \mathbb{G}_m^{\tau}.$$

*where $\tau$ is the number of tacnodes and the superscript $\circ$ denotes the connected component of the identity.*

*Proof of Proposition 11.6.* Assume that $C'$ is a strictly semistable curve with closed orbit. Let $C$ be a pseudostable reduction and $C^{\star}$ the curve specified in Proposition 11.5, so the Chow point of $C'$ is in the basin of attraction of the Chow point of $C^{\star}$. Since $C^{\star}$ is Chow semistable, we conclude that $C' = C^{\star}$.

Conversely, suppose $C'$ is a curve satisfying the three conditions of Proposition 11.6. Again, let $C$ be a pseudostable reduction of $C'$ and $C^{\star}$ the curve obtained in Proposition 11.5, so that $C'$ is in the basin of attraction of $C^{\star}$ for some one-parameter subgroup $\rho'$. Note that $C^{\star}$ also satisfies the conditions of Proposition 11.6. The basin of attraction analysis in Section 10.2 implies that any nontrivial deformation of $C^{\star}$ in $A_{\rho'}(\mathrm{Ch}(C^{\star}))$ induces a nontrivial deformation of at least one of the singularities of $C^{\star}$ sitting in an open rosary.

There are three cases to consider: First, we could deform the tacnode on one of the rosaries $R_i$. However, then the rosary $R_i$ deforms to an elliptic bridge in $C'$ that is not a length two rosary, which yields a contradiction. Therefore, we may assume that none of the tacnodes in $C^{\star}$ is deformed in $C'$. Second, we could smooth a node where length two rosaries meet. However, this would yield a rosary in $C'$ of length $> 2$. Finally, we could smooth a node where a length two rosary $R_i$ meets a component not contained in an rosary. However, the tacnode of $R_i$ then deforms to a tacnode of $C'$ not on any length two rosary. □

11.3. **Hilbert semistability of h-semistable curves.** Suppose that $C$ is a h-semistable bicanoncal curve. By definition it is also c-semistable and thus Chow-semistable by the analysis of Section 11.2. Of course, strictly Chow-semistable

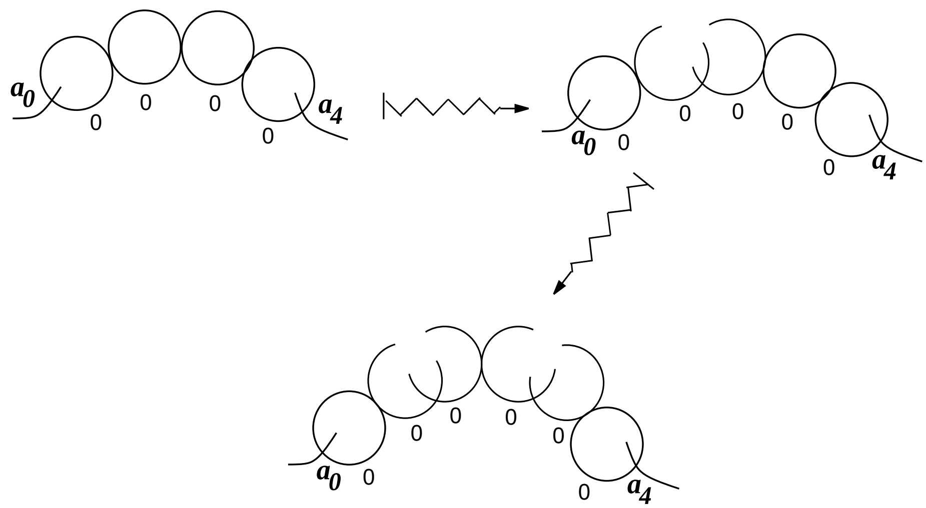


FIGURE 15. Degeneration to the c-semistable closed orbit curve

points can be Hilbert unstable, and we classify these in two steps. First, we enumerate the curves $C_0$ with strictly semistable Chow point such that there exists a one-parameter subgroup $\rho : \mathbb{G}_m \hookrightarrow \mathrm{Aut}(C_0)$ destabilizing the Hilbert point of $C_0$, i.e., with $\mu([C_0]_m, \rho) < 0$ for $m \gg 0$. Second, we list the curves that are in the basins of attraction $A_\rho([C_0]_m)$, which are also guaranteed to be Hilbert unstable by the Hilbert-Mumford one-parameter subgroup criterion. Proposition 4.4 shows these are *all* the Hilbert unstable curves: If $C$ is a c-semistable bicanonical curve that is Hilbert unstable then its Chow point is contained in the basin of attraction $A_\rho(\mathrm{Ch}(C_0))$ of a Chow semistable curve $C_0$ with closed orbit such that $\mu([C_0]_m, \rho) < 0$.

If the genus is odd and $C_0$ is a closed rosary (without broken beads) then $C_0$ is Hilbert semistable with respect to any 1-PS coming from $\mathrm{Aut}(C_0)$ (Proposition 10.5).

Suppose that $C_0$ has open rosaries $S_1, \dots, S_\ell$. Each contributes $\mathbb{G}_m$ to the automorphism group of $C_0$ and $\mathrm{Aut}(C_0)^\circ \simeq \mathbb{G}_m^{\times \ell}$. Let $p_i, q_i$ denote the nodes in the intersection $S_i \cap \overline{C_0 - S_i}$. The automorphism coming from $S_i$ gives rise to a one-parameter subgroup

$$\rho_i : \mathbb{G}_m \xrightarrow{\simeq} \{1\} \times \cdots \times \underbrace{\mathbb{G}_m}_{i\text{th}} \times \cdots \times \{1\} \hookrightarrow \mathbb{G}_m^{\times \ell} \simeq G^\circ_{Ch(C_0)}$$

where the second $\mathbb{G}_m$ means the $i$th copy in the product $\mathbb{G}_m^{\times \ell}$ and $G_{Ch(C_0)}$ is the stabilizer group. We assume that $S_1, \ldots, S_k$ are open rosaries of even length and $S_{k+1}, \ldots, S_\ell$ are of odd length. For $i \le k$, the weights of $\rho_i$ on the versal deformation spaces of $p_i$ and $q_i$ have the same sign (see §10.2). We normalize $\rho_i$ so that this weight is negative.

Given our one-parameter subgroups $\rho : \mathbb{G}_m \to \mathrm{Aut}(C_0)^\circ$ with $\mu([C_0]_m, \rho) < 0$, we can expand

$$\rho = \prod_{i=1}^{k} \rho_i^{a_i} \times \prod_{i=k+1}^{\ell} \rho_i^{b_i}, \quad a_i, b_i \in \mathbb{Z}$$

so that

$$\mu([C_0]_m, \rho) = \sum_{i=1}^{k} a_i\, \mu([C_0]_m, \rho_i) + \sum_{i=k+1}^{\ell} b_i\, \mu([C_0]_m, \rho_i) < 0.$$

We have already computed these terms: Proposition 10.1 implies that $\mu([C_0]_m, \rho_i) = 0$ for $i = k+1, \ldots, \ell$ and $\mu([C_0]_m, \rho_i) = 1 - m$ for $i = 1, \ldots, k$. Thus in order for the sum to be negative, we must have $a_i > 0$ for some $i = 1, \ldots, k$. In particular, there is at least one rosary of even length. Proposition 10.3 implies that the basin of attraction $A_\rho([C_0]_m)$ contains curves with elliptic chains, which are not h-semistable.

We are left with the case of a closed rosary $C_r$ of even genus with one broken bead. There is a unique one-parameter subgroup $\rho$ of the automorphism group, and we choose the sign so that it destabilizes $C_r$ (cf. Proposition 10.7). The basin of attraction analysis in Proposition 10.8 again shows that the curves with unstable Hilbert points admit elliptic chains.

Thus curves with unstable Hilbert points are not h-semistable, which completes our proof that h-semistable curves are Hilbert semistable.

We shall now prove that if $C$ is h-stable then it is Hilbert stable. If $C$ is Hilbert strictly semistable, then it belongs to a basin of attraction $A_\rho([C_0]_m)$ where $C_0$ is a Hilbert semistable curve with infinite automorphisms and $\rho$ is a 1-PS coming from $\mathrm{Aut}(C_0)$. By Corollary 8.9, $C_0$ admits an open rosary of odd length $\ge 3$ or is a closed rosary of even length $\geq 4$. But we showed in Propositions 10.3 and 10.6 that any curve in the basin of such $C_0$ has a weak elliptic chain and hence is not h-stable.

Finally, we characterize the closed orbits of strictly h-semistable curves. These do not admit elliptic chains, and in particular, do not admit open rosaries of even length (see Remark 8.4).

**Proposition 11.8.** *A strictly h-semistable curve has a closed orbit if and only if*

- *it is a closed rosary of odd genus; or*
- *each weak elliptic chain is contained in a chain of open rosaries of length three.*

Since each length three open rosary has two tacnodes and contributes $\mathbb{G}_m$ many automorphisms to $\mathrm{Aut}(C)$,

**Corollary 11.9.** *If* $C$ *is a strictly h-semistable curve with closed orbit then*

$$\mathrm{Aut}(C)^\circ \simeq \mathbb{G}_m^{\tau/2}$$

*where* $\tau$ *is the number of tacnodes.*

*Proof of Proposition 11.8.* Suppose $C'$ is strictly h-semistable. We shall show that there exists a curve $C^*$ satisfying the conditions of Proposition 11.8 and a one-parameter subgroup $\rho'$ of $\mathrm{Aut}(C^*)$ such that $[C']_m \in A_{\rho'}([C^*]_m)$ for $m \gg 0$ and $\mu([C^*]_m, \rho') = 0$.

Assume first that $C'$ is a closed weak elliptic chain with $r$ components, with arithmetic genus $2r+1$. Let $C^*$ denote a closed rosary with beads $L_1, \ldots, L_{2r}$ and tacnodes $a_1, \ldots, a_{2r}$. Proposition 8.5 implies $\mathrm{Aut}^\circ(C^*) \simeq \mathbb{G}_m$, generated by a one-parameter subgroup $\rho$ acting on the versal deformation spaces of the $a_{2j}$ with positive weights and the $a_{2j-1}$ with negative weights. Proposition 10.6 implies $A_\rho([C^*]_m)$ contains the closed weak elliptic chains of length $r$.

Now assume that $C'$ is not a closed weak elliptic chain but contains maximal closed weak elliptic chains $C''_1, \ldots, C''_s$ of lengths $\ell_1, \ldots, \ell_s$. Let $p_j$ (resp. $q_j$) denote the node (resp. tacnode) where $C''_j$ meets the rest of the curve. Let $C^*$ be the curve obtained from $C'$ by replacing each $C''_j$ with a chain of $\ell_j$ open rosaries of length three. Precisely, write

$$D = C' \setminus (\bigcup_{j=1}^{s} C''_j \setminus \{p_j, q_j\})$$

and let $S_j, j = 1, \ldots, s$ denote a chain of $\ell_j$ open rosaries of length three joined end-to-end. Then $C^*$ is obtained by gluing $S_j$ to $D$ via nodes at $p_j$ and $q_j$. One special case requires further explanation: If $C'$ admits an irreducible component $\simeq \mathbb{P}^1$ meeting the rest of $C'$ at two points $q_i$ and $q_j$ then we contract this component in $C^*$.

**Example 11.10.** There are examples where the construction of $C^*$ involves components being contracted. Let $C_1$ and $C_2$ be smooth and connected of genus $\geq 2$ and let $E_1$ and $E_2$ be elliptic. Consider the curve $C'$

$$C_1 -_{p_1} E_1 =_{q_1} \mathbb{P}^1 =_{q_2} E_2 -_{p_2} C_2.$$

Replacing the weak elliptic chains with rosaries of length three yields

$$C_1 -_{p_1} \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 -_{q_1} \mathbb{P}^1 -_{q_2} \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 -_{p_2} C_2$$

which is not h-semistable. Contracting the middle $\mathbb{P}^1$, we obtain $C^*$:

$$C_1 -_{p_1} \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 - \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 -_{p_2} C_2.$$

There are examples where $D$ fails to be pure-dimensional. Start with the curve $C'$

$$C_1 =_{q_1} E_1 -_{p_1 = p_2} E_2 =_{q_2} C_2$$

with the $C_i$ and $E_i$ as above. Then $C^*$ is equal to

$$C_1 -_{q_1} \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 -_{p_1 = p_2} \mathbb{P}^1 = \mathbb{P}^1 = \mathbb{P}^1 -_{q_2} C_2.$$

We return to our proof: The curve $C^*$ has

$$\mathrm{Aut}(C^*)^\circ \simeq \mathbb{G}_m^N, \quad N = \sum_{j=1}^{s} \ell_j.$$

Essentially repeating the argument of Proposition 11.5, using the one-parameter subgroup analysis of Proposition 10.1 and the basin-of-attraction analysis of Proposition 10.3 (or Proposition 10.7 and 10.8 in the generate case), we obtain a one-parameter subgroup $\rho'$ in the automorphism group such that $[C']_m \in A_{\rho'}([C^*]_m)$ for $m \gg 0$ and $\mu([C^*]_m, \rho') = 0$.

We now show that the curves enumerated in Proposition 11.8 all have closed orbits. Due to [Kem78, Theorem 1.4], it suffices to show none of these are contained in the basin of attraction of any other. Suppose that $C_1^*$ and $C_2^*$ are such that

$$[C_2^*]_m \in A_\rho([C_1^*]_m)$$

for some one-parameter subgroup $\rho$ of $\mathrm{Aut}(C_1^*)^\circ$. A nontrivial deformation of $C_1^*$ necessarily deforms one of the singularities of $C_1^*$. If the singularity is a tacnode on a length-three open rosary, the resulting deformation admits a weak elliptic chain that is not contained in a chain of length-three rosaries. If the singularity is a node where two length-three open rosaries meet then the deformation admits a weak elliptic chain *not* contained in a chain of length-three open rosaries.

However, there is one case that requires special care: Suppose that $C_1^*$ is a closed chain of $r$ rosaries of length three

$$R_1 -_{p_{12}} R_2 -_{p_{23}} \cdots -_{p_{r-2\,r-1}} R_{r-1} -_{p_{r-1\,r}} R_r -_{p_{r1}} R_1$$

where $R_1$ and $R_r$ meet at a node $p_{r1}$; this has arithmetic genus $2r+1$. Let $C_2^*$ denote a closed rosary of genus $2r+1$, which is a deformation of $C_1^*$. We need to insure that

$$[C_2^*]_m \notin A_\rho([C_1^*]_m) \tag{11.1}$$

for any one-parameter subgroup $\rho$ of $\mathrm{Aut}([C_1^*]_m)^\circ$. We can express

$$\rho = \prod_{j=1}^{r} \rho_j^{e_j}$$

where $\rho_j$ acts trivially except on $R_j$ and has weights $+1$ and $-1$ on $T_{p_{j-1\,j}}R_j$ and $T_{p_{j\,j+1}}R_j$. (Here $\rho_r$ acts with weights $+1$ and $-1$ on $T_{p_{r-1\,r}}R_r$ and $T_{p_{r,1}}R_r$.) However, assuming $\rho$ is nontrivial, one of the following differences

$$e_1 - e_2, \ldots, e_r - e_1$$

is necessarily negative; for simplicity, assume $e_1 - e_2 < 0$. It follows that $\rho$ acts with negative weight on the versal deformation of the node $p_{12}$, thus deformations in $A_\rho([C_1^*]_m)$ cannot smooth $p_{12}$. We conclude that deformations in the basin of attraction of $C_1^*$ cannot smooth each node, which yields (11.1) □

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