HODGE THEORY AND LAGRANGIAN PLANES ON GENERALIZED KUMMER FOURFOLDS

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1. Introduction

Suppose X is a smooth projective complex variety. Let $\mathrm{N}_1(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$ and $\mathrm{N}^1(X,\mathbb{Z}) \subset \mathrm{H}^2(X,\mathbb{Z})$ denote the group of curve classes modulo homological equivalence and the Néron-Severi group respectively. The monoids of effective classes in each group generate cones $\mathrm{NE}_1(X) \subset \mathrm{N}_1(X,\mathbb{R})$ and $\mathrm{NE}^1(X) \subset \mathrm{N}^1(X,\mathbb{R})$ with closures $\overline{\mathrm{NE}}_1(X)$ and $\overline{\mathrm{NE}}^1(X)$, the *pseudoeffective cones*. These play a central rôle in the birational geometry of X.

Let X be an irreducible holomorphic symplectic variety, i.e., a smooth projective simply-connected manifold admitting a unique non-degenerate holomorphic two-form. Let $(,)$ denote the Beauville-Bogomolov form on the cohomology group $\mathrm{H}^2(X,\mathbb{Z})$, normalized so that it is integral and primitive. Duality gives a $\mathbb{Q}$-valued form on $\mathrm{H}_2(X,\mathbb{Z})$, also denoted $(,)$. When X is a K3 surface these coincide with the intersection form. In higher-dimensions, the form induces an inclusion

$$\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z}),$$

which allows us to extend $(,)$ to a $\mathbb{Q}$-valued quadratic form.

Now suppose that X contains a Lagrangian projective space $\mathbb{P}^{\dim(X)/2}$; let $\ell \in \mathrm{H}_2(X,\mathbb{Z})$ denote the class of a line in $\mathbb{P}^{\dim(X)/2}$, and $\lambda = N\ell \in \mathrm{H}^2(X,\mathbb{Z})$ a positive integer multiple; we can take N to be the index of $\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$. Hodge theory [29, 35] shows that the deformations of X containing a deformation of the Lagrangian space coincide with the deformations of X for which $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ remains of type $(1,1)$. Infinitesimal Torelli implies this is a divisor on the deformation space, i.e.,

$$\lambda^{\perp} \subset \mathrm{H}^1(X,\Omega^1_X) \simeq \mathrm{H}^1(X,T_X).$$

Our goal is to establish intersection theoretic properties of ℓ for various deformation-equivalence classes of holomorphic symplectic varieties. Previous results in this direction include

(1) If X is a K3 surface then $(\ell,\ell) = -2$.

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(2) If X is deformation equivalent to the Hilbert scheme of length n subschemes of a K3 surface then $(\ell, \ell) = -(n+3)/2$, provided $n = 2$ [17], $n = 3$ [15], or $n = 4$ [1].

Here we prove

Theorem 1.1. *If X is deformation equivalent to the generalized Kummer manifold of dimension four and ℓ the class of a line on a Lagrangian plane in X then*

$$(\ell, \ell) = -3/2.$$

We refer the reader to Theorem 13.1 for a more precise description of the class of a Lagrangian plane in such a manifold.

This is part of a program described in [18] to characterize numerically classes of extremal rays in holomorphic symplectic manifolds, with a view towards characterizing ample divisors in terms of the intersection properties of the Beauville-Bogomolov form on the divisor/curve classes. We expect that geometric properties of birational contractions should be encoded in the self-intersections of their extremal rational curves. In the examples we have considered, extremal rays associated to Lagrangian projective spaces have *smallest* self-intersection. See [16, 17] for a detailed discussion of Hilbert schemes of length-two subschemes of K3 surfaces. Markman [25] addresses *divisorial* contractions and the associated reflections. Bayer and Macri [2] introduce additional constraints on the classes representing extremal rays in manifolds deformation equivalent to Hilbert schemes; their analysis is grounded in the study of Bridgeland stability conditions.

The main ingredients in our proof include an analysis of automorphisms of generalized Kummer varieties, their fixed-point loci, and the resulting 'tautological' Hodge classes in middle cohomology. (Unlike the case of length-two Hilbert schemes, the middle cohomology is not generated by the second cohomology.) In particular, these tautological classes arise from explicit complex surfaces (see Theorem 4.4). We analyze the saturation of the lattice generated by these tautological classes in the middle cohomology, and integrality properties of the quadratic form associated with the cup product. Computing the orbit of a Lagrangian plane under the automorphism group, we obtain a precise characterization of the homology classes that may arise.

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2. Automorphisms of holomorphic symplectic manifolds

Let X be an irreducible holomorphic symplectic manifold with symplectic form ω. (We use 'manifold' in the Kähler case and 'variety' in the projective context.) Let

$$\mathrm{Aut}^{\circ}(X) \subset \mathrm{Aut}(X)$$

denote the subgroup of holomorphic automorphisms of X acting trivially on $\mathrm{H}^1(T_X)$ and preserving ω. Since X has no vector fields, this is a discrete group. This is equivalent to the automorphisms acting trivially on $\mathrm{H}^2(X, \mathbb{C})$, as

$$\begin{aligned} \mathrm{H}^2(X, \mathbb{C}) &= \mathrm{H}^2(\mathcal{O}_X) \oplus \mathrm{H}^1(\Omega^1_X) \oplus \mathrm{H}^0(\Omega^2_X) \\ &= \overline{\mathbb{C}\omega} \oplus (\omega \otimes \mathrm{H}^1(T_X)) \oplus \mathbb{C}\omega. \end{aligned}$$

Let X' be deformation equivalent to X, i.e., there exists a connected complex manifold B, with distinguished points b and b', and a proper family of complex manifolds $\pi : \mathcal{X} \to B$ with $\mathcal{X}_b := \pi^{-1}(b) = X$ and $\mathcal{X}_{b'} = \pi^{-1}(b') = X'$.

Theorem 2.1. $\mathrm{Aut}^{\circ}(X)$ *is a deformation invariant of* X*, i.e., there exists a local system of groups*

$$\mathrm{Aut}^{\circ}(\mathcal{X}/B) \to B$$

acting on $\mathcal{X} \to B$*, such that for each* $b' \in B$ *the fiber is isomorphic to* $\mathrm{Aut}^{\circ}(X')$*.*

Proof. Consider a local universal deformation space of X

$$\psi : \mathcal{U} \to \Delta,$$

where Δ is a small polydisk and $\mathcal{U}_0 = X$ [22]. The completeness of this family implies we can construct this equivariantly for the action of $\mathrm{Aut}(X)$, i.e., $\mathrm{Aut}(X)$ acts on $\mathcal{U}$ and Δ and ψ is equivariant with respect to these actions. However, $\mathrm{Aut}^{\circ}(X)$ acts trivially on the tangent space $T_0\Delta = \mathrm{H}^1(T_X)$, thus acts trivially on Δ as well. It follows that $\mathrm{Aut}^{\circ}(X)$ acts fiberwise on $\mathcal{U} \to \Delta$.

Let $\mathrm{Aut}^{\circ}(\mathcal{X}/B)$ denote the group over B classifying automorphisms acting trivially on $\mathrm{H}^2(\mathcal{X}_b)$ for each $b \in B$. The previous analysis shows $\mathrm{Aut}^{\circ}(\mathcal{X}/B) \to B$ is a local homeomorphism. It remains to show this is universally closed.

We first show that each automorphism specializes to a bimeromorphic mapping $\phi : X' \dashrightarrow X'$. The reasoning is identical to the proof of

[19, 4.3], i.e., that two non-separated points in the moduli space correspond to bimeromorphic holomorphic symplectic manifolds. Consider a convergent sequence $b_n \to b'$ and a sequence $\alpha_n \in \mathrm{Aut}^{\circ}(\mathcal{X}_{b_n})$. The sequence of graphs

$$\Gamma_n := \Gamma_{\alpha_n} \subset \mathcal{X}_{b_n} \times \mathcal{X}_{b_n}$$

admits a convergent subsequence. Let $\Gamma' \subset X' \times X'$ denote the resulting limit cycle, which necessarily includes a component Z that maps bimeromorphically to each factor. We take ϕ to be the bimeromorphic map associated with Z.

The second step is to show that

$$\phi_* : \mathrm{H}^2(X', \mathbb{Z}) \to \mathrm{H}^2(X', \mathbb{Z})$$

is the identity. This essentially follows from the description of the 'birational Kähler cone' in [20, §4], but we offer a proof here. Since Γ' is the specialization of a correspondence acting as the identity on H^2, we know that $\Gamma'_* = \mathrm{Id}$. Express

$$\Gamma' = Z + \sum_i Y_i \subset X' \times X',$$

where the Y_i map to proper analytic subsets of each factor; let π_1 and π_2 denote the projections. It suffices to show that $\pi_j(Y_i), j = 1, 2$, has codimension > 1; then Y_i acts trivially on H^2 and $\phi_* = \Gamma'_* = \mathrm{Id}$ on H^2. Since ϕ is a bimeromorphic map from a holomorphic symplectic manifold to itself, it is an isomorphism in codimension one and ϕ_* is an isomorphism of H^2. Enumerate the Y_i such that $\pi_1(Y_i)$ has codimension one, i.e., $Y_1, \dots, Y_k$; since ϕ is an isomorphism in codimension one, these coincide with the components such that $\pi_2(Y_i)$ has codimension one (cf. the proofs of [20, 2.5,4.2]). Furthermore, ϕ respects these divisors in the sense that

$$\phi_*(\sum_{i=1}^k \pi_1(Y_i)) = \sum_{i=1}^k \pi_2(Y_i);$$

indeed, Y_i is ruled over both $\pi_1(Y_i)$ and $\pi_2(Y_i)$. In this situation, we have [20, p. 508]

$$\sum_{i=1}^k [Y_i]_* (\sum_{i=1}^k \pi_1(Y_i)) = -\sum_{i=1}^k b_i \pi_2(Y_i), \quad b_i > 0.$$

The previous two equations contradict the fact that Γ' introduces the identity on the second homology.

Thus ϕ is a bimeromorphic mapping of K-trivial varieties respecting Kähler cones. A result of Fujiki [11] implies ϕ is an isomorphism. □

Remark 2.2. This is related to unpublished results of Kaledin and Verbitsky [21, §6], where such automorphisms of generalized Kummer varieties $K_n(A)$ were used to exhibit non-trivial trianalytic subvarieties of $K_n(A)$.

3. Application to generalized Kummer varieties

Suppose that X is a holomorphic symplectic manifold of dimension $2n$, deformation equivalent to a generalized Kummer variety $K_n(A)$, defined as follows: Given an abelian surface A with length-$(n+1)$ Hilbert scheme $A^{[n+1]}$, $K_n(A)$ is the fiber over 0 of the addition map $\alpha : A^{[n+1]} \to A$. The Beauville-Bogomolov form is given by

$$\mathrm{H}^2(K_n(A),\mathbb{Z}) = \mathrm{H}^2(A,\mathbb{Z}) \oplus_\perp \mathbb{Z}e, \quad (e,e) = -2(n+1), \tag{1}$$

where $2e$ is the class E of the non-reduced subschemes [36, §4.3.1]. Each class $\eta \in \mathrm{H}^2(A,\mathbb{Z})$ yields a class in $\mathrm{H}^2(K_n(A),\mathbb{Z})$, i.e., the subschemes with some support along η. We use $(,)$ to embed $\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$ and extend $(,)$ to a $\mathbb{Q}$-valued form on $\mathrm{H}_2(X,\mathbb{Z})$. Let $e^\vee \in \mathrm{H}_2(K_n(A),\mathbb{Z})$ denote the class of a general ruling of E, i.e., where $n-1$ of the points are fixed and the tangent vector at the nth point varies. We have

$$e = 2(n+1)e^\vee, \quad (e^\vee, e^\vee) = -\frac{1}{2(n+1)}.$$

The following data can be extracted from Göttsche's generating series for the cohomology of generalized Kummer varieties [13, p. 50]. The table displays the Betti numbers $\beta_\nu(K_n(A))$ for $n \leq 4$:

ν \ n	1	2	3	4
0	1	1	1	1
1	0	0	0	0
2	22	7	7	7
3	0	8	8	8
4	1	108	51	36
5		8	56	64
6		7	458	168
7		0	56	288
8		1	51	1046

Consider the group

$$G_n = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/n\mathbb{Z})^4$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{Z}/n\mathbb{Z})^4$ via ± 1. Let A be an abelian surface and identify $A[n] = (\mathbb{Z}/n\mathbb{Z})^4$. Then $A[n]$ acts on A via translation and

$\mathbb{Z}/2\mathbb{Z}$ acts on A via ± 1. Altogether, we get an action

$$G_n \times A \to A.$$

Note that the induced action is trivial on $\mathrm{H}^2(A)$. Conversely, any finite group acting on A such that the induced action on $\mathrm{H}^2(A)$ is trivial is a subgroup of G_n for some n. Indeed, actions fixing 0 are determined by the associated representation on $\mathrm{H}^1(A)$, but the only group acting on $\mathrm{H}^1(A)$ such that the induced action on $\bigwedge^2(\mathrm{H}^1(A)) = \mathrm{H}^2(A)$ is trivial is ± 1.

Theorem 2.1 and the description of the second cohomology of generalized Kummer manifolds yields:

Proposition 3.1. *For each $n > 2$, the action of G_n on A induces a natural action on $K_{n-1}(A)$, which is trivial on $\mathrm{H}^2(K_{n-1}(A))$. This action extends to a natural action on any deformation X on $K_{n-1}(A)$.*

The action of G_n on $K_{n-1}(A)$ has been considered previously, e.g., [30, 5].

4. Analysis of the cohomology of $K_2(A)$

We recall the structure of the cohomology ring of $K_2(A)$, following [33], [23, §4], [28], and [30]:

Proposition 4.1. [23, 4.6] *Let X be deformation equivalent to $K_2(A)$ for A an abelian surface. The Lie algebra $\mathfrak{so}(4,5)$ acts on the cohomology $\mathrm{H}^*(X)$ which admits a decomposition*

$$\mathrm{H}^*(X) = \mathrm{Sym}(\mathrm{H}^2(X)) \oplus \mathbf{1}_X^{80} \oplus (\mathrm{H}^3(X) \oplus \mathrm{H}^5(X)),$$

where the middle term is a trivial $\mathfrak{so}(4,5)$-representation consisting of Hodge cycles in $\mathrm{H}^4(X)$ and the last term carries the structure of a sixteen-dimensional spinor representation.

Furthermore, the induced decomposition on middle cohomology

$$\mathrm{H}^4(X) = \mathrm{Sym}^2(\mathrm{H}^2(X)) \oplus_\perp \mathbf{1}_X^{80} \tag{2}$$

is orthogonal under the intersection form.

Decomposition (2) is not the only natural decomposition of the middle cohomology. The second Chern class $c_2(X)$ is also an invariant element of $\mathrm{H}^4(X)$ but is not orthogonal to $\mathrm{Sym}^2(\mathrm{H}^2(X))$ [23, 4.8]; indeed, we have the Fujiki relation

$$c_2(X) \cdot D_1 \cdot D_2 = a_X\,(D_1, D_2)\,, \quad D_1, D_2 \in H^2(X),$$

where a_X is a non-zero constant (which will be computed explicitly below). Writing

$$\mathrm{Sym}^2(\mathrm{H}^2(X))^\circ = c_2(X)^\perp \cap \mathrm{Sym}^2(\mathrm{H}^2(X)),$$

we obtain an alternate decomposition

$$\mathrm{H}^4(X) = \mathrm{Sym}^2(\mathrm{H}^2(X))^\circ \oplus_\perp \mathbf{1}_X^{81}. \tag{3}$$

By the infinitesimal Torelli theorem, the first summand contains no Hodge classes for general deformations X. Indeed, since the period map is a local diffeomorphism, $\mathrm{H}^2(X)$ carries a general weight-two Hodge structure with the relevant numerical invariants; such a Hodge structure is *Mumford-Tate generic* [34, §2]. Hodge cycles in $\mathrm{Sym}^2(\mathrm{H}^2(X))$ yield invariants for the action of the Hodge group of $\mathrm{H}^2(X)$, which is the special orthogonal group for $\mathrm{H}^2(X)$ under the Beauville-Bogomolov form. The only such invariants in $\mathrm{Sym}^2(\mathrm{H}^2(X))$ are multiples of the dual to the Beauville-Bogomolov form.

Integral classes in the summand $\mathbf{1}_X^{81}$ are called *canonical Hodge classes*, because they remain of type $(2,2)$ under arbitrary deformations.

We construct 81 distinguished rational surfaces in $X = K_2(A)$, whose classes span an 81-dimensional subspace in $\mathrm{H}^4(X,\mathbb{Q})$ which contains the summand $\mathbf{1}_X^{80}$ above. This 81-dimensional space is different from the subspace indicated in (3), but we will describe explicitly how they are related.

For each $\tau \in A$, let W_τ denote the subschemes in $A^{[3]}$ supported entirely at τ, with the induced reduced scheme structure. Briançon [6, p. 76] gives explicit equations for the corresponding subscheme of the Hilbert scheme, via local coordinates. Eliminating embedded components from these equations, we find that $W_\tau \simeq \mathbb{P}(1,1,3)$, realized as a cone over a twisted cubic in $\mathbb{P}^4$.

In the case where $\tau \in A[3]$, we have $W_\tau \subset K_2(A)$. This yields 81 disjoint copies of $\mathbb{P}(1,1,3)$. We recover some of their intersection properties:

- $W_\tau^2 = 3$;
- $W_\tau e^2 = 3$;
- $W_\tau c_2(K_2(A)) = -1$.

We prove these assertions. Using the fibration

$$\begin{array}{ccc} K_2(A) & \to & A^{[3]} \\ & & \downarrow \Sigma \\ & & A \end{array}$$

we can reduce the computation of the first number to the main result of [10]. For the second number, it suffices to check that

$$\mathcal{O}_{K_2(A)}(e)|W_\tau = \mathcal{O}_{\mathbb{P}(1,1,3)}(-H)$$

where H is the hyperplane class associated with $\mathbb{P}(1,1,3) \hookrightarrow \mathbb{P}^4$. Let R denote the extremal ray corresponding to the generic fiber of the diagonal divisor (in $K_2(A)$) over the diagonal of the symmetric product. The diagonal divisor has class $2e$ and $2e \cdot R = -2$, as the symmetric product has A_1-singularities at the generic point of the diagonal. Thus $e \cdot R = -1$. However, R specializes to some cycle of rational curves $R_0 \subset W_\tau$ and $e|W_\tau$ is a non-trivial Cartier divisor, i.e., $e = -nH$ for some n. Thus

$$-1 = e \cdot R = -nH \cdot R_0,$$

hence $n = 1$ and R_0 is a ruling of $\mathbb{P}(1,1,3)$. For the third number, consider the induced morphism

$$\phi : \mathbb{F}_3 \to \mathbb{P}(1,1,3) \simeq W_\tau \hookrightarrow K_2(A)$$

from the Hirzebruch surface resolving W_τ; let Σ_0 denote the (-3)-curve contracted by ϕ. We have the exact sequence

$$0 \to T_{\mathbb{F}_3} \to \phi^* T_{K_2(A)} \to N_\phi \to 0$$

which implies

$$c_2(\phi^* T_{K_2(A)}) = c_2(N_\phi) - 4.$$

We may interpret

$$c_2(N_\phi) + \{\text{contribution of } \Sigma_0 \text{ to the double point formula}\} = W_\tau^2 = 3,$$

and an excess-intersection computation [12, ch. 9] shows that contribution of Σ_0 is zero. We conclude that $c_2(\phi^* T_{K_2(A)}) = -1$.

Remark 4.2 (contributed by M. Nieper-Wisskirchen)**.** These computations can also be put into the framework of Nakajima's [27] description of the cohomology of the Hilbert scheme.

Consider the Galois cover

$$\begin{array}{rcl} v : A \times K_{n-1}(A) & \to & A^{[n]} \\ (a, \xi) & \mapsto & a + \xi \end{array}$$

where $a + \xi$ denotes the translate of ξ by a. The Galois group is $A[n]$. Let B_0 be the Briançon variety in $A^{[n]}$ of those subschemes whose support is 0. Then $v^* B_0$ is the same as $0 \times W$ in the cohomology ring of $A \times K_{n-1}(A)$, where $W = \sum_{\tau \in A[n]} W_\tau$. Let B be the Briançon variety in $A^{[n]}$ of those subschemes that are supported at a single arbitrary point. Then $v^* B = A \times W$.

Now assume $n = 3$. To show that $W_\tau^2 = 3$, we have to show $(v^* B_0) \cdot (v^* B) = 81 \cdot 3$, as the different W_τ are orthogonal. As $\deg(v) = 81$, this is the same as showing that $B_0 \cdot B = 3$ on the Hilbert scheme. Now B_0 happens to be $q_3(\omega)|\mathbf{0}\rangle$, where ω is the codimension-four class of a point in A, and B is $q_3(1)|\mathbf{0}\rangle$, using the standard notation for the cohomology classes in the Nakajima basis [4, §4]. Using the well-known commutation relations between these operators, one gets that the Poincaré duality pairing of $q_3(\omega)|\mathbf{0}\rangle$ and $q_3(1)|\mathbf{0}\rangle$ on the Hilbert scheme is exactly 3.

Intersections with c_2 can be obtained using the techniques of [3, 4].

Consider the averaged class

$$W := \sum_{\tau \in A[3]} W_\tau$$

with intersection properties

- $W^2 = 81 \cdot 3 = 243$;
- $We^2 = 243$;
- $Wc_2(K_2(A)) = -81$.

It follows that

$$W = \frac{3}{8}(c_2(X) + 3e^2).$$

Given $p \in A$ consider the locus

$$Y_p = \{(a_1, a_2, p) : a_1 + a_2 + p = 0\} \subset K_2(A).$$

Exchanging the first two terms induces

$$\iota : a_1 \mapsto -p - a_1$$

on the first factor, which is conjugate to the sign involution. (Set $a_1 = q + \alpha$ for $2q = -p$; then $\iota(\alpha) = -\alpha$.) In addition, Y_p meets the boundary at $\{(-2p) + p + p\}$. Thus Y_p is isomorphic to the Kummer surface $K_1(A)$ blown up at one point, i.e., the images of $\alpha = \pm 3q$.

The diagonal divisor meets Y_p along the 16 distinguished (-2)-curves and with multiplicity two along the (-1)-curve over the center, thus we have

$$e^2 Y_p = \frac{1}{4}(-4 + 16 \cdot (-2)) = -9.$$

Evidently, Y_p is disjoint from W_τ for $p \notin A[3]$, thus

$$Y_p \cdot W_\tau = 0.$$

Using the computations for W_τ done previously, we find

$$Y_p = \frac{1}{72}(3c_2(K_2(A)) + e^2).$$

In particular, $Y_p^2 = 1$ and given $f, g \in \mathrm{H}^2(A, \mathbb{C})$ we have

$$f \cdot g \cdot Y_p = 2f \cdot g.$$

Assume $p \notin A[3]$ and write

$$Z_\tau = Y_p - W_\tau.$$

As a consequence, we deduce

$$Z_\tau \cdot D_1 \cdot D_2 = 2\,(D_1, D_2)$$

for all $D_1, D_2 \in \mathrm{H}^2(K_2(A), \mathbb{Z})$. By the orthogonal decompositions (2) and (3), we conclude $Z_\tau \in \mathbf{1}^{81}_{K_2(A)}$, i.e., they are canonical Hodge cycles. We summarize this analysis as follows:

Proposition 4.3. *Let X be deformation equivalent to $K_2(A)$, for A an abelian surface. Consider the lattice of canonical Hodge classes*

$$(\mathbf{1}^{81}_X \cap \mathrm{H}^4(X, \mathbb{Z})) \subset \mathrm{H}^4(X, \mathbb{Z}),$$

associated with the decomposition (3). The classes $\{Z_\tau\}_{\tau \in A[3]}$ in this lattice span $(\mathbf{1}^{81}_X \cap \mathrm{H}^4(X, \mathbb{Q}))$ and satisfy

$$Z_\tau^2 = 4, \quad Z_\tau \cdot Z_{\tau'} = 1 \text{ if } \tau \neq \tau', \quad c_2(X) Z_\tau = 28. \tag{4}$$

This raises a question: How do we interpret these geometrically? One approach is to analyze the limit $\lim_{p \to \tau} Y_p$, to determine whether it contains W_τ as an irreducible component, with residual intersection in the summand $\mathbf{1}^{81}_X$.

The approach we take is to analyze the action of G_3 on $K_2(A)$ and its deformations. Regard $G_3 \subset \mathrm{Sp}(A[3]) \ltimes A[3]$, the canonical semidirect product associated to the action of the symplectic group on $A[3]$.

Theorem 4.4 (Hodge conjecture for canonical classes)**.** *Consider the action of G_3 on $K_2(A)$. For each $\tau \in A[3] \subset G_3$, let $\iota_\tau \in G_3$ denote the involution fixing τ.*

1. *The fixed point locus of ι_τ has two irreducible components. First, there is an isolated point corresponding to the vertex of W_τ. Second, there is a Kummer surface that is an irreducible component of $\lim_{p \to \tau} Y_p$ and has class Z_τ. For instance, we have*
$$Z_0 = \overline{\{(a_1, a_2, a_3) : a_1 = 0, a_2 = -a_3, a_2 \neq 0\}}$$
and the other Z_τ are translates of Z_0 via the action of $A[3]$.
2. *For each deformation X of $K_2(A)$, the Z_τ deforms to a submanifold of X.*

Proof. Each ι_τ is conjugate to the involution induced by the sign involution on A; its fixed points are clearly the stratum Z_0 and the vertex of the exceptional divisor W_0.

Proposition 3.1 shows each ι_τ carries over to deformations of X; thus the fixed-point loci carry over as well. □

Definition 4.5. Let X be an irreducible holomorphic symplectic manifold, with its natural hyperkähler structure. A submanifold $Z \subset X$

is *trianalytic* if Z is analytic with respect to each of the associated complex structures.

These have been studied systematically by Verbitsky; see, for example, [32, 34]. One general result is that analytic subvarieties representing canonical Hodge classes are automatically trianalytic [32, Thm. 4.1]. In particular, *all* analytic subvarieties of (Mumford-Tate) general irreducible holomorphic symplectic manifolds have this property. The deformations of the Z_τ in X are thus examples of trianalytic submanifolds. We lack a clear picture of what these generic deformations look like. However, their middle cohomology should have a piece that is isogenous to $\mathrm{H}^2(X)$. For more discussion of trianalytic subvarieties of generalized Kummer varieties, see [21, §6].

5. The lattice of canonical classes

Proposition 5.1. *The lattice* $\Pi = \langle Z_\tau : \tau \in A[3]\rangle$ *under the intersection form is positive definite of discriminant* $2^2 \cdot 3^{81} \cdot 7$. *We have*

$$c_2(X) = \frac{1}{3} \sum_{\tau \in A[3]} Z_\tau,$$

which is non-divisible.

Proof. From Equation (4), we see that the intersection form of Π is

	Z_{τ_1}	Z_{τ_2}	Z_{τ_3}	$\cdots$
Z_{τ_1}	4	1	1	$\cdots$
Z_{τ_2}	1	4	1	$\cdots$
Z_{τ_3}	1	1	4	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

so the corresponding matrix has eigenvalues 84 (with multiplicity one) and 3 (with multiplicity eighty).

The non-divisibility follows from

$$c_2(X) \cdot Z_j = 28, \quad c_2(X)^2 = 756, \quad c_2(X) \cdot \mathbb{P}^2 = -3,$$

where $\mathbb{P}^2 \subset X$ is a Lagrangian plane. □

Proposition 5.2. *Consider the generalized Kummer varieties deformation equivalent to* $K_2(A)$. *The image of the monodromy representation on*

$$\Pi = \langle Z_\tau : \tau \in A[3]\rangle \subset \mathrm{H}^4(X, \mathbb{Z})$$

contains the semidirect product

$$\mathrm{Sp}(A[3]) \ltimes A[3],$$

where $\mathrm{Sp}(A[3])$ *is the symplectic group. These groups act on the* $Z_\tau, \tau \in A[3]$ *via permutation.*

Proof. The monodromy representation for abelian surfaces acts on their three-torsion via the symplectic group $\mathrm{Sp}(A[3])$; this group acts on Π as well via the permutation representation on the Z_τ. In addition, translation by a three-torsion element $\tau' \in A[3]$ induces a non-trivial action on $K_2(A)$; the action on Π corresponds to the permutation

$$Z_\tau \mapsto Z_{\tau+\tau'}.$$

□

6. Comparison between Hilbert schemes and generalized Kummer varieties

The middle cohomology of the Hilbert scheme Y of length-two subschemes of a K3 surface is generated by the second cohomology. Thus the class of a Lagrangian plane in Y can be written as a quadratic polynomial in $\mathrm{H}^2(Y)$. After suitable deformation, it can be written as a linear combination of $c_2(Y)$ and λ^2, where λ is proportional to the class of a line. This *Ansatz* allowed us to compute (λ, λ) in this case [17, §4].

In our situation, the presence of 80 additional classes complicates the algebra. The following proposition shows these must be included in any formula for the class of a Lagrangian plane:

Proposition 6.1. *Let* X *be deformation equivalent to* $K_2(A)$ *and* $P \subset X$ *a Lagrangian plane. Let* $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$ *be* $6[\ell]$ *where* $\ell \subset P$ *is a line. Then we cannot write*

$$[P] = a\lambda^2 + bc_2(X)$$

for any rational $a, b \in \mathbb{Q}$.

Proof. Recall that [31, p. 123]

$$c_2(X)^2 = 756 \tag{5}$$

and [8, §5],[7, Thm. 6]

$$\chi(\mathcal{O}_X(L)) = 3\binom{\frac{(L,L)}{2}+2}{2} = L^4/4! + L^2c_2(X)/24 + \chi(\mathcal{O}_X)$$

for any divisor L. Thus we find

$$L^4 = 9\,(L,L)^2\,, L^2c_2(X) = 54\,(L,L)\,. \tag{6}$$

The condition $[P]^2 = 3$ translates into

$$3 = 9a^2\,(\lambda,\lambda)^2 + 108ab\,(\lambda,\lambda) + 756b^2. \tag{7}$$

The condition $c_2(X)|\mathbb{P}^2 = -3$ yields

$$-3 = 54a\,(\lambda,\lambda) + 756b. \tag{8}$$

Finally, the fact that $\lambda|\mathbb{P}^2 = (\lambda,\ell)\,\ell$ implies

$$(\lambda,\lambda)^2/36 = 9a\,(\lambda,\lambda)^2 + 54b\,(\lambda,\lambda). \tag{9}$$

These equations admit no solutions for $a, b, (\lambda,\lambda) \in \mathbb{Q}$. □

7. A key special case

In order to formulate a revised *Ansatz*, we analyze a specific example: Let $A = E_1 \times E_2$ and consider the planes $P \subset K_2(A)$ associated with the linear series $3 \cdot (0)$ in $E_1 \times 0$. Let $\Lambda' = E_1[3] \times 0 \subset A[3]$, ℓ the class of a line in P, and

$$\lambda = 6\ell = 6E_1 - 3e$$

the corresponding class in $\mathrm{H}^2(K_2(A),\mathbb{Z})$.

Proposition 7.1.

$$[P] = \frac{1}{216}\lambda^2 + \frac{1}{8}c_2(K_2(A)) - \frac{1}{3}\sum_{\tau\in\Lambda'} Z_\tau$$

Proof. We start with the Ansatz

$$[P] = a\lambda^2 + bc_2(X) + \widehat{Z}$$

where $\widehat{Z} \in c_2(X)^\perp \cap \langle Z_\tau : \tau \in A[3]\rangle$.

We apply Equations (5) and (6) above, as well as the constraints

$$[P]^2 = 3, \quad c_2(X)|\mathbb{P}^2 = -3, \lambda|\mathbb{P}^2 = (\lambda,\ell)\,\ell.$$

Under our revised Ansatz, Equation (7) changes but Equations (8) and (9) remain the same:

$$3 = 9a^2\,(\lambda,\lambda)^2 + 108ab\,(\lambda,\lambda) + 756b^2 + \widehat{Z}\cdot\widehat{Z},$$
$$-3 = 54a\,(\lambda,\lambda) + 756b$$
$$(\lambda,\lambda)^2/36 = 9a\,(\lambda,\lambda)^2 + 54b\,(\lambda,\lambda).$$

Given that $(\lambda,\lambda) = -54$ in this particular example, the last two equations allow us to solve for a and b, i.e., we find $a = 1/216$ and $b = 1/72$. Thus we have

$$[P] = \frac{1}{216}\lambda^2 + \frac{1}{72}c_2(K_2(A)) + \widehat{Z}.$$

It follows that

$$\begin{aligned}[P]\cdot e^2 &= \tfrac{1}{216}\lambda^2 e^2 + \tfrac{54}{72}\,(e,e)\\ &= \tfrac{3}{216}((\lambda,\lambda)\,(e,e) + 2\,(\lambda,e)^2) - 9/2\\ &= 9.\end{aligned}$$

We apply the formulas from Section 4: We know that

$$P \cdot [W] = P \cdot \frac{3}{8}(c_2(X) + 3e^2) = -9/8 + 81/8 = 9$$

and

$$P \cdot [Y_p] = P \cdot \frac{1}{72}(3c_2(X) + e^2) = 0.$$

The geometry of P shows it is disjoint from W_τ for $\tau \notin \Lambda'$. By symmetry, we conclude that

$$P \cdot [W_\tau] = 1, \quad \tau \in \Lambda'.$$

It follows that

$$P \cdot Z_\tau = \begin{cases} 0 & \text{if } \tau \notin \Lambda' \\ -1 & \text{if } \tau \in \Lambda', \end{cases}$$

which reflects the fact that P fails to intersect these trianalytic varieties transversally.

Thus for $\tau \in \Lambda'$ we have

$$\begin{array}{rcl} \widehat{Z} \cdot Z_\tau & = & \left([P] - \frac{1}{216}\lambda^2 - \frac{1}{72}c_2(X)\right) \cdot Z_\tau \\ & = & -1 - \frac{1}{216}2\,(\lambda, \lambda) - \frac{1}{72}28 \\ & = & -8/9. \end{array}$$

Since the $\tau \in \Lambda'$ appear symmetrically in $\widehat{Z}$, we conclude that

$$\widehat{Z} = \alpha \left(\frac{1}{3}c_2(X) - \sum_{\tau \in \Lambda'} Z_\tau \right)$$

for some constant α. We saw in Section 5 that

$$Z_\tau Z_{\tau'} = \begin{cases} 4 & \text{if } \tau = \tau' \\ 1 & \text{if } \tau \neq \tau', \end{cases}$$

thus $\alpha = 1/3$ and we conclude

$$[P] = \frac{1}{216}\lambda^2 + \frac{1}{72}c_2(X) + \frac{1}{3}(c_2(X)/3 - \sum_{\tau \in \Lambda'} Z_\tau),$$

which yields the desired formula. □

8. Integrality

The example in Section 7 allows us to obtain further integrality results:

Proposition 8.1. *Let $\Lambda' \subset A[3]$ be a translate of a non-isotropic two-dimensional subspace. Then*

$$\frac{1}{8}c_2(X) - \frac{1}{3}\sum_{\tau \in \Lambda'} Z_\tau$$

intersects each class of Π integrally.

This statement follows from the computation in Proposition 7.1 and Proposition 5.2, which implies that the monodromy acts transitively on translates of non-isotropic subspaces. These are precisely the projections of planes described above (and their orbits under the monodromy action of $\mathrm{Sp}(A[3])$ and translation by $A[3]$) into Π.

Proposition 8.2. *Let Π' denote the dual of the lattice Π, with the $\mathbb{Q}$-valued intersection form induced by $\Pi' \subset \Pi \otimes \mathbb{Q}$. Then Π' is generated by*

- $\frac{1}{28}c_2(X) = \frac{1}{84}\sum_\tau Z_\tau$;
- $\frac{1}{3}\sum_\tau \beta_\tau Z_\tau$, *where the $\beta_\tau \in \mathbb{Z}$ and satisfy $\sum_\tau \beta_\tau \equiv 0 \pmod 3$.*

Proof. It is evident from Proposition 5.1 that these classes generate a subgroup containing Π and intersect each of the Z_τ integrally. Thus if M is the lattice they generate, we have

$$\Pi \subset M \subset \Pi'.$$

We compute the index of Π in M. The sublattice $M_2 \subset \Pi'$ generated by the classes $\frac{1}{3}\sum_\tau \beta_\tau Z_\tau$ as above factors

$$\Pi \subset M_2 \subset \frac{1}{3}\Pi.$$

The image of M_2 in $\frac{1}{3}\Pi/\Pi \simeq (\mathbb{Z}/3\mathbb{Z})^{81}$ is a hyperplane, thus $\Pi \subset M_2$ has index 3^{80}. Note that $c_2(X) \in M_2$ as a primitive vector, thus

$$M_2 \subset M_2 + \frac{1}{28}c_2(X) = M$$

has index 28. Hence the index of $\Pi \subset M$ is $3^{80} \cdot 28$, which is the discriminant. ☐

Proposition 8.3. *Consider the vectors $Z'_\tau := \frac{1}{3}(\frac{c_2(X)}{28} - Z_\tau)$ for each τ, which have intersections*

$$Z'_{\tau_1} Z'_{\tau_2} = \begin{cases} \frac{83}{28\cdot 9} & \text{if } \tau_1 = \tau_2 \\ \frac{-1}{28\cdot 9} & \text{if } \tau_1 \neq \tau_2, \end{cases} . \tag{10}$$

We have

$$\Pi' = \{\sum_\tau \beta_\tau Z'_\tau : \beta_\tau \in \mathbb{Z}, \sum_\tau \beta_\tau \equiv 0 \pmod 3\}.$$

The intersection form on Π' takes values in $\frac{1}{28\cdot 9}\mathbb{Z}$; squares of elements are in $\frac{1}{28\cdot 3}\mathbb{Z}$.

Proof. Only the last statement requires verification: It suffices to observe that the matrix

$$\Psi = \begin{pmatrix} 83 & -1 & -1 & \cdots \\ -1 & 83 & -1 & \ddots \\ -1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \cdots & 83 \end{pmatrix} \tag{11}$$

is congruent to

$$\begin{pmatrix} -1 & -1 & -1 & \cdots \\ -1 & -1 & -1 & \ddots \\ -1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \cdots & -1 \end{pmatrix}$$

modulo three. Elements summing to zero modulo three are in the kernel. □

9. Computing (λ, λ)

We complete the proof of Theorem 1.1 and extract additional information on how classes of Lagrangian planes project onto the tautological classes.

We formulate a new *Ansatz* in light of the example in Section 7. Let P be a class of a plane, which we now assume can be written

$$P = a\lambda^2 + bc_2(X) + \widehat{Z} \tag{12}$$

as above and in the proof of Proposition 7.1. Note that $\widehat{Z}^2 \geq 0$ as the lattice Π is positive definite, thus

$$(a\lambda^2 + bc_2(X))^2 \leq 3.$$

Using Equations (8) and (9) to eliminate a and b, we find

$$(a\lambda^2 + bc_2(X))^2 = \frac{1}{46656}(7\,(\lambda,\lambda)^2 + 108\,(\lambda,\lambda) + 972).$$

This is always positive, and is ≤ 3 only when

$$-149 < (\lambda,\lambda) < 134. \tag{13}$$

Thus there are only finitely many possibilities for (λ, λ).

From now on, we shall use decomposition (3) in order to take advantage of the action of the automorphism group on the canonical Hodge classes. Let $\mu \in \mathbb{Z}\lambda^2 + \mathbb{Z}c_2(X)$ denote a class orthogonal to $c_2(X)$ such that λ^2 appears with a positive coefficient. By construction, μ is also orthogonal to each of the Z_τ and Z'_τ. This is unique up to a positive scalar; we'll take

$$\mu = 7\lambda^2 - \frac{(\lambda,\lambda)}{2} c_2(X).$$

Using the intersection numbers computed above, we find

$$\mu^2 = 49 \cdot 9\,(\lambda,\lambda)^2 - 7 \cdot 54\,(\lambda,\lambda)^2 + 756/4\,(\lambda,\lambda)^2 = 252\,(\lambda,\lambda)^2 \geq 0.$$

We write the class of a Lagrangian plane in terms of decomposition (3). Since $[P]$ is integral, its projection onto each summand lies in the dual to the underlying lattice. In particular, we have

$$[P] = \alpha\mu + \sum_\tau \beta_\tau Z'_\tau, \quad \beta_\tau \in \mathbb{Z}, \alpha \in \mathbb{Q}, \tag{14}$$

and $\sum_\tau \beta_\tau \equiv 0 \pmod 3$. If a is the constant in Equation (12) we have $a = 7\alpha$. Combining the equations above with these, we find

$$(\alpha\mu)^2 = a^2\,(\lambda,\lambda)^2\,\frac{36}{7}.$$

Write

$$S = \left(\sum_\tau \beta_\tau Z'_\tau\right)^2 = P^2 - (\alpha\mu)^2 = 3 - a^2\,(\lambda,\lambda)^2\,\frac{36}{7}$$

and use the equations above to eliminate a. We find

$$\begin{aligned} S &= \frac{-49\,(\lambda,\lambda)^2 - 756\,(\lambda,\lambda) + 976860}{326592} \\ &= \frac{-49\,(\lambda,\lambda)^2 - 756\,(\lambda,\lambda) + 2^2 \cdot 3^6 \cdot 5 \cdot 67}{2^6 \cdot 3^6 \cdot 7} \end{aligned}$$

The lattice $\langle Z'_\tau \rangle$ is positive definite, so $S > 0$.

Lemma 9.1 (Analysis of Short Vectors I)**.** *Let $\sum_\tau \beta_\tau Z'_\tau$ denote the projection of P onto the sublattice Π', as described in (14). Then we have $\sum_\tau \beta_\tau = 9$.*

Proof. Recall that $c_2(X)[P] = -3$ and $c_2(X)\mu = 0$, so that

$$c_2(X)(\sum_\tau \beta_\tau Z'_\tau) = -3.$$

On the other hand, we have

$$c_2(X) Z'_\tau = c_2(X)(c_2(X)/84 - Z_\tau/3) = -1/3,$$

so the equation follows. □

Let $\Psi = 9 \cdot 28 \cdot \Pi'$, i.e., the dual to Π renormalized to an integral form as in (11). Then Lemma 9.1 implies

$$\begin{array}{rcl}(\sum_\tau \beta_\tau Z'_\tau)^2_\Psi & = & 84(\sum_\tau \beta_\tau^2) - (\sum_\tau \beta_\tau)^2 \\ & & 84(\sum_\tau \beta_\tau^2) - 81.\end{array}$$

This means that $9{\cdot}28{\cdot}S{+}81 \equiv 0 \pmod{84}$, which yields the congruence

$$-49\,(\lambda,\lambda)^2 - 756\,(\lambda,\lambda) + 2^2\cdot 3^6\cdot 7\cdot 53 \equiv 0 \pmod{2^6\cdot 3^5\cdot 7}.$$

We obtain that

$$(\lambda,\lambda) \equiv 2 \pmod 8, \quad (\lambda,\lambda) \equiv 0 \pmod{27}.$$

Combining this with the bounds (13), we conclude that

$$(\lambda,\lambda) = -54.$$

Since $\lambda = 6\ell$, we deduce that $(\ell,\ell) = -3/2$, which completes the proof of Theorem 1.1.

We extract a bit more information about the β_τ:

Lemma 9.2 (Analysis of Short Vectors II)**.** *Retain the notation of Lemma 9.1. We have $\beta_\tau = 0$ for all but nine values of τ, for which $\beta_\tau = 1$.*

Proof. Since $(\lambda,\lambda) = -54$ the expression above for S in terms of (λ,λ) gives $S = 75/28$, which means that

$$84\sum_\tau \beta_\tau^2 = 81 + 9\cdot 28\cdot S = 756$$

whence

$$\sum_\tau \beta_\tau^2 = \sum_\tau \beta_\tau = 9.$$

Since the β_τ are integers, at most 9 of them are non-zero, and we may as well restrict to that subspace.

By basic calculus, the maximal value of the function $x_1 + \cdots + x_n$ on the unit sphere is $\sqrt{n}$, achieved precisely at $x_1 = x_2 = \cdots = x_n = 1/\sqrt{n}$. Applying this to our nine-dimensional subspace gives the result. □

We summarize the conclusions of this section:

Proposition 9.3. *Let X be deformation equivalent to $K_2(A)$, $P \subset X$ a Lagrangian plane, $\ell \in \mathrm{H}_2(X,\mathbb{Z})$ the class of a line on the plane, and $\lambda = 6\ell \in \mathrm{H}^2(X,\mathbb{Z})$. Then we have*

$$[P] = \frac{1}{216}\lambda^2 + \frac{1}{56}c_2(X) + \sum_{\tau\in\Lambda} Z'_\tau = \frac{1}{216}\lambda^2 + \frac{1}{8}c_2(X) - \frac{1}{3}\sum_{\tau\in\Lambda} Z_\tau, \tag{15}$$

where $(\lambda,\lambda) = -54$ and $\Lambda \subset A[3]$ is a set of cardinality nine.

10. Monodromy and intersection analysis

We continue to identify the set of all τ with $A[3]$, the three-torsion of an abelian surface. Recall the following geometric facts:

- the group G_3 acts on X, and thus on $\{Z_\tau : \tau \in A[3]\}$ and Π, as described in Proposition 3.1;
- the Z_τ are canonical trianalytic submanifolds, and they deform as X deforms (see the end of Section 4);
- the monodromy group acts on $\{Z_\tau\}$ and Π, as described in Proposition 5.2; given a distinguished base point τ_0, the intersection of the monodromy group with the stabilizer of τ_0 contains the symplectic group $\mathrm{Sp}(A[3])$, where τ_0 is interpreted as 0.

Theorem 10.1. *Suppose $P \subset X$ is a Lagrangian plane as in Proposition 9.3. Then $\Lambda \subset A[3]$ is a translate of a two-dimensional subspace.*

Note that the example presented in Section 7 and the facts on monodromy/automorphisms quoted above imply that every translate Λ of a two-dimensional non-isotropic subspace arises from a plane in a deformation of X. In Sections 12 and 13, we will address whether *isotropic* subspaces arise from planes.

Corollary 10.2. *Suppose that $P \subset X$ is a plane contained in a manifold deformation equivalent to $K_2(A)$. Then the orbit of P under* $\mathrm{Aut}(X)$ *contains nine distinct planes; the stabilizer of P has order divisible by nine.*

Proposition 10.3. *Let $\Lambda \subset A[3]$ denote a set of points arising from a Lagrangian plane, as in Proposition 9.3. Then*

$$\#\{\Lambda \cap (\Lambda + \tau_0)\} \pmod 3$$

is constant as $\tau_0 \in A[3]$ varies.

Proof. Let $P_2 \subset X$ denote the plane obtained by letting τ_0 act on P. Since

$$P^2 - P \cdot P_2 \in \mathbb{Z}$$

we have

$$\left(\sum_{\tau\in\Lambda} Z'_\tau\right)^2 \equiv \left(\sum_{\tau\in\Lambda} Z'_\tau\right)\left(\sum_{\tau\in(\Lambda+\tau_0)} Z'_\tau\right) \pmod{\mathbb{Z}}.$$

The corresponding vectors have pairing (with respect to Ψ) divisible by $9 \cdot 28$

$$\left(\sum_{\tau\in\Lambda} Z'_\tau\right) \cdot_\Psi \left(\sum_{\tau\in(\Lambda+\tau_0)} Z'_\tau\right) \equiv 0 \pmod{9 \cdot 28}.$$

Modulo $9 \cdot 28$ we find

$$83\#\{\tau \in \Lambda \cap (\Lambda + \tau_0)\} - \#\{(\tau, \widetilde{\tau}) : \tau \neq \widetilde{\tau}, \tau \in \Lambda, \widetilde{\tau} \in (\Lambda + \tau_0)\} \equiv 0$$

and

$$84\#\{\tau \in \Lambda \cap (\Lambda + \tau_0)\} - \#\{(\tau, \widetilde{\tau}) : \tau \in \Lambda, \widetilde{\tau} \in (\Lambda + \tau_0)\} \equiv 0.$$

Since $|\Lambda| = |\Lambda + \tau_0| = 9$, we have

$$84\#\{\tau \in \Lambda \cap (\Lambda + \tau_0)\} \equiv 81 \pmod{9 \cdot 28}$$

and

$$\#\{\tau \in \Lambda \cap (\Lambda + \tau_0)\} \equiv 0 \pmod{3}.$$

□

Remark 10.4. This is insufficient to characterize translates of two-dimensional subspaces in $A[3] \simeq (\mathbb{Z}/3\mathbb{Z})^4$. For instance, consider the set

$$\{e_2, e_2 + e_1, e_2 - e_1, e_3, e_3 + e_1, e_3 - e_1, e_4, e_4 + e_1, e_4 - e_1\}.$$

Every element of the orbit of this set under G_3 meets the set in 0 or 3 points. Thus our intersection condition is insufficient to establish Theorem 10.1.

We strengthen the analysis above. Suppose P and $\widetilde{P}$ are planes of the form

$$[P] = \alpha\mu + \sum_{\tau} \beta_\tau Z'_\tau, \quad [\widetilde{P}] = \alpha\mu + \sum_{\tau} \widetilde{\beta}_\tau Z'_\tau,$$

so the difference is an *integral* class

$$[P] - [\widetilde{P}] = \sum_{\tau} (\beta_\tau - \widetilde{\beta}_\tau) Z'_\tau. \tag{16}$$

The non-zero coefficients are all ± 1. Applying the symplectic group $\mathrm{Sp}(A[3])$, we get additional integral classes. In particular, the classes

$$\sum_{\tau \in \widetilde{\Lambda}} Z'_\tau - Z'_{\tau + \tau_0}$$

are integral for each non-isotropic $\widetilde{\Lambda} \subset A[3]$ and $\tau_0 \in A[3]$.

Proposition 10.5. *Let $\widetilde{\Lambda}$ denote a non-isotropic subspace and Λ a set of points arising from a Lagrangian plane. Then*

$$\#\{\Lambda \cap (\widetilde{\Lambda} + \tau_0)\} \pmod{3}$$

is constant as τ_0 varies.

Proof. Consider the classes $\sum_{\tau\in\Lambda} Z'_\tau$, $\sum_{\tau\in\widetilde{\Lambda}} Z'_\tau$, and $\sum_{\tau\in\Lambda+\tau_0} Z'_\tau$. We have

$$\left(\sum_{\tau\in\Lambda} Z'_\tau\right)\cdot\left(\sum_{\tau\in\widetilde{\Lambda}+\tau_0} Z'_\tau-\sum_{\tau\in\widetilde{\Lambda}} Z'_\tau\right)\in\mathbb{Z}$$

which translates into

$$\left(\sum_{\tau\in\Lambda} Z'_\tau\right)\cdot_\Psi\left(\sum_{\tau\in\widetilde{\Lambda}+\tau_0} Z'_\tau-\sum_{\tau\in\widetilde{\Lambda}} Z'_\tau\right)\equiv 0 \pmod{9\cdot 28}.$$

Thus we have

$$\begin{aligned}&83\#\{\tau\in\Lambda\cap(\widetilde{\Lambda}+\tau_0)\}-\#\{(\tau,\widetilde{\tau}):\tau\neq\widetilde{\tau},\tau\in\Lambda,\widetilde{\tau}\in\widetilde{\Lambda}+\tau_0\}\\ &\equiv 83\#\{\tau\in\Lambda\cap\widetilde{\Lambda}\}-\#\{(\tau,\widetilde{\tau}):\tau\neq\widetilde{\tau},\tau\in\Lambda,\widetilde{\tau}\in\widetilde{\Lambda}\}\pmod{9\cdot 28}\end{aligned}$$

and

$$\begin{aligned}&84\#\{\tau\in\Lambda\cap(\widetilde{\Lambda}+\tau_0)\}-\#\{(\tau,\widetilde{\tau}):\tau\in\Lambda,\widetilde{\tau}\in(\widetilde{\Lambda}+\tau_0)\}\\ &\equiv 84\#\{\tau\in\Lambda\cap\widetilde{\Lambda}\}-\#\{(\tau,\widetilde{\tau}):\tau\in\Lambda,\widetilde{\tau}\in\widetilde{\Lambda}\}\pmod{9\cdot 28}.\end{aligned}$$

Since $|\Lambda|=|\widetilde{\Lambda}|=|\widetilde{\Lambda}+\tau_0|=9$, we obtain

$$\#\{\tau\in\Lambda\cap(\widetilde{\Lambda}+\tau_0)\}\equiv\#\{\tau\in\Lambda\cap\widetilde{\Lambda}\}\pmod{3},$$

which is what we sought to prove. □

To complete the proof of Theorem 10.1, we shall need the following result on finite geometries, which should be understood in the context of Radon transforms over finite fields [37]:

Proposition 10.6. *Let V be a four-dimensional vector space over a finite field with q elements, with q odd. Suppose that V admits a symplectic form. Suppose that $\Lambda\subset V$ is a subset with q^2 elements such that, for each affine non-isotropic plane $\widetilde{\Lambda}\subset V$, the function*

$$\begin{array}{ccl} V & \to & \mathbb{Z}/q\mathbb{Z}\\ \tau & \mapsto & \#\{\Lambda\cap(\widetilde{\Lambda}+\tau)\}\pmod{q}\end{array}$$

is constant. Then Λ is an affine plane in V.

Proof. It suffices to show that Λ is 'convex', in the sense that for any pair of distinct $\tau_1,\tau_2\in\Lambda$, the affine line $\mathfrak{l}(\tau_1,\tau_2)$ is contained in Λ. Suppose this is not the case, so in particular

$$\#\{\mathfrak{l}(\tau_1,\tau_2)\cap\Lambda\}<q.$$

For simplicity, assume that $\tau_1=0$ so that every affine plane containing $\mathfrak{l}(\tau_1,\tau_2)$ is a subspace. Let $\mathrm{Gr}(2,V)$ denote the Grassmannian (a smooth quadric hypersurface in $\mathbb{P}^5$), $\mathrm{IGr}(2,V)$ the isotropic Grassmannian (a smooth hyperplane section of $\mathrm{Gr}(2,V)$), and $\Sigma\subset\mathrm{Gr}(2,V)$

the Schubert variety of planes containing $\mathfrak{l}(\tau_1, \tau_2)$ (which is isomorphic to $\mathbb{P}^2$). Note that $\Sigma \not\subset \mathrm{IGr}(2, V)$, since the latter is a smooth quadric threefold. Thus

$$\Sigma_\circ := \Sigma \cap (\mathrm{Gr}(2, V) \setminus \mathrm{IGr}(2, V)) \simeq \mathbb{A}^2$$

which has q^2 points over our finite field.

The planes parametrized by $\Sigma_\circ$ are disjoint away from $\mathfrak{l}(\tau_1, \tau_2)$. Hence the pigeon-hole (Dirichlet) principle guarantees there exists at least one such plane $\widetilde{\Lambda}$ that contains no points of Λ outside $\mathfrak{l}(\tau_1, \tau_2)$. Otherwise, Λ would have more than q^2 points. Consequently,

$$m := \#\{\Lambda \cap \widetilde{\Lambda}\} = \#\{\Lambda \cap \mathfrak{l}(\tau_1, \tau_2)\}$$

is between 2 and $q - 1$.

Consider the affine translates $\widetilde{\Lambda} + \tau$, for τ taken from a set of coset representatives of $V/\widetilde{\Lambda}$. These are q^2 disjoint affine planes, each with at least m points of Λ. Thus Λ has cardinality at least $mq^2 > q^2$, a contradiction. □

11. Further remarks on saturation

Problem 11.1. Characterize the saturation of Π, i.e., the intersection

$$\Pi^{sat} = (\Pi \otimes \mathbb{Q}) \cap \mathrm{H}^4(X, \mathbb{Z}).$$

Example 11.2. Let $S = K_1(A)$ be a Kummer surface, Z_τ the (-2)-classes associated with $A[2]$, and $\Pi = \langle Z_\tau : \tau \in A[2] \rangle$ which has discriminant 2^{16}. The saturation of Π in $\mathrm{H}^2(S, \mathbb{Z})$ is computed in [24, §3]; an element

$$\frac{1}{2} \sum_{\tau} \epsilon(\tau) Z_\tau, \quad \epsilon(\tau) = 0, 1 \in \mathrm{H}^2(S, \mathbb{Z})$$

if and only if $\epsilon : A[2] \to \mathbb{Z}/2\mathbb{Z}$ is affine linear. In particular, the saturation has discriminant $2^{16-2\cdot 5} = 2^6$, since the affine linear functions have dimension five over $\mathbb{Z}/2\mathbb{Z}$.

We exhibit generators admitting geometric interpretations. For each $\Lambda \subset A[2]$ non-isotropic of dimension two, the class

$$\frac{1}{2} \left(\sum_{\tau \in \Lambda} Z_\tau - \sum_{\tau \in (\Lambda + \tau_0)} Z_\tau \right) \tag{17}$$

is integral for geometric reasons. Analogously to the example in Section 7, we can consider $A = E_1 \times E_2$ and $\mathbb{P}^1$'s corresponding to degree-two linear series on the factors. Taking difference of one such $\mathbb{P}^1$ and its translate, we obtain classes of the form (17).

We claim that the extension of Π corresponding to these geometric classes agrees with the extension associated with affine linear forms. Suppose that Λ is non-isotropic through the origin, defined by $x = y = 0$; over $\mathbb{Z}/2\mathbb{Z}$ it can be defined as $x^2+xy+y^2 = 0$. Taking the difference of this quadric and a translate, we find

$$x^2+xy+y^2-(x+a)^2-(x+a)(y+b)-(y+b^2) = -xb-ya-a^2-ab-b^2,$$

which span *all* affine linear forms involving x and y. Varying over all such Λ, we get the space of all affine linear forms.

The same construction is applicable to $K_2(A)$ as well: The additional classes (16) give an extension of $\Pi = \langle Z_\tau \rangle$ by a subgroup isomorphic to the $\mathbb{Z}/3\mathbb{Z}$-vector space of affine linear forms on $A[3]$. The resulting lattice has discriminant $2^2 \cdot 3^{71} \cdot 7$. However, this is far from the full saturation, because the discriminant of the lattice

$$\mathrm{Sym}^2(\mathrm{H}^2(X,\mathbb{Z})) \cap c_2(X)^\perp$$

is much smaller. Indeed, using the formula

$$\begin{aligned} D_1D_2D_3D_4 \; &= \; 3\left((D_1,D_2)\,(D_3,D_4) + (D_1,D_3)\,(D_2,D_4)\right. \\ &\qquad \left. + (D_1,D_4)\,(D_2,D_3)\right) \end{aligned}$$

for $D_1, D_2, D_3, D_4 \in \mathrm{H}^2(A,\mathbb{Z}) \subset \mathrm{H}^2(K_2(A),\mathbb{Z})$, we can show that $\mathrm{Sym}^2(\mathrm{H}^2(K_2(A)))$ has discriminant $2^{14} \cdot 3^{38}$. Taking the orthogonal complement to $c_2(X)$ can only increase the exponent of 3 by the power of 3 appearing in $c_2(X)^2 = 756$.

12. A negative result on isotropic subspaces

Theorem 10.1 yields affine subspaces $\Lambda \subset A[3]$ but does not specify whether they are isotropic or non-isotropic (up to translation). In Section 7, we exhibited examples of non-isotropic subspaces. Here we identify obstructions to the appearance of isotropic subspaces:

Proposition 12.1. *Let X and $\widetilde{X}$ denote manifolds deformation equivalent to $K_2(A)$. Assume there exist Lagrangian planes $P \subset X$ and $\widetilde{P} \subset \widetilde{X}$ with*

$$\begin{aligned} [P] \; &= \; \frac{1}{216}\lambda^2 + \frac{1}{56}c_2(X) + \sum_{\tau\in\Lambda} Z'_\tau, \\ [\widetilde{P}] \; &= \; \frac{1}{216}\widetilde{\lambda}^2 + \frac{1}{56}c_2(X) + \sum_{\tau\in\widetilde{\Lambda}} Z'_\tau, \end{aligned}$$

where $\widetilde{\Lambda} \subset A[3]$ is a two-dimensional non-isotropic affine subspace. Assume $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ and $\widetilde{\lambda} \in \mathrm{H}^2(\widetilde{X},\mathbb{Z})$ are equivalent under the monodromy action. Then Λ is not a translate of an isotropic subspace.

Proof. Suppose that Λ is a translate of a two-dimensional isotropic subspace; after applying an automorphism of X, we may assume $0 \in \Lambda$. Let γ be any element of the monodromy group of X acting on $A[3]$ via a symplectic transformation. Note that

$$\gamma\Lambda = \Lambda \text{ or } \gamma\Lambda \cap \Lambda = \{0\}$$

which implies

$$(\sum_{\tau\in\Lambda} Z'_\tau)\cdot(\sum_{\tau\in\gamma\Lambda} Z'_\tau) = \frac{75}{28} \text{ or } \frac{1}{84}.$$

Since $P \cdot \gamma(P) \in \mathbb{Z}$ we find that

$$(\frac{1}{216}\lambda^2 + \frac{1}{56}c_2(X))\cdot\gamma(\frac{1}{216}\lambda^2 + \frac{1}{56}c_2(X)) \equiv \frac{9}{28} \text{ or } \frac{-1}{84} \pmod{\mathbb{Z}}.$$

On the other hand, it is possible to produce γ where $\gamma(\widetilde{\Lambda}) \cap \widetilde{\Lambda}$ is one-dimensional (see Proposition 5.2). Here, it is crucial that $\widetilde{\Lambda}$ be non-isotropic. Then we have

$$(\sum_{\tau\in\widetilde{\Lambda}} Z'_\tau)\cdot(\sum_{\tau\in\gamma\widetilde{\Lambda}} Z'_\tau) = 1 - \frac{1}{28\cdot 9},$$

which combined with the fact that $\widetilde{P}\cdot\gamma(\widetilde{P}) \in \mathbb{Z}$ yields

$$(\frac{1}{216}\widetilde{\lambda}^2 + \frac{1}{56}c_2(X))\cdot\gamma(\frac{1}{216}\widetilde{\lambda}^2 + \frac{1}{56}c_2(X)) \equiv \frac{1}{28\cdot 9} \pmod{\mathbb{Z}}.$$

Since λ and $\widetilde{\lambda}$ are in the same orbit under the monodromy representation, they share common intersection properties. Thus this is incompatible with the first equation above. □

13. Divisibility properties and monodromy

Let X be deformation equivalent to $K_n(A)$. Note that

$$\mathrm{H}^2(A,\mathbb{Z}) \simeq U^{\oplus 3}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the Beauville-Bogomolov form on $\mathrm{H}^2(X,\mathbb{Z})$ (see (1)) has discriminant group $d(X) = \mathrm{Hom}(\mathrm{H}^2(X,\mathbb{Z}),\mathbb{Z})/\mathrm{H}^2(X,\mathbb{Z}) \simeq \mathbb{Z}/2(n+1)\mathbb{Z}$.

Let Γ denote the subgroup of the orthogonal group of $\mathrm{H}^2(X,\mathbb{Z})$ acting trivially on $d(X)$. Consider equivalence classes of primitive vectors, i.e., primitive $D, D' \in \mathrm{H}^2(X,\mathbb{Z})$ are equivalent if

(1) $(D,D) = (D',D')$,
(2) $(D,\mathrm{H}^2(X,\mathbb{Z})) = (D',\mathrm{H}^2(X,\mathbb{Z})) = \langle d \rangle$, and
(3) $\frac{1}{d}D = \frac{1}{d}D'$ in $d(X)$.

By [9, §10] and [14, Lemma 3.5], Γ-orbits of primitive vectors are equal to these equivalence classes. Thus each vector is in the Γ-orbit of a vector in the sublattice

$$U \oplus (-2(n+1)) \subset U^{\oplus 3} \oplus (-2(n+1)) .$$

Let Γ' denote the subgroup of the orthogonal group of $\mathrm{H}^2(X, \mathbb{Z})$ satisfying the following conditions

(1) Γ' is orientation preserving, i.e., it respects orientations on positive definite three-dimensional subspaces $L \subset \mathrm{H}^2(X, \mathbb{R})$;
(2) elements $\gamma \in \Gamma'$ with $\det(\gamma) = (-1)^{\epsilon(\gamma)}$, $\epsilon(\gamma) = 0, 1$, act on $d(X)$ via $(-1)^{\epsilon(\gamma)}$.

Each vector is fixed by an orientation reversing element of Γ; this is evident for vectors in $U \oplus (-2(n+1))$, which represent all Γ-orbits. Thus orbits of primitive $D \in \mathrm{H}^2(X, \mathbb{Z})$ are determined by

$$(D, D) , \quad \big(D, \mathrm{H}^2(X, \mathbb{Z})\big) = \langle d \rangle , \quad \frac{\pm D}{d} \in d(X).$$

Recently, Markman [26, Thm. 4.2] has announced that the image of the monodromy representation in the orthogonal group associated with $\mathrm{H}^2(X, \mathbb{Z})$ contains the group Γ'.

Assume again that X is deformation equivalent to $K_2(A)$. Thus the class ℓ of a line on a Lagrangian plane would have to be in one of the following equivalence classes

$$E - 3e^\vee \text{ or } 3e^\vee,$$

where $E \in U^{\oplus 3}$ is primitive with $(E, E) = 0$. Only the primitive class occurs because classes of type $e^\vee$ arise from rulings of the exceptional divisor in $K_2(A)$ for some complex torus A (cf. [26, Lemma 4.9]). Combining Markman's result with Proposition 9.3, Theorem 10.1, and Proposition 12.1, we obtain

Theorem 13.1. *Let X be deformation equivalent to $K_2(A)$. If $P \subset X$ is a Lagrangian plane then*

$$[P] = \frac{1}{216}\lambda^2 + \frac{1}{56}c_2(X) + \sum_{\tau \in \Lambda} Z'_\tau$$

where $\lambda/3 \in \mathrm{H}^2(X, \mathbb{Z})$, $(\lambda, \lambda) = -54$, and $\Lambda \subset A[3]$ is a two-dimensional non-isotropic affine subspace.

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