MOVING AND AMPLE CONES OF HOLOMORPHIC SYMPLECTIC FOURFOLDS

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Abstract. We analyze the ample and moving cones of holomorphic symplectic manifolds, in light of recent advances in the minimal model program. As an application, we establish a numerical criterion for ampleness of divisors on fourfolds deformation-equivalent to punctual Hilbert schemes of K3 surfaces.

1. Introduction

Let X be a complex smooth projective variety. Its Néron–Severi group $\mathrm{N}^1(X,\mathbb{Z})$, i.e., the group of divisors modulo homological equivalence, embeds in the cohomology group $H^2(X,\mathbb{Z})$. A divisor is *ample* if some nonnegative multiple is in the class of a hyperplane section of some projective embedding of X. A divisor is *numerically effective (nef)* if it intersects every curve on X nonnegatively; ample divisors are clearly nef. A divisor is *effective* if it is linearly equivalent to a sum of irreducible divisors with nonnegative coefficients.

In many geometric applications, e.g., explicit factorization of rational maps, it is important to identify the classes of ample divisors. By Kleiman's criterion, a divisor is ample if and only if its class is in the interior of the *nef cone*, i.e., the closed convex cone in $\mathrm{N}^1(X,\mathbb{R}) = \mathrm{N}^1(X,\mathbb{Z}) \otimes \mathbb{R} \subset H^2(X,\mathbb{R})$ spanned by the nef divisor classes. This description does not suffice to characterize ample divisors, since it transfers the problem to the description of effective classes in the group $\mathrm{N}_1(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ of curves modulo homological equivalence. The ample cone is known explicitly for very few classes of varieties, e.g., varieties with rank-one Néron–Severi group (like complete intersections of dimension ≥ 3), abelian varieties [1], and toric varieties.

The case of a surface S is somewhat special: curve and divisor classes reside in the same group, which carries the intersection form. A basic example is the smooth cubic surface $S_3 \subset \mathbb{P}^3$. Every smooth cubic surface contains 27 lines with self-intersection -1, which span the *cone*

Date: July 11, 2008.

of effective curves in $\mathrm{N}_1(S_3, \mathbb{R}) = \mathrm{N}_1(S_3, \mathbb{Z}) \otimes \mathbb{R}$, i.e., every curve on S_3 is equivalent to a sum of lines with nonnegative coefficients. The nef cone is dual to the effective cone with respect to the intersection pairing, and thus is controlled by (-1)-classes. The Néron–Severi group and the cones are preserved under small deformations of S_3.

A different situation arises for quartic surfaces $S \subset \mathbb{P}^3$ and their deformations, i.e., *K3 surfaces*. The intersection form $(,)$ can be expressed

$$H^2(S, \mathbb{Z})_{(,)} \simeq U^{\oplus 3} \oplus_{\perp} (-E_8)^{\oplus 2}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is the positive-definite unimodular quadratic form corresponding to the Dynkin diagram E_8. The sublattice $\mathrm{N}^1(S, \mathbb{Z}) \subset H^2(S, \mathbb{Z})$ depends on the surface. For general S, the Néron–Severi group $\mathrm{N}^1(S, \mathbb{Z}) \simeq \mathbb{Z}$ and every divisor class has positive self-intersection. However, deformations of S may contain new algebraic cycles, so that $\mathrm{N}^1(S, \mathbb{Z}) \simeq \mathbb{Z}^r$ with $r \in [1, \ldots, 20]$. For example, when S is a quartic surface containing a line the self-intersection of the line equals -2. The induced intersection form on $\mathrm{N}^1(S, \mathbb{Z})$ is nondegenerate and hyperbolic, but fails to be unimodular in general. Hyperbolicity means that the set of elements in $\mathrm{N}^1(S, \mathbb{R})$ with positive self-intersection splits into two convex cones $\mathcal{C}_S$ and $-\mathcal{C}_S$, where $\mathcal{C}_S$ is the component containing the ample divisors.

Select an ample divisor class $g \in \mathcal{C}_S$. The ample cone of (S, g) is controlled by (-2)-classes: a divisor h is ample if and only if for each curve C with $(C, C) \geq -2$ and $(g, C) > 0$ we have $(h, C) > 0$ [17, §2]. Thus the set of ample classes can be described simply using the intersection form on $\mathrm{N}^1(S, \mathbb{Z})$; this has numerous applications in the theory of K3 surfaces.

One main result of this article is that a similar statement holds for fourfolds F obtained by deforming the complex structure on a symmetric square of a K3 surface S blown-up along the diagonal. The self-intersection form on $H^2(F, \mathbb{Z})$ has degree four. However, this is proportional to the square of a quadratic form $(,)$ derived from the intersection form on S:

$$H^2(F, \mathbb{Z})_{(\,,\,)} \simeq U^{\oplus 3} \oplus_{\perp} (-E_8)^{\oplus 2} \oplus_{\perp} (-2), \tag{1}$$

Restrict this quadratic form to the lattice $\mathrm{N}^1(F, \mathbb{Z})$. Our main result gives *sufficient* conditions for a divisor h on F to be ample in terms of the quadratic form (1):

Theorem 1. *Let F be a projective algebraic variety deformation equivalent to the blowup of a symmetric square of a K3 surface along the* *diagonal. Fix an ample divisor g on F. A divisor h on F is ample if $(h, \rho) > 0$ for each divisor class ρ satisfying $(g, \rho) > 0$ and either*

(1) $(\rho, \rho) \geq -2$, *or*
(2) $(\rho, \rho) = -10$ *and* $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$.

This result is part of a conjectural characterization of the ample cones of such varieties [6]. The other half of the conjecture–that these conditions are *necessary* for ampleness–remains open.

Our proof of this theorem uses the fact that F is an *irreducible holomorphic symplectic variety*, i.e., a smooth projective simply-connected variety admitting a unique (up to scalar) non-degenerate holomorphic two-form. It is an interesting problem to construct new higher-dimensional irreducible holomorphic symplectic varieties. All known examples are among the following: deformations of Hilbert schemes of punctual subschemes on K3 surfaces, generalized Kummer varieties, and certain moduli spaces of simple sheaves on K3 surfaces [2],[19],[22],[21]. The geometry of these varieties is much richer than that of K3 surfaces. For example, they may admit nontrivial birational transformations which are isomorphisms on complements of subvarieties of codimension ≥ 2. Ample divisors on such models yield *moving* divisors on F, i.e., effective divisors D such that the complete linear system of some positive multiple of D has no fixed components. The *moving cone* of F, i.e., the cone in $\mathrm{N}^1(F, \mathbb{R})$ spanned by classes of moving divisors, is a birational invariant of F.

In all dimensions, we provide a *Symplectic interpretation of moving divisors* (Theorem 7): The closure of the moving cone equals the closure of the union of the nef cones of all the *nonsingular* irreducible holomorphic symplectic varieties birational to F.

The paper is organized as follows: in Section 2 we recall basic notation and constructions relating to holomorphic symplectic fourfolds. Section 3 outlines applications of the minimal model program to our situation. Section 4 offers an analysis 'from first principles' of cohomology classes of extremal rays. Finally, Section 5 contains the proof of the main theorem.

Acknowledgments: We are grateful to Daniel Huybrechts and Dmitry Kaledin for valuable conversations. The first author was partially supported by National Science Foundation Grants 0554491 and 0134259. He appreciates the hospitality of the University of Göttingen. The second author was partially supported by National Science Foundation Grants 0554280 and 0602333.

2. Generalities on ample cones of holomorphic symplectic manifolds

Let F be a irreducible holomorphic symplectic Kähler manifold and $\mathrm{N}^1(F,\mathbb{Z}) \subset H^2(F,\mathbb{Z})$ its group of divisor classes. A divisor D on F is *big* if there exists a positive constant c such that $\dim H^0(F,nD) \sim cn^{\dim(F)}$ as $n \to \infty$.

Let $\mathrm{N}_1(F,\mathbb{Z})$ denote the group of one-cycles (up to numerical equivalence), $\mathrm{NE}_1(F) \subset \mathrm{N}_1(F,\mathbb{R}) = \mathrm{N}_1(F,\mathbb{Z}) \otimes \mathbb{R}$ the cone generated by classes of effective curves, and $\overline{\mathrm{NE}}_1(F)$ its closure. Recall that $\mathbb{R}_{\geq 0}\varrho \subset \overline{\mathrm{NE}}_1(F)$ is an *extremal ray* if whenever $\varrho = c_1C_1 + c_2C_2$ for $C_1, C_2 \in \overline{\mathrm{NE}}_1(F)$ and $c_1, c_2 > 0$ then $C_1, C_2 \in \mathbb{R}_{\geq 0}\varrho$.

We recall some general facts:

- The cohomology group $H^2(F,\mathbb{Z})$ admits a quadratic form $(,)$, the *Beauville–Bogomolov form*, with signature $(3, \dim H^2(F,\mathbb{R})-3)$ on $H^2(F,\mathbb{R})$ and signature $(1, \dim H^2(F,\mathbb{R})-3)$ on $H^2(F,\mathbb{R}) \cap H^{1,1}(F,\mathbb{C})$ [8, 1.9]. We normalize this form so that it is integral but not divisible.
- There is an integral formula for the Beauville–Bogomolov form [8, §1.9] [2]. Given $\sigma \neq 0 \in H^0(F,\Omega^2_F)$ there exists a positive real constant c such that

$$(\alpha,\beta) = c\int_F \alpha\beta(\sigma\bar{\sigma})^{\dim(F)-1} \tag{2}$$

  for all $\alpha,\beta \in H^{1,1}(F,\mathbb{C})$.
- Using the duality between $H^2(F,\mathbb{Z})$ and $H_2(F,\mathbb{Z})$, the Beauville–Bogomolov form extends to a $\mathbb{Q}$-valued form on $H_2(F,\mathbb{Z})$.
- For each Chern class $c_i(F)$ there exists a constant c_i such that

$$c_i(F)\alpha^{\dim(F)-i} = c_i\,(\alpha,\alpha)^{(\dim(F)-i)/2}\,.$$

- Each divisor class D with $(D,D) > 0$ is big [8, 3.10] [9].

Let $\mathcal{C}_F$ denote the connected component of the positive cone of F

$$\{\alpha \in H^2(F,\mathbb{R}) \cap H^{1,1}(F,\mathbb{C}) : (\alpha,\alpha) > 0\}$$

containing the Kähler class. Let

$$\mathcal{K}_F \subset \mathcal{C}_F, \quad \overline{\mathcal{K}}_F \subset \overline{\mathcal{C}}_F$$

denote the *Kähler cone* of F and its closure. The intersection $\mathcal{K}_F \cap H^2(F,\mathbb{Z})$ is the set of ample divisors on F; *nef divisors* on F are defined as elements of $\overline{\mathcal{K}}_F \cap H^2(F,\mathbb{Z})$. Recall the following results of Boucksom [4] and Huybrechts [10, §3], [8, §5]:

Theorem 2. *Let F be an irreducible holomorphic symplectic Kähler manifold. A class $\alpha \in \mathcal{C}_F$ (resp. $\overline{\mathcal{C}}_F$) is in $\mathcal{K}_F$ (resp. $\overline{\mathcal{K}}_F$) if and only if $\alpha.C > 0$ (resp. $\alpha.C \geq 0$) for each rational curve $C \subset F$.*

Theorem 3. *Let F be an irreducible holomorphic symplectic Kähler manifold and $\alpha \in \mathcal{C}_F$ a 'very general' class, e.g., not orthogonal to any integral class, cf.* [8, 5.9]. *Then there exist an irreducible holomorphic symplectic Kähler manifold F' and a correspondence $\Gamma \subset F \times F'$ inducing a birational map $\phi : F' \dashrightarrow F$ such that*

- *$\Gamma_* : H^2(F, \mathbb{Z}) \to H^2(F', \mathbb{Z})$ is an isomorphism respecting the Beauville–Bogomolov forms;*
- *$\Gamma_* \alpha \in \mathcal{K}_{F'}$.*

The correspondence Γ is the specialization of the graph of an isomorphism $F'_t \xrightarrow{\sim} F_t$, where F'_t and F_t are fibers of small deformations

$$\mathcal{F}', \mathcal{F} \to \mathbf{D} := \{t : |t| < 1\}$$

of F' and F respectively.

Example 4. The simplest nontrivial example is the Atiyah flop: Let F be a K3 surface containing a (-2)-curve E and $\mathcal{F} \to \mathbf{D}$ a general deformation of F, so the cohomology class $[E]$ does not remain algebraic. Let $\mathcal{F}' \to \mathbf{D}$ denote the flop of E; the fiber F' over $t = 0$ contains a (-2)-curve E'. Note that in this case $\phi : F' \xrightarrow{\sim} F'$ but

$$\Gamma = \mathrm{Graph}(\phi) + E \times E' \subset F \times F'.$$

Remark 5. From our example, it is evident that

$$\phi^* \alpha \neq \Gamma_* \alpha$$

in general. Equality holds iff

$$\Gamma = \mathrm{Graph}(\phi) + \sum_i Z_i$$

where each Z_i maps to a codimension ≥ 2 subvariety in each factor.

Let

$$\overline{\mathcal{BK}}_F \subset \overline{\mathcal{C}}_F \subset H^2(F, \mathbb{R}) \cap H^{1,1}(F, \mathbb{C})$$

denote the closure of the *birational Kähler cone* $\mathcal{BK}_F$, i.e.,

$$\mathcal{BK}_F = \cup_f f^* \mathcal{K}_{F'}$$

where the union is taken over all birational maps $f : F \dashrightarrow F'$ to an *irreducible holomorphic symplectic* Kähler manifold F'. This has the following numerical interpretation:

Proposition 6. [10, 4.2] *Let F be an irreducible holomorphic symplectic Kähler manifold. A class $\alpha \in \overline{\mathcal{C}}_F$ lies in $\overline{\mathcal{BK}}_F$ if and only if $(\alpha, D) \geq 0$ for each uniruled divisor $D \subset F$.*

Theorem 7 (Symplectic interpretation of moving divisors)**.** *Let F be an irreducible holomorphic symplectic Kähler manifold. Each moving divisor is contained in the closure of the birational Kähler cone $\overline{\mathcal{BK}}_F$.*

Remark 8. Corollary 19 below is a partial converse to this result.

Proof. We are grateful to Professor D. Huybrechts for his help with this argument.

Suppose that M is moving. To show that $M \in \overline{\mathcal{BK}}_F$, it suffices to prove that $(M, D) \geq 0$ for each irreducible uniruled divisor $D \subset F$. (Our argument below only requires that D be effective.) We write $2n = \dim(F)$.

Replacing M by a suitable multiple if necessary, we may assume that M has no fixed components, i.e., its base locus has codimension at least two. There exists a diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & F' \\ {\scriptstyle q}\downarrow & & \\ F & & \end{array}$$

where $Z \to F$ is a smooth projective resolution of the base locus of $|M|$ and p is the resulting morphism. Thus there exists an ample divisor H on F' such that

$$q^*M = \sum_i c_i E_i + p^*H,$$

where each $c_i \geq 0$ and E_i is a q-exceptional divisor in Z.

Compute the Beauville–Bogomolov form by pulling back to Z:

$$\begin{array}{rcl} (M, D) & = & c\int_F [M][D](\sigma\bar{\sigma})^{n-1} \\ & = & c\int_Z q^*[M]q^*[D]q^*((\sigma\bar{\sigma})^{n-1}) \\ & = & c\int_Z (\sum_i c_i[E_i] + p^*[H])(q^*[D])q^*((\sigma\bar{\sigma})^{n-1}). \end{array}$$

First, note that

$$\int_Z [E_i]q^*[D]q^*((\sigma\bar{\sigma})^{n-1}) = 0.$$

Indeed, any degree-$(4n-2)$ form pulled back from F integrates to zero along E_i because $\mathrm{codim}_{\mathbb{R}}(q(E_i), F) \geq 4$. To evaluate the second term

$$\int_Z p^*[H]q^*[D]q^*((\sigma\bar{\sigma})^{n-1}),$$

observe that the intersection $p^*[H]\cap q^*[D]$ involves a semiample divisor and an effective divisor. In particular, we can express

$$p^*[H]\cap q^*[D]=\sum_j n_j[W_j], \quad n_j>0,$$

where each W_j is a $(2n-2)$-dimensional subvariety of Z. Thus we have

$$\int_Z p^*[H]q^*[D]q^*((\sigma\bar{\sigma})^{n-1})=\sum_j n_j\int_{W_j} q^*((\sigma\bar{\sigma}))^{n-1}.$$

Let $\tilde{W}_j\to W_j$ denote a resolution of singularities and $r:\tilde{W}_j\to F$ the induced morphism. We have

$$\int_{W_j} q^*((\sigma\bar{\sigma})^{n-1})=\int_{\tilde{W}_j}(r^*\sigma\overline{r^*\sigma})^{n-1}\geq 0$$

because the integrand is a nonnegative multiple of the volume form on $\tilde{W}_j$. □

From now on, we assume that F is projective; in this case, we call F an irreducible holomorphic symplectic *variety*. Recall that:

- An irreducible holomorphic symplectic Kähler manifold F is projective if and only if there exists a divisor D on F with $(D,D)>0$ [8, 3.11].
- Let D be a nef and big divisor class on F. By Kawamata–Viehweg vanishing [15, Theorem 2.64], D has no higher cohomology. By basepoint-freeness [15, Theorem 3.3], ND is globally generated for some $N\gg 0$.

Note that

$$\overline{\mathcal{BK}}_F\cap \mathrm{N}^1(F,\mathbb{R})=\overline{\mathcal{BK}_F\cap \mathrm{N}^1(F,\mathbb{R})},$$

i.e., the closure of the *birational ample cone*, which is the union of the pull-backs of the ample cones of all *irreducible holomorphic symplectic* varieties birational to F. Indeed, whether a class $\alpha\in\mathcal{C}_F$ is Kähler is determined by its image under the projection

$$H^{1,1}(F,\mathbb{C})\cap H^2(F,\mathbb{R})\to \mathrm{N}^1(F,\mathbb{R})$$

dual to the inclusion

$$N_1(F,\mathbb{R})\hookrightarrow H_2(F,\mathbb{R})\cap H_{1,1}(F,\mathbb{C});$$

this follows from Theorem 2. Thus taking closures is compatible with restricting to $\mathrm{N}^1(F,\mathbb{R})$.

Remark 9. In the projective case, Theorem 7 implies that the closure of the birational ample cone is the closure of the moving cone. Indeed, elements of $H^2(F,\mathbb{Z}) \cap \mathcal{BK}_F$ correspond to ample divisors on some model of F and thus are moving divisors.

Definition 10. Let $\overline{\mathcal{C}} \subset \mathbb{R}^n$ be a closed convex cone with nonempty interior. We say that $\overline{\mathcal{C}}$ is *locally finite rational polyhedral* at $M \in \overline{\mathcal{C}}$ if there exists an open neighborhood V of M such that $\overline{\mathcal{C}} \cap V$ is defined in V by a finite number of rational linear inequalities.

Theorem 2 shows that the Kähler cone is controlled by classes of rational curves, but this does not imply *a priori* that these classes determine a locally finite rational polyhedral cone, nor does it provide a geometric interpretation of these rational curves. Even for K3 surfaces the cone of curves can be quite intricate [16]. The Cone Theorem sheds some further light on this:

Proposition 11 (Cone Theorem for K-trivial varieties)**.** [15, 3.7] *Let Y be a smooth projective variety with $K_Y = 0$ and Δ an effective $\mathbb{Q}$-divisor on Y. Then the closed cone of effective curves $\overline{\mathrm{NE}}_1(Y)$ can be expressed*

$$\overline{\mathrm{NE}}_1(Y) = \overline{\mathrm{NE}}_1(Y)_{\Delta.C\geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j], \quad \Delta.C_j < 0,$$

where the C_j are extremal and represent rational curves collapsed by contractions of Y. This is locally finite in the following sense: Given an ample divisor A and $\epsilon > 0$, there are a finite number of C_j with $C_j.(\Delta + \epsilon A) < 0$.

Remark 12. This differs slightly from the standard statement of the Cone Theorem in that we are making no assumptions on the singularities of (Y, Δ). Normally, one assumes that the pair has Kawamata log terminal singularities (see [13, 2.13] for the definition). However, we can always choose $\epsilon > 0 \in \mathbb{Q}$ such that $(Y, \epsilon\Delta)$ is Kawamata log terminal. Indeed, since Y is smooth if we choose ϵ such that

$$1/\epsilon > \max_{x\in Y}\{\mathrm{mult}_x(\Delta)\}$$

then the singularities are Kawamata log terminal by [14, 8.10].

Which parts of $\overline{\mathrm{NE}}_1(F)$ can be analyzed using the Cone Theorem?

Proposition 13. *Let F be an irreducible holomorphic symplectic variety and D a big divisor class on F. Then there exist a finite collection of rational hyperplanes separating D from $\overline{\mathcal{K}}_F$, i.e., if $\langle D, \overline{\mathcal{K}}_F\rangle$ is the* *cone generated by D and $\overline{\mathcal{K}}_F$ then*

$$\overline{\mathcal{K}}_F \subset \langle D, \overline{\mathcal{K}}_F \rangle$$

is determined by a finite number of rational linear inequalities. Thus $\overline{\mathcal{K}}_F$ is locally finite rational polyhedral at divisors in $\mathcal{C}_F$.

Proof. Express $D = \Delta + \epsilon A$ for Δ effective, A ample, and $\epsilon > 0$ a small rational number. The Cone Theorem (Proposition 11) asserts that D intersects the generators of the cone of curves positively, except for a finite number of extremal rays $C_1, \ldots, C_n$ with $D.C_j < 0$. Theorem 2 shows that $\overline{\mathcal{K}}_F$ is the subcone of $\overline{\mathcal{C}}_F$ dual to $\overline{\mathrm{NE}}_1(F)$. Thus a class

$$tD + (1-t)A \in \langle D, \overline{\mathcal{K}}_F \rangle, \quad t \in [0,1], A \in \overline{\mathcal{K}}_F,$$

is in the closure of the Kähler cone if and only if

$$tD.C_j + (1-t)A.C_j \geq 0, \quad j = 1, \ldots, n.$$

The last assertion follows from the fact that divisors in $\mathcal{C}_F$ are automatically big. □

The extremal rays described in the Cone Theorem have negative Beauville-Bogomolov form:

Proposition 14. *Let F be an irreducible holomorphic symplectic variety. Suppose there exists a Kawamata log terminal pair (F, Δ) such that*

$$(K_F + \Delta).R = \Delta.R < 0.$$

Then $(R, R) < 0$ and the extremal contraction associated with R is birational.

Proof. The extremal contraction $\beta : F \to F'$ associated with R is discussed in [15, Theorem 3.7(3)]. This is characterized as a projective morphism with connected fibers contracting precisely the curves with classes proportional to R. First suppose that β is birational. Choose an ample divisor A' on F' and consider its pull-back $A = \beta^* A'$, which is nef and big on F. We have $(A, A) > 0$ and $A.R = 0$, and the Beauville-Bogomolov form has signature $(1, \dim \mathrm{N}^1(F, \mathbb{R}) - 1)$ on the Néron-Severi group. It follows that $(R, R) < 0$.

Now suppose that β has positive-dimensional fibers, in which case it is almost an abelian fibration, in the sense that the generic fiber Z admits an étale covering $\gamma : \tilde{Z} \to Z$ by an abelian variety [18]. For each curve $C \subset Z$, the class $[C] \in H_2(F, \mathbb{Z})$ equals rR for some $r > 0$ Since β is a fibration we have

$$\Delta.R = \frac{1}{r}(\Delta \cap Z).C = \frac{1}{r \deg(\gamma)} \gamma^*(\Delta \cap Z).\gamma^* C.$$

However, the last intersection number cannot be negative; a curve and an effective divisor on an abelian variety meet with nonnegative intersection number. This contradicts our hypothesis. □

3. Application of the log minimal model program

We will use the following consequence of the log minimal model program:

Theorem 15. *Let Y be a smooth projective variety with K_Y trivial. Suppose that $D_1, \ldots, D_r$ are big divisors on Y. Then the ring*

$$\oplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}}\Gamma(F, \mathcal{O}_F(n_1D_1 + \ldots + n_rD_r))$$

is finitely generated.

Proof. There exists a positive $\epsilon \in \mathbb{Q}$ such that each ϵD_i has divisorial log terminal singularities (see [13, 2.13] for the definition). Indeed, if we choose ϵ such that

$$1/\epsilon > \max_{y\in Y, i=1,\ldots,r}\{\mathrm{mult}_y(D_i, y)\}$$

then [14, 8.10] guarantees the singularities have the desired property. It follows from [3, 1.1.9] that the graded ring

$$\oplus_{(m_1,\ldots,m_r)\in\mathbb{Z}^r_{\geq 0}}\Gamma(F, \mathcal{O}_F(\lfloor\sum_i m_i\epsilon D_i\rfloor))$$

is finitely generated. It remains finitely generated when we restrict to the multidegrees such that each $m_i\epsilon \in \mathbb{Z}$. □

Definition 16. Assume $\overline{\mathcal{C}} \subset \mathbb{R}^n$ is a closed convex cone. A *rational chamber decomposition* of $\overline{\mathcal{C}}$ is a stratification by locally closed subcones or *chambers*

$$\overline{\mathcal{C}} = \sqcup_i \mathcal{C}_i$$

induced by a finite collection $\{H_j\}_{j\in J}$ of rational codimension-one linear subspaces $H_j \subset \mathbb{R}^n$. Precisely, consider the stratification of $\mathbb{R}^n$ into locally closed subsets characterized as the points contained in some of the hyperplanes but not contained in others. The *chambers* $\mathcal{C}_i$ are defined as the connected components of the intersections of these strata with $\overline{\mathcal{C}}$. Thus the (relatively) open chambers are the connected components of $\overline{\mathcal{C}} \setminus \cup_{j\in J}H_j$.

Let $\mathcal{D} \subset \mathbb{R}^n$ be a convex cone with nonempty interior. A *locally finite rational chamber decomposition* of $\mathcal{D}$ is a decomposition as a disjoint union of connected subcones

$$\mathcal{D} = \sqcup_i \mathcal{D}_i$$

such that, for each rational polyhedral subcone $\overline{\mathcal{C}} \subset \mathcal{D}$, the induced decomposition

$$\overline{\mathcal{C}} = \sqcup_i (\mathcal{D}_i \cap \overline{\mathcal{C}})$$

is a rational chamber decomposition.

Proposition 17. *Let F be a (projective) irreducible holomorphic symplectic variety. Consider the collection of open subcones*

$$\sqcup_{F''} \mathcal{K}_{F''} \subset \mathcal{BK}_F, \tag{3}$$

where the union is taken over irreducible holomorphic symplectic birational models of F, and the corresponding collection

$$\sqcup_{F''} (\mathcal{K}_{F''} \cap \mathrm{N}^1(F,\mathbb{R})) \ \subset \ \mathcal{BK}_F \cap \mathrm{N}^1(F,\mathbb{R}). \tag{4}$$

These are the open chambers of locally finite rational chamber decompositions of $\overline{\mathcal{BK}}_F \cap \mathcal{C}_F$ and $\overline{\mathcal{BK}}_F \cap \mathcal{C}_F \cap \mathrm{N}^1(F,\mathbb{R})$ respectively.

Kawamata [11, Theorem 2.6] has similar results for Calabi-Yau threefolds.

Proof. We first analyze the chamber decomposition of $\overline{\mathcal{BK}}_F \cap \mathcal{C}_F \cap$ $\cap \mathrm{N}^1(F,\mathbb{R})$. Recall that divisors $D \in \mathcal{C}_F$ are big. Thus each element of $\mathcal{C}_F \cap N^1(F,\mathbb{R})$ is contained in a polyhedral cone

$$\langle D_1, \ldots, D_r \rangle$$

where $D_1, \ldots, D_r$ are big divisors generating $\mathrm{N}^1(F,\mathbb{Z})$.

Consider the associated graded ring

$$\mathrm{R}(D_1,\ldots,D_r) := \oplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}} \Gamma(F, \mathcal{O}_F(n_1D_1 + \ldots + n_rD_r)),$$

which is finitely generated by Theorem 15. As discussed in [7, 2.9], this finite generation has implications for the birational geometry of F:

- the subcone
$$\overline{\mathcal{K}}_F \cap \langle D_1, \ldots, D_r \rangle \subset \langle D_1, \ldots, D_r \rangle$$
is determined by a finite number of linear rational inequalities;
- the intersection of the closure of the moving cone with $\langle D_1, \ldots, D_r \rangle$ admits a chamber decomposition
$$\overline{\mathcal{BK}}_F \cap \langle D_1, \ldots, D_r \rangle \ = \ \sqcup_{F'} (\mathcal{K}_{F'} \cap \langle D_1, \ldots, D_r \rangle), \tag{5}$$
where each $F \overset{\sim}{\dashrightarrow} F'$ is a small birational modification.

Indeed, the chambers correspond to the various Geometric Invariant Theory quotients of $\mathrm{R}(D_1, \ldots, D_r)$ under the $\mathbb{G}_m^r$-action associated with the multigrading. We consider linearizations of the action corresponding to positive characters of $\mathbb{G}_m^r$.

Now assume F' is a small modification corresponding to an open chamber. *A priori*, F' might be very singular. However, each polarization on F' pulls back to a moving divisor M on F; the 'Symplectic interpretation of moving divisors' (Theorem 7) implies M is contained in the closure of the birational Kähler cone. The finiteness analysis above implies that a sufficiently general M is actually contained in $\mathcal{BK}_F$. Thus there exists a birational modification $F \dashrightarrow F''$ to a polarized holomorphic symplectic variety such that the polarization pulls back to M. Consequently, F' and F'' are isomorphic. This proves that the open chambers of (5) are parametrized by smooth holomorphic symplectic varieties; thus the collection of open subcones (4) induces a chamber decomposition of $\overline{\mathcal{BK}}_F \cap \mathcal{C}_F \cap \mathrm{N}^1(X, \mathbb{R})$.

The hyperplanes inducing the chamber decompositions (5) correspond to extremal rays contracted by moving divisors on birational models F'' of F. Recall (Theorem 2) that each $\overline{\mathcal{K}}_{F''}$ is dual to the cone generated by rational curves of F''. The extremal rays determine hyperplanes in $H^2(F, \mathbb{R}) \cap H^{1,1}(X, \mathbb{C})$ which induce a chamber decomposition of the birational Kähler cone

$$\cup_{F''} \mathcal{K}_{F''} \subset \mathcal{BK}_F,$$

where the union is over the holomorphic symplectic models F'' or F. Thus the open subcones (3) induce a chamber decomposition of $\overline{\mathcal{BK}}_F \cap \mathcal{C}_F$. □

Recall that the Beauville-Bogomolov form on $H^2(F, \mathbb{R})$ induces a form $(,)$ on $H_2(F, \mathbb{R})$ by duality. The following result should be read with the Cone Theorem and Proposition 14 in mind:

Corollary 18. *Let F be an irreducible holomorphic symplectic variety. Then the intersection*

$$\overline{\mathrm{NE}}_1(F) \cap \{R \in H_2(F, \mathbb{R}) : (R, R) < 0\}$$

is locally finite rational polyhedral.

Proof. Supporting hyperplanes to $\overline{\mathrm{NE}}_1(F)$ in the region

$$\{R : (R, R) < 0\}$$

correspond to divisor classes M with $(M, M) > 0$, and Proposition 17 applies. □

Corollary 19. *Let F be an irreducible holomorphic symplectic variety. Each divisor $M \in \overline{\mathcal{BK}}_F \cap \mathcal{C}_F$ is moving.*

Proof. Proposition 17 implies that M corresponds to a nef and big divisor M' on some small birational modification $F \overset{\sim}{\dashrightarrow} F'$, where

F' is a projective irreducible holomorphic symplectic variety. Thus basepoint-freeness implies that some multiple of M' is basepoint free. Since F and F' are isomorphic in codimension one, we conclude that M is moving on F. □

Remark 20. This analysis only applies to divisor classes with *positive* Beauville–Bogomolov form. The case where the form is zero remains open (cf. Conjecture 25).

Remark 21. The underlying techniques here are reminiscent of those used in the proof that 'minimal models are connected by flops' [12] [3, 1.1.3].

4. Deriving (−2) and (−10)-classes from first principles

Let F be an irreducible holomorphic symplectic variety deformation equivalent to $S^{[2]}$, the Hilbert scheme of length-two subschemes on a K3 surface. The Beauville–Bogomolov form can be written [23, §2]:

$$H^2(F,\mathbb{Z})_{(\,,\,)} \simeq U^{\oplus 3} \oplus_{\perp} (-E_8)^{\oplus 2} \oplus_{\perp} (-2), \tag{6}$$

where U is the hyperbolic plane and E_8 the positive-definite integral lattice associated to the corresponding root system. We have

$$\alpha^4 = 3\,(\alpha,\alpha)^2$$

for each $\alpha \in H^2(F,\mathbb{Z})$. By duality, there is an induced $\frac{1}{2}\mathbb{Z}$-valued quadratic form on $H_2(F,\mathbb{Z})$:

$$H_2(F,\mathbb{Z})_{(\,,\,)} \simeq U^{\oplus 3} \oplus_{\perp} (-E_8)^{\oplus 2} \oplus_{\perp} (-1/2).$$

We recall additional properties of the cohomology ring $H^*(F,\mathbb{Z})$ (see [23, §2]):

- The intersection product induces an isomorphism
  $$\mathrm{Sym}^2 H^2(F,\mathbb{Q}) \xrightarrow{\sim} H^4(F,\mathbb{Q})$$
  and the intersection form on the middle cohomology is given by the formula
  $$\alpha_1\alpha_2.\alpha_3\alpha_4 = (\alpha_1,\alpha_2)\,(\alpha_3,\alpha_4) + (\alpha_1,\alpha_3)\,(\alpha_2,\alpha_4) + (\alpha_1,\alpha_4)\,(\alpha_2,\alpha_3)$$
  for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in H^2(F,\mathbb{Z})$.
- There is a distinguished class $q^\vee \in H^4(F,\mathbb{Q}) \cap H^{2,2}(F,\mathbb{C})$ such that
  $$q^\vee.\alpha_1.\alpha_2 = 25\,(\alpha_1,\alpha_2)$$
  for all $\alpha_1,\alpha_2 \in H^2(F,\mathbb{Z})$. This is a rational multiple of the dual Beauville–Bogomolov form induced on $H_2(F,\mathbb{Z})$ via Poincaré duality.

- We have the formulas

$$c_2(F) = \frac{6}{5} q^\vee, \quad q^\vee . q^\vee = 23 \cdot 25.$$

Theorem 22 (Classification of Extremal Rays)**.** *Let F be an irreducible holomorphic symplectic fourfold such that there exists an isomorphism*

$$\psi : H^2(F, \mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]}, \mathbb{Z}),$$

with $\psi(\alpha)^4 = \alpha^4$ for each $\alpha \in H^2(F, \mathbb{Z})$, where S is a K3 surface and $S^{[2]}$ its Hilbert scheme of length-two subschemes. Suppose $R \in \mathrm{N}_1(F, \mathbb{Z})$ is an extremal ray such that there exists a Kawamata log terminal effective divisor $\Delta \subset F$ with $\Delta . R < 0$. Then we have

$$(R, R) = -1/2, -2, -5/2.$$

Moreover, $\mathrm{N}^1(F, \mathbb{Z})$ contains an element ρ satisfying one of the following:

- $(\rho, \rho) = -2$ *and* $(\rho, H^2(F, \mathbb{Z})) = \mathbb{Z}$*;*
- $(\rho, \rho) = -2$ *and* $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$*;*
- $(\rho, \rho) = -10$ *and* $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$*.*

Proof. Proposition 14 guarantees that $(R, R) < 0$. Again, consider the extremal contraction $\beta : F \to F'$ with $\beta_* R = 0$ and $\mathrm{Pic}(F/F') \simeq \mathbb{Z}$.

We use the partial description of extremal contractions [25], [26, 1.1], [20, 1.4,1.11], [5]. The morphism $\beta : F \to F'$ satisfies one of the following alternatives:

- β is a divisorial contraction taking the exceptional divisor to a surface $T \subset F'$. At each smooth point of T, β is locally a contraction to a two-dimensional rational double point.
- β is a small contraction, taking a smooth Lagrangian $\mathbb{P}^2 \subset F$ to an isolated singularity of F'.

In the divisorial case, the smooth locus of T has codimension ≥ 2 complement and admits a holomorphic symplectic form.

Consider first the divisorial case. Suppose that D is the exceptional divisor of β; the generic fiber of $\beta | D : D \to T$ is an ADE-configuration of $\mathbb{P}^1$'s. Since β is extremal, the fundamental group of T^{sm} acts transitively on the components of $\beta^{-1}(t)$ for $t \in T$ generic. An analysis of intersection numbers implies that only A_1 and A_2 configurations may occur (see [25, 5.1]).

Let $\tilde{D}$ denote the normalization of D and

$$\tilde{D} \xrightarrow{\gamma} \tilde{T} \to T$$

the Stein factorization of $\beta | \tilde{D}$. Then the generic fiber $C = \gamma^{-1}(t)$ is isomorphic to $\mathbb{P}^1$. However, the classification of rational double points yields

$$\mathcal{N}_{\tilde{D}/F}|C \simeq \mathcal{O}_{\mathbb{P}^1}(-2),$$

hence

$$D.C = -1, -2.$$

This analysis only requires that F is an irreducible holomorphic symplectic fourfold.

We will now use integrality properties of the Beauville–Bogomolov form. Let $\rho \in \mathrm{N}^1(F, \mathbb{Z})$ denote the primitive class identified with a positive multiple of the extremal ray R via the Beauville–Bogomolov form. Precisely, for each $A \in H^2(F, \mathbb{Z})$ we have

$$A.R = r\,(A, \rho)$$

with $r = 1, 1/2$ depending on whether $(R, H_2(F, \mathbb{Z})) = \mathbb{Z}, \frac{1}{2}\mathbb{Z}$. Since C and D are contracted under the extremal $\beta : F \to F'$, $C = mR$ and $D = n\rho$ for $m, n \in \mathbb{N}$, and we have

$$D.C = mnR.\rho = mnr\,(\rho, \rho)\,.$$

The following cases may occur:

- (I) $D.C = -1$:
  - (a) $r = 1$: Here $m = n = 1$ and $R.\rho = -1$, hence $(\rho, \rho) = -1$ which is impossible because $(,)$ is even-valued.
  - (b) $r = 1/2$: Here $mn\,(\rho, \rho) = -2$ and thus $(\rho, \rho) = -2$. We conclude that $(R, R) = -1/2$.
- (II) $D.C = -2$:
  - (a) $r = 1$: Here we have $(\rho, \rho) = -2/mn$ which forces $m = n = 1$ and $(\rho, \rho) = -2$. We conclude that $(R, R) = -2$.
  - (b) $r = 1/2$: Here $(\rho, \rho) = -4/mn$ so $mn = 1$ or 2. However, the lattice (6) does not admit primitive vectors ρ of length four with $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$. Indeed, if we had

$$\rho = 2v + a\delta, \quad 2 \nmid a$$

with

$$v \in U^{\oplus 3} \oplus (-E_8)^{\oplus 2}, (\delta, \delta) = -2, (v, \delta) = 0,$$

then it would follow that

$$(\rho, \rho) = 4\,(v, v) - 2a^2 \equiv -2 \pmod 8. \tag{7}$$

We conclude that $mn = 2$, $(\rho, \rho) = -2$, and $(R, R) = -1/2$.

This completes the proof in the divisorial case.

We turn to the case where $\beta : F \to F'$ is a small contraction of a Lagrangian $\mathbb{P}^2$. Some multiple of the extremal ray R is necessarily the class L of a line in $\mathbb{P}^2$. We shall show that $(L, L) = -5/2$ which implies that $R = L$, completing the proof of the theorem.

Suppose that $\lambda \in H^2(F, \mathbb{Z})$ is the unique class with

$$2A.L = (A, \lambda)$$

for all $A \in H^2(F, \mathbb{Z})$. Note that $(\lambda, \lambda) < 0$ because λ is nonzero and proportional to R. We do not assume *a priori* that λ is primitive. Consider a deformation F_t of F for which $[L] \in H_2(F_t, \mathbb{Z})$ (or equivalently, λ) remains a Hodge class. The Lagrangian plane also deforms in F_t (see [24] and [6]). For a general deformation F_t, the only Hodge classes in $H^4(F_t, \mathbb{Z})$ are rational linear combinations of $q^\vee$ and λ^2. Indeed, the Torelli map is locally an isomorphism and $q^\vee, \lambda^2 \in H^4(F_t, \mathbb{Q})$ are the only Hodge classes in $\mathrm{Sym}^2 H^2(F_t, \mathbb{Z})$ for generic Hodge structures on $H^2(F_t, \mathbb{Z})$ (see [23, §3] for a detailed proof).

We may put

$$[\mathbb{P}^2] = aq^\vee + b\lambda^2. \tag{8}$$

Geometric properties of the Lagrangian plane translate into algebraic conditions on the coefficients a, b; we use the intersection properties listed above:

- The normal bundle to any Lagrangian submanifold is equal to its cotangent bundle. Thus we have

  $$[\mathbb{P}^2].[\mathbb{P}^2] = c_2(\Omega^1_{\mathbb{P}^2}) = 3$$

  which implies

  $$25 \cdot 23a^2 + 50ab\,(\lambda, \lambda) + 3b^2\,(\lambda, \lambda)^2 = 3.$$

- Using the exact sequence

  $$0 \to \mathcal{T}_{\mathbb{P}^2} \to \mathcal{T}_F|\mathbb{P}^2 \to \mathcal{N}_{\mathbb{P}^2/F} \to 0$$

  we compute that $c_2(T_F)|\mathbb{P}^2 = -3$. It follows that

  $$-3 = \frac{6}{5}(25 \cdot 23a + 25b\,(\lambda, \lambda)).$$

- We know that $\lambda|\mathbb{P}^2$ is some multiple of the hyperplane class, i.e., $\lambda.[\mathbb{P}^2] = \big(\lambda.L\big)\,L$. We deduce that

  $$\lambda.\lambda.[\mathbb{P}^2] = \big(\lambda.L\big)^2 = (\lambda, \lambda)^2/4.$$

  Using formula (8) to evaluate $\lambda.\lambda.[\mathbb{P}^2]$ we obtain

  $$(\lambda, \lambda)^2/4 = 25a\,(\lambda, \lambda) + 3b\,(\lambda, \lambda)^2.$$

Altogether, we obtain three Diophantine equations in the variables $(\lambda, \lambda), a$, and b. Eliminating a and b and solving for (λ, λ) we obtain the quadratic equation

$$23\,(\lambda, \lambda)^2 + 20\,(\lambda, \lambda) - 2100 = 0$$

with solutions $(\lambda, \lambda) = -10, 210/23$. Only the first solution makes sense. We conclude that $(L, L) = -5/2$, λ is primitive, and $(\lambda, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$. $\square$

5. Applications to ample divisors

Theorem 23 (Main theorem). *Let (F, g) be a polarized irreducible holomorphic symplectic variety deformation equivalent to the Hilbert scheme of length-two subschemes on a K3 surface. A divisor h on F is ample if $(h, \rho) > 0$ for each divisor class ρ such that $(\rho, g) > 0$ and one of the following holds:*

1. $(\rho, \rho) \geq 0$;
2. $(\rho, \rho) = -2$ *and* $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$;
3. $(\rho, \rho) = -2$ *and* $(\rho, H^2(F, \mathbb{Z})) = \mathbb{Z}$;
4. $(\rho, \rho) = -10$ *and* $(\rho, H^2(F, \mathbb{Z})) = 2\mathbb{Z}$.

Equivalently, $h.R > 0$ for each curve class R such that $R.g > 0$ and one of the following holds:

1. $(R, R) \geq 0$;
2. $(R, R) = -\frac{1}{2}$;
3. $(R, R) = -2$,
4. $(R, R) = -\frac{5}{2}$.

Let $\mathrm{N}_E(F, g) \subset \mathrm{N}_1(F, \mathbb{R})$ denote the smallest real cone containing these four types of classes. Classes of the last three types that are extremal in the closure of $\mathrm{N}_E(F, g)$ will be called *nodal classes* (cf. [17, 1.4]).

Proof of Theorem 23. Let h be a divisor satisfying the hypotheses, so in particular $(h, g) > 0$. We claim that $(h, h) > 0$, which guarantees $h \in \mathcal{C}_F$. Indeed, if $(h, h) \leq 0$ then the lattice generated by g and h is hyperbolic with respect to $(,)$. However, for each

$$\rho \in (\mathbb{R}h + \mathbb{R}g) \cap \overline{\mathcal{C}}_F,$$

$(h, \rho) \geq 0$ with strict inequality whenever ρ is integral. This happens only when $(h, h) > 0$, contradicting our assumption.

Suppose that h fails to be ample. After a small perturbation of g, the line segment

$$th + (1 - t)g, \quad t \in [0, 1]$$

meets the boundary of the ample cone of F in the interior of some facet (codimension-one face) of the nef cone. Indeed, Proposition 13 shows that the nef cone is locally finite rational polyhedral at big divisors. Thus the value

$$\tau := \sup\{t : th + (1-t)g \text{ is ample}\} \in (0,1]$$

is rational.

Let R be the (primitive, integral) generator of the extremal ray corresponding to our facet. We have

$$(\tau h + (1-\tau)g).R = 0 \quad \text{and} \quad g.R > 0,$$

so $h.R \leq 0$. The Classification of Extremal Rays (Theorem 22) implies that

$$(R,R) = -1, -2, -5/2$$

whence R is a nodal class. This contradicts our assumption that $h.R > 0$ for each such class. □

In general, we conjecture that each nodal class arises as the extremal ray associated with a birational contraction:

Conjecture 24 (Nodal classes conjecture)**.** Each nodal class R represents a rational curve contracted by a birational morphism β given by sections of $\mathcal{O}_F(m\lambda), m \gg 0$, where λ is any nef and big divisor class with $R.\lambda = 0$.

(1) If $(R,R) = -\frac{1}{2}, -2$ (i.e., the corresponding ρ is a (-2)-class) then ρ is represented by a family of rational curves parametrized by a K3 surface, which blow down to rational double points.
(2) If $(R,R) = -\frac{5}{2}$ (i.e., the corresponding ρ is a (-10)-class) then ρ is represented by a family of lines contained in a $\mathbb{P}^2$ contracted to a point.

The remaining generators of the cone of curves are given by:

Conjecture 25 (Square-zero class conjecture)**.** [6, 3.8] Let λ be a primitive class on the boundary of the nef cone with $(\lambda,\lambda) = 0$. Then the corresponding line bundle $\mathcal{O}_F(\lambda)$ has no higher cohomology and its sections yield a morphism

$$F \to \mathbb{P}^2$$

whose generic fiber is an abelian surface.

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